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Selection of Weights in Optimal Control

G S Virk and J M Tahir

Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield S1 3JD
U K

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Abstract

A method to design the weighting matrices in the optimal control of an aircraft is presented. The method is equally suitable for any general multivariable application provided some prior knowledge is available to enable the rankings and the determination of the important terms.

1 Introduction

In the design of control systems the performance assessment of a particular design is usually very subjective. For example the assessment may take factors such as percentage overshoot, steady state errors, settling times and rise-times into account but using these, it is difficult to categorise the "best" design. In addition, the design phase normally takes an iterative form where the controller design and its assessment are repeated several times essentially following a trial and error design procedure until a satisfactory control law is reached. In many applications the design is dependent upon multiple performance and constraint requirements, and so the general trial and error can prove exceedingly difficult. The situation is compounded if an application calls for time varying control action forcing the control engineer to select not only the controller configuration but also the time varying gains using an ad hoc approach. For such cases optimal control techniques can offer distinct advantages since the designs are reduced to the selection of weighting factors in a performance index. The commonest form of cost function in current use is a quadratic function taking the form

$$\min_{u} \frac{1}{2}\int_0^T \left\{x^T(t)Qx(t) + u^T(t)Ru(t)\right\} dt \tag{1}$$

where Q is an $n \times n$ symmetric positive semi-definite matrix, R is a $m \times m$ symmetric positive definite matrix for a system of state $x(t)$ with dimension n, and m inputs $u_1, u_2, \cdots, u_m$.

It is well known that the x^TQx term relates to system errors and u^TRu relates to control effort. By varying Q and R the emphasis given to the individual terms can be varied for different objectives. For example, as Q is increased tighter regulatory performance is sought, which usually requires larger controlling signals. Similarly as R is increased, control effort is penalised more and so large control signals are avoided, but this generally leads to worsening of the errors. It is therefore clear that the proper balance between Q and R needs to be reached so that good regulation is obtained without using excessive control.

Now although the theory of optimal control is well documented, see Banks [1] and Bryson and Ho [2], and leads to the determination of optimal solutions for general problems, the proper formulation of such problems is not. The Q and R matrices in equation (1) are usually set to be diagonal matrices and the design follows the classical procedure where the weighting elements are changed using a trial and error method until an adequate response is obtained.

The weights selection procedure can get very complicated when multivariable systems are considered where there is significant cross-coupling between the various modes. To handle such cases efficiently Q and R ideally need to be general matrices and not restricted to be of diagonal form. This however leads to the need for selection of even more weighting terms which are tightly coupled to others, and so the above trial and error approach is not feasible. It is difficulties such as these, and the level of mathematical analysis required, that have restricted the wide spread adoption of optimal control techniques in practical applications and so a better design procedure is required.

In this paper a suitable approach is suggested based on a proper analysis of the system dynamics under consideration. We will concentrate on an aircraft optimal autopilot to illustrate the approach but the method is equally applicable to any multivariable problem. A receding/moving horizon optimal control problem is considered suitable for handling the time-varying aircraft equations in a real-time on-line situation, see Kwon [3]. We start by giving a short introduction to the receding horizon problem.

2 Receding Horizon Optimal Control

Consider the time varying linear system described by

$$\begin{aligned} \dot{e}(t) &= A(t)e(t) + B(t)\Delta u(t) \qquad \text{for a.a. } t \in [t_0, t_f] \qquad (2)\\ e(t_0) &= e_0 \qquad (3) \end{aligned}$$

where $e(t)$ is the state error vector of dimension n, $\Delta u(t)$ is the control deviation vector of dimension m, A is the $n \times n$ system matrix, B is the $n \times m$ input matrix, e_0 is the initial error at time t_0, and a.a. stands for almost all.

Let us minimise a quadratic cost function V_f over a fixed interval $[t_0, t_f]$ where $t_f = t_0 + T$ for some horizon length T seconds, and

$$V_f^u(x) = \frac{1}{2}\int_{t_0}^{t_f} \left\{e^T(t)Qe(t) + \Delta u^T(t)R\Delta u(t)\right\} dt + \frac{1}{2}e^T(t_f)Fe(t_f) \qquad (4)$$

Here Q, R and F are symmetric matrices which contain the designed weighting parameters. We will assume that Q and F are $n \times n$ positive semi-definite matrices and R is a $m \times m$ positive definite diagonal matrix.

This is the standard regulator problem where the Q, F and R matrices need to be selected so that the minimising control will drive the initial error e_0 towards the origin in an optimal manner. It is well known that the optimal control input that minimises V_f is provided by the following state-feedback law

$$\Delta u^o(t) = -R^{-1}B^T(t)K(t)e(t) \qquad \text{for } t \in [t_0, t_f] \qquad (5)$$

where $K(t)$ is a matrix valued function calculated from solving the following Riccati equation backwards

$$\begin{aligned} -\dot{K}(t) &= K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}B^T(t)K(t) + Q \quad \text{for a.a. } t \in [t_0, t_f] \quad (6)\\ K(t_f) &= F \qquad (7) \end{aligned}$$

We now introduce the receding horizon problem for system (2), (3) by calculating the control input $\Delta u^*(t)$ at time t that minimises a quadratic cost function V over $[t, t+T]$, where

$$V^u(x,t) = \frac{1}{2}\int_{t}^{t+T} \left\{e^T(t)Qe(t) + \Delta u^T(t)R\Delta u(t)\right\} dt + \frac{1}{2}e^T(t+T)Fe(t+T) \qquad (8)$$

In a similar way as above this leads to the state feedback control law

$$\Delta u^*(t) = -R^{-1}B^T(t)P(t)e(t) \qquad \text{for all t} \qquad (9)$$

where $P(t)$ is computed backwards using the following Riccati equation

$$-\dot{P}(\tau) = P(\tau)A(\tau) + A^T(\tau)P(\tau) - P(\tau)B(\tau)R^{-1}B^T(\tau)P(\tau) + Q \quad \text{for a.a. } \tau \in [t, t+T] \tag{10}$$

$$P(t+T) = F \tag{11}$$

Hence at each moment t, $\Delta u^*(t)$ is chosen as if the final objective is to minimise V over $[t, t+T]$, that is over a moving interval of length T. It is readily seen that this is clearly not the situation and so Δu^* does not in fact minimise any obvious cost function of the type (4) on any given interval $[t_0, t_{final}]$. We may therefore ask ourselves, "Why should we use such a strategy?" There are in fact several reasons for its use and popularity, and why we choose to use it here; these are mainly related to its practicality and computational efficiency. It is also conceptionally a reasonable approach, see Kwon[3], and Banks [1] for further discussion.

In principle to implement the control law (9) one might solve at all moments in time the Riccati equations (10), (11). This is clearly not feasible computationally but we immediately realise that in the case of time invariant systems, the receding horizon method yields a constant feedback gain. The control law in this case, is simply given by

$$\Delta u^*(t) = -R^{-1}B^T P_T e(t) \tag{12}$$

where $P_T = P(T)$ can be obtained from solving

$$\dot{P}(t) = P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q \quad \text{for a.a. } t \in [0, T] \tag{13}$$

$$P(0) = F \tag{14}$$

Such a simplifying linear time invariance assumption over short time intervals is used for our aircraft control problem and forms the basis of the QR selection procedure.

3 Nonlinear Optimal Control

Consider a general nonlinear control system

$$\dot{x}(t) = f(x(t), u(t) \quad \text{for a.a. } t \in [t_0, t_f] \tag{15}$$

$$x(t_0) = x_0 \tag{16}$$

where $x \in R^n$, $u \in R^m$ and $f : R^n \times R^m \to R^n$ is a continuous and bounded function. For practical optimal control of such systems it is normal practice to linearise about some point and formulate a simpler problem that can be solved on-line. Assuming we wish to linearise about some operating control input u_0 and its corresponding state x_0, then we can obtain

$$\dot{e}(t) = A(t)e(t) + B(t)\Delta u(t) \quad \text{for a.a. } t \in [t_0, t_f] \tag{17}$$

where $e(t) = x(t) - x_0$ is the error in the state, $\Delta u(t) = u(t) - u_0$ is the control deviation, and the $A(t)$ and $B(t)$ matrices have elements

$$a_{ij}(t) = \frac{\partial f_i}{\partial x_j}(x_0(t), u_0(t)) \quad \text{for } i, j = 1, 2, \cdots, n \tag{18}$$

$$b_{ij}(t) = \frac{\partial f_i}{\partial u_j}(x_0(t), u_0(t)) \quad \text{for } i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m \tag{19}$$

Hence the system (17) is a local representation to the nonlinear system (15).

The optimal control of linear time-varying systems can be computationally demanding, see Tahir and Virk [4], [5] and [6]. It is normal to assume time invariance for short intervals T_l so that the processing is realistic while maintaining adequate description of the time-varying system. Under these assumptions we have

$$\dot{e}(t) = Ae(t) + B\Delta u(t) \qquad \text{for a.a. } t \in [t_1, t_2] \tag{20}$$

where $t_2 - t_1 = T_l$. At time t consider a cost function V over a receding horizon of length T seconds for this time invariant linear system where

$$V^u(x) = \frac{1}{2}\int_t^{t+T} \left\{ e^T(\tau)Qe(\tau) + \Delta u^T(\tau)R\Delta u(\tau)\right\} dt + \frac{1}{2}e^T(t+T)Fe(t+T) \tag{21}$$

and Q, F and R are as assumed in section 2. We assume that the horizon length is greater than the linearising interval, i.e, $T > T_l$, to ensure a good representation of the system for all time. Hence several linearisations are performed within a single horizon duration. The receding optimal control law is given by

$$\Delta u^*(t) = R^{-1}B^T P_T e(t) \qquad \text{for all } t \tag{22}$$

where P_T satisfies equations (13), (14). The normal procedure is to apply the control given by equation (22) for time T_l seconds and then the nonlinear equations are relinearised at the new operating point. The new A and B matrices give a new control problem and P_T is recalculated thus giving Δu^* for the next T_l interval, etc.

What follows now is a procedure that allows a proper selection of the Q and R weightings so that the overall control performance is as required.

From equation (22) we have

$$\Delta u^*(t) = Ge(t) \qquad \text{for all } t \tag{23}$$

where G is an $m \times n$ matrix given by

$$G = -R^{-1}B^T P_T \tag{24}$$

The components of the control input are therefore defined by

$$\Delta u_i^*(t) = g_{i1}e_1(t) + g_{i2}e_2(t) + \cdots + g_{in}e_n(t) \qquad \text{for all } t \tag{25}$$

Hence each control input Δu_i^* has n gains $g_{i1}, g_{i,2}, \cdots, g_{in}$ associated with it. These gains are multiplied by the state error vector elements to give contributions to the overal control signals that need to be applied.

As can be seen from equations (13), (14), (21) and (22) the gain elements in the matrix G are affected by the weighting matrices Q, R and F. Here we will assume $F = 0$ and concentrate on Q and R. The important points to bear in mind when Q and R are selected include the following:

(a) Each gain g_{ij} must possess the correct sign so that when it is multiplied by the corresponding state error, the resulting control leads to a reduction of the error in that state. If this is not so the system will face a stability problem.

(b) Each gain g_{ij} should be of "reasonable" magnitude so that when it is multiplied by the state error, the control signal determined is within the constraint limits for Δu_i^*. Hence the g_{ij}'s magnitudes must be inversely proportional to the expected sizes of the corresponding state errors.

(c) How much authority should be given to each individual gain g_{ij}? The answer to this is system dependent and on the performance requirements sought, but to assist in this selection the gains corresponding to each Δu_i^* can be divided into two groups, which are

- Group 1 which contains the gains related to the states which can be directly controlled by the Δu_i.
- Group 2 which contains the gains resulting from the cross-coupling between the system states. This situation usually corresponds to the case where these states are controlled by some other control element of Δu.

The gains in Group 1 are given a higher authority, and then the relative importance of all the gains in Groups 1 and 2 are decided.

(d) The required control effort, i.e. the size of the Δu_i^* terms define the size of the control effort and usually specifies the error tolerance levels in the system performance. Usually, larger controls imply lower error levels and faster system response requirements. If Δu_i^* is too excessive it can be reduced by increasing the corresponding R matrix element r_{ii}.

(e) What time varying characteristics do the gains g_{ij} possess? To answer this it is necessary to establish which states have the major time variation influences the system. The dominant states can be found by analysing the time varying properties of elements in the A and B matrices. For example in aircraft systems high velocities give a higher rate of change in the pitching moment with respect to elevator deflection, that is some of the elements of the B matrix will have higher values, and so the $PBR^{-1}B^TP$ term in equation (13) will lead to a reduction in the gains g_{ij}. This means that lower control effort will be needed to remedy an error when the aircraft is travelling at high velocity than when it is travelling slower. From equation (13), if Q and R are too large, this automatic adjustment could be limited. The design arrived at, should therefore be tested at least 3 operating points that cover the entire operational range of the system.

(f) When all the above aspects have been considered and the Q and R elements have been set, it is advisable to study the system response if assess the performance. Very slight changes maybe needed to fine tune the behaviour.

In the next section we shall show how the above analysis can yield a suitable procedure for designing Q and R, (F is assumed to be zero without loss in generality). Although the procedure is system dependent, it can be used as a guideline for other systems if prior knowledge concerning the application is available.

4 Optimal Aircraft Longitudinal Control

In this section we shall apply the techniques presented in the previous sections to design a suitable optimal control law for an aircraft whose engineering data was supplied by British

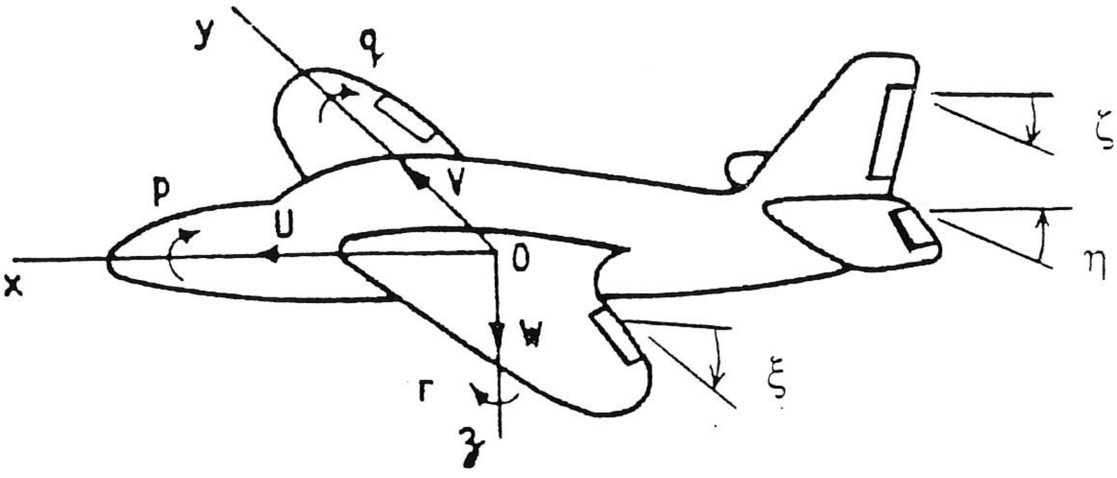

Figure 1: Aircraft in flight with notation

Aerospace, Brough [7]. The longitudinal motion of an aircraft system, shown if Figure 1, is defined by the following equations, see [8], [9],

$$\dot{U}(t) = \frac{X_f(t)}{M} - Q(t)W(t) \tag{26}$$

$$\dot{W}(t) = \frac{Z_f(t)}{M} + Q(t)U(t) \tag{27}$$

$$\dot{Q}(t) = \frac{P_m(t)}{I_y} \tag{28}$$

$$\dot{\Theta}(t) = Q(t) \tag{29}$$

$$\dot{H}(t) = U(t)\sin\Theta(t) - W(t)\cos\Theta(t) \tag{30}$$

$$\dot{E}_s(t) = f(E_s(t), V_r(t), H(t), \gamma(t)) \tag{31}$$

where the notation is also shown in Figure 1. This nonlinear system can be linearized about an operating point (x_0, u_0) to give

$$\dot{e}(t) = A(t)e(t) + B(t)\Delta u(t) \qquad \text{for a.a. } t \tag{32}$$

where the matrices $A(t)$ and $B(t)$ are given by

$$A(t) = \begin{bmatrix} X_u & X_w - Q & x_q - W & X_\theta & 0 & X_{E_s} \\ Z_u & Z_w & Z_q + U & Z_\theta & 0 & 0 \\ P_{mu} & P_{mw} & P_{mq} & P_{m\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & H_\theta & 0 & 0 \\ E_{su} & 0 & 0 & 0 & 0 & Es_{E_s} \end{bmatrix}, \quad B(t) = \begin{bmatrix} X_\eta & 0 \\ Z_\eta & 0 \\ P_{m\eta} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & Es_\gamma \end{bmatrix} \tag{33}$$

and

$$e(t) = x(t) - x_0 \tag{34}$$

$$\Delta u(t) = u(t) - u_0 \tag{35}$$

$$X_a = \left[\frac{\partial X}{\partial a}\right] \text{evaluated at } x_0, u_0, \text{ etc.} \tag{36}$$

$x = [U, W, Q, \Theta, H, E_s]^T$ is the state vector, $u = [\eta, \gamma]^T$ is the control input vector. Note that for convenience we have not always shown the time dependence of the various functions - this should not cause any confusion.

From equations (22)-(25) we have

$$\Delta\eta = g_{11}e_1 + g_{12}e_2 + \cdots + g_{16}e_6 \quad (37)$$

$$\Delta\gamma = g_{21}e_1 + g_{22}e_2 + \cdots + g_{26}e_6 \quad (38)$$

where the gains g_{ij}'s are of the form

$$g_{1i} = \frac{-1}{r_{11}}(X_\eta P_{1i} + Z_\eta P_{2i} + P_{m\eta} P_{3i}) \qquad \text{for } i = 1, 2, \cdots, 6 \quad (39)$$

$$g_{2i} = \frac{-1}{r_{22}} E s_\gamma P_{i6} \qquad \text{for } i = 1, 2, \cdots, 6 \quad (40)$$

Also $e_1, e_2, \cdots, e_6$ are the errors in the states $U, W, \cdots, E_s$ respectively, P_{ij} are the elements of the P_T Riccati gain matrix obtained by calculating equations (13) and (14), and r_{ii} are the diagonal elements of R. Bearing in mind the aspects stated above the following procedure is undertaken to select the weighting matrices for our aircraft application:

1. Linearize the aircraft about some operating point in the middle of its working range.

2. Set the diagonal elements of Q in the opposite order of the corresponding expected state error magnitudes:

   (i) The expected errors in the states Q and Θ have the smallest expected values (0 to ∓ 0.5 rad). Therefore the diagonal elements q_{33} and q_{44} must have the highest values, say $100 - 1000$.

   (ii) The error in the engine speed has the highest possible values ($100 - 1000\ rev/min$), hence its corresponding Q element q_{66} must have the lowest value (10^{-4} or 10^{-5}).

   (iii) The errors in the U and W states may have values such as 10 or 20, and the aircraft height H takes values such as $100m$. Therefore q_{11}, q_{22} and q_{55} may be set to "medium" values such as 0.1, 1 or 2.

   The diagonal elements of R, i.e. r_{11} and r_{22} are set to some reasonable value such as 1 or 10 with $r_{11} = r_{22}$, and the resulting g_{ij} gains checked. If these do not satisfy points (a) and (b) in section 3, then some of the Q elements and/or the R elements maybe changed to achieve reasonable results. For example if an increase is required in some g_{ij} element then one may look to equations (39) and (40) and see the required changes in the P_{ij} elements. Then from equation (13) we can see which of the diagonal elements of Q should be changed. If the required result is not possible using diagonal element changes, it may be necessary to set off-diagonal elements of the Q matrix. That is if some of the gains cannot be changed without effecting the other gains (in cases of high cross-coupling) then it is possible to use equations (13), (39) and (40) to find which P_T elements and hence which of the Q off diagonal elements needs to be modified while maintaining the positive definiteness of Q.

   At the end of this stage the g_{ij} gains will have been allocated their correct signs as well as having been set to reasonable values. These gains will contribute approximately equal amounts in the formation of Δu_i when each gain is multiplied by its equivalent error such as 5% of the operating state.

3. Setting of the authority for each gain. From equation (37), the state errors that are directly controlled by the elevator (η) are e_2, e_3, e_4, and e_5, and it is not required to control the forward velocity error e_1 and engine speed error e_6 by pitching the aircraft up and down by deflecting the elevator. Hence the gains $g_{12}, g_{13}, g_{14}, g_{15}$ must have higher authority than gains g_{11}, and g_{16}. In fact g_{11} and g_{16} are used only to limit or to assist $\Delta\eta$ if there are large errors in velocity and/or engine speed.

   If we decided that the first group $g_{12}, g_{13}, g_{14}, g_{15}$ must have an authority 10 times higher than the second group g_{11}, g_{16}, then we may multiply $q_{22}, q_{33}, q_{44}, q_{55}$ and r_{11} by 10. This will lead to a reduction of approximately 10 times in g_{11} and g_{16} while the others are kept roughly the same value.

   In the same way from equation (38) we also find two groups, the first being g_{21} and g_{26} corresponding to the state errors e_1 and e_6 which are directly controlled by the throttle control $\Delta\gamma$, and the second group $g_{22}, g_{23}, g_{24}, g_{25}$ which are used to alter the throttle control corresponding to the state errors for W, Q, Θ, H. That is if the height is less than the desired value then g_{25} will lead to an increase in engine speed to keep the forward speed close to the desired value when the aircraft is pitched up to gain height. Here the errors e_4 and e_5 are required to have similar effect on the formation of $\Delta\gamma$, and so we may give equal authority to the two groups of gains. In this case we do not need to do anything as this has already been achieved in step 2. However we may increase q_{11} and/or q_{66} to increase g_{21} and/or g_{26} or vica versa. If this leads to a considerable change in the Group 2 gains of equation (37) then we may multiply $q_{22}, q_{33}, q_{44}, q_{55}$ and r_{11} by any factor to keep the balance achieved for the g_{1i} gains.

   The correct relative authority within each group may be achieved by slight changes in the corresponding Q diagonal elements. In the case of high cross-coupling some of the off diagonal elements may be set non-zero as stated in step 2(iii).

4. The system response is checked, and if the control effort is too excessive it can be reduced by increasing the corresponding R element without changing the relative authority set in step 3. That is, if it is required to increase r_{22} by multiplying it by some factor, then $q_{22}, q_{33}, q_{44}, q_{55}$ and r_{11} must be multiplied by the same factor. It is also possible to see from the response whether the relative authorities need to be reconsidered. If so step 3 above may be repeated.

5. If a good response is obtained then it is necessary to determine the states which cause the major time variation effects. Clearly these will be application dependent; for our aircraft application the important states are the U, and W, or the aircraft speed and angle of attack for the elevator control, and U, E_s and H for the engine control. The gains for these states for maximum and minimum possible errors are checked to assess if the gain changes are reasonable (see point (e) in section 3). To establish this requires good knowledge of the system under consideration. If not it may be possible to determine another combination of Q and R that is more suitable, but again following the procedure outlined above.

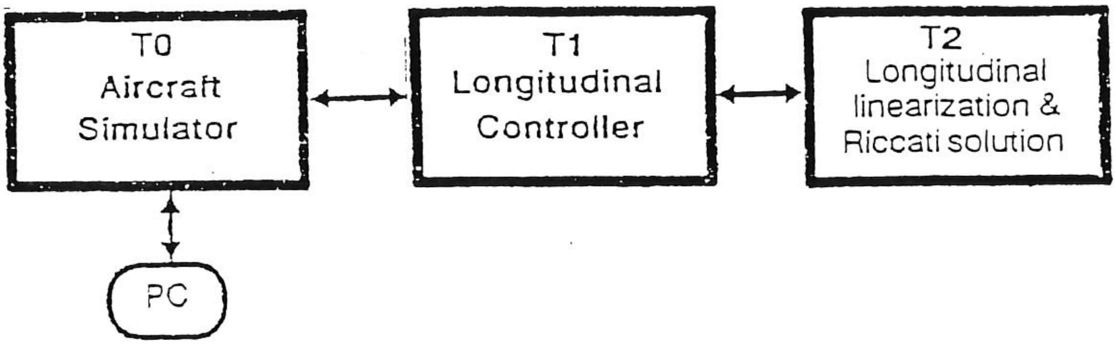

Figure 2: Transputer system for autopilot design implementation

5 Results and Conclusion

Following the procedure stated in the previous sections we arrived at the following values for Q and R:

$$Q = diag\left[5, 0, 500, 1000, 0.6, 5 \times 10^{-5}\right] \quad (41)$$

$$R = diag\,[5, 1] \quad (42)$$

$$F = 0 \quad (43)$$

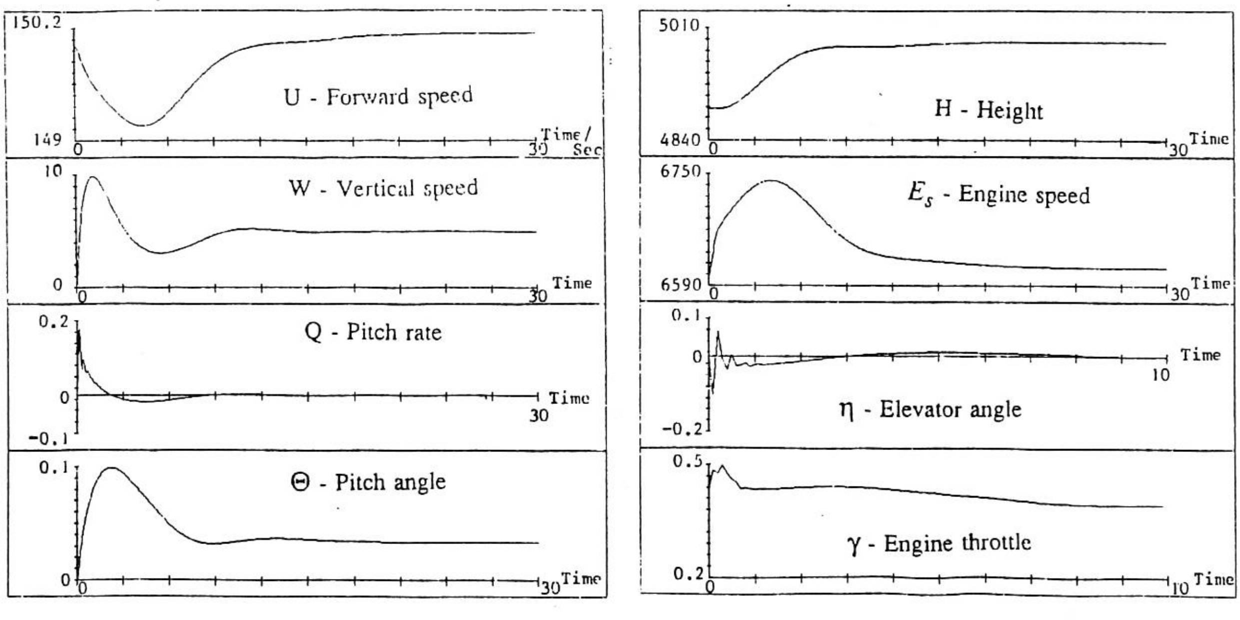

Figure 3: Longitudinal optimal autopilot results

The receding horizon control law presented in Tahir and Virk [4] is implemented in real-time using the T800 transputer system shown in Figure 2. The state vector $x = [U, W, Q, \Theta, H, E_s]^T$

and assuming an initial state

$$x_{ic} = [150, 0, 0, 0, 4960, 6600]^T$$

and a desired state

$$x_d = [150, 5, 0, 0.033, 5000, 6615]^T$$

the optimal trajectories are shown in Figure 3. The results are deemed to be satisfactory indicating that the settings of Q and R are adequate. It is felt that the above procedure enables a systematic way of selecting the weighting matrices that is an improvement upon the traditional trial and error approach. Although our results are specific to aircraft control systems it is hoped the other applications engineers will find the approach useful in formulating their optimal control problems adequately.

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