Monograph:
Billings, S.A. and Peyton Jones, J.C. (1989) Mapping Nonlinear Integro-Differential Equations into the Frequency Domain. Research Report. Acse Report 366 . Dept of Automatic Control and System Engineering. University of Sheffield

MAPPING NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS INTO THE FREQUENCY DOMAIN

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Research Report No. 366

July 1989

July 12, 1989

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Abstract: A recursive algorithm is derived which computes the generalised frequency response functions for a large class of nonlinear integro-differential equations. Applications to Duffings equation, and the Van-der-Pol model are discused.

1. Introduction

The analysis and design of linear systems is a well established discipline which finds wide application in many branches of science and engineering. Linear systems can be described either in the time domain using a differential equation or impulse response representation, in the s domain by a transfer function model or in the frequency domain. The relationship between each of these descriptors is well understood and standard algorithms are available which can be employed to convert any one representation into a model in any of the other domains.

Unfortunately the situation is much more complex when the system under investigation is nonlinear. Models have been developed in the time, s and frequency domains [Barrett 1963], [Schetzen 1980], but the relationship and more specifically the transformation between domains is far from striaghtforward in the nonlinear case. The classical representation in the time domain is the Volterra functional series [Barrett 1963],[Schetzen 1980], which has been used extensively in the analysis and identification of nonlinear systems [Billings 1980]. By introducing multidimensional Laplace and Fourier transforms the Volterra kernals can be mapped into multidimensional transfer functions and generalised frequency response functions respectively. Numerous authors have studied the properties of the generalised frequency response functions [Bedrosian,Rice 1971], [Bussgang,Ehrman,Graham 1974] , but little work has been done on the interpretation of these as a function of physical parameters in nonlinear differential equation models which are so often used in nonlinear analysis.

In the present study an analytical relationship between nonlinear integro differential equations and the generalised frequency response functions is derived. This avoids the complexity that occurs with previous methods as the order of the nonlinearity increases, provides a great deal of insight into the relationships between the time and frequency domain representations of nonlinear systems and exposes the sensitivity of the frequency domain characteristics to parameters in the differential equation model. The frequency domain effects of changing physical characteristics such as the mass, damping and stiffness in Duffings equation for example can therefore be readily investigated to provide engineering insight into system characteristics. The results represent a natural extension of previous work [Peyton-Jones,Billings 1989] which showed how to generate all the generalised frequency response functions from the coefficients of estimated NARMAX (Nonlinear AutoRegressive Moving Average model with eXogenous inputs) models.

The paper begins in Section 2 with definitions of the Volterra series and generalised frequency response functions. Because the main results of the paper are an extension of previous work for discrete time systems Section 3 provides a brief overview of these results. Methods of computing the generalised frequency response functions are introduced in Section 4. A relationship which maps the parameters of nonlinear integro differential equations directly to the generalised frequency response functions is derived in Sections 5 and 6, and examples of applying the algorithm to both a Duffing and Van der Pol model are given in Section 7. These results are extended to time lagged nonlinear integro-differential equations in Section 8.

2. Generalised frequency response functions: The Volterra model

Linear systems posses the equivalent properties of superposition and homogeneity of degree 1. For nonlinear systems however, superposition does not hold, and the output may be expressed as the sum of N components $y_n(t)$,

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{2.1}$$

each of which are homogeneous of degree n. These 'n-th order outputs' are themselves defined by an extension of the familiar convolution integral of linear systems theory to higher orders,

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i \qquad n > 0 \tag{2.2}$$

where the function $h_n(\tau_1, \cdots, \tau_n)$ would be recognised as the system impulse response for the (linear) case $n = 1$. More generally then $h_n(\cdot)$ is called the 'n-th order impulse response' or 'n-th order Volterra kernel' of the system, and (2.1),(2.2) define the time domain representation of the Volterra model.

Alternatively the Volterra model may be expressed in the frequency domain by defining an 'n-th order transfer function' $H_n(j\omega_1, \cdots j\omega_n)$ as the multi-dimensional Fourier transform of the n-th order impulse response,

$$H_n(j\omega_1, \cdots j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n)\, e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, d\tau_1 \cdots d\tau_n \quad (2.3)$$

Notice that (2.3) reduces to the standard linear transfer function definition for the case $n = 1$. Indeed just as in the linear case, the n-th order transfer function and its impulse response form a Fourier pair; so the relationship (2.3) may also be written as an inverse Fourier transform, namely,

$$h_n(\tau_1, \cdots, \tau_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots j\omega_n)\, e^{j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, d\omega_1 \cdots d\omega_n \quad (2.4)$$

Substituting (2.4) into (2.2) and carrying out the multiple integration on $\tau_1, \cdots, \tau_n$ yields,

$$y_n(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots j\omega_n) \prod_{i=1}^{n} U(j\omega_i)\, e^{j(\omega_1 + \cdots + \omega_n)t}\, d\omega_i \quad (2.5)$$

where $U(j\omega_i)$ denotes the input spectrum.

Note that $y_n(t)$ is dependent on the value of the integral (2.5), and that this value may not change even though changes are made to the function being integrated, (such as changing the order of any two arguments in $H_n(\cdot)$). For this reason the n-th order transfer function is not unique in yielding $y_n(t)$ unless it has values which are independent of the order of its arguments. This 'symmetric' property is obtained by summing an asymmetric function over all possible permutations of its arguments and dividing by their number, such that,

$$H_n^{sym}(j\omega_1, \cdots j\omega_n) = \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ \text{of } \omega_1 \cdots \omega_n}} H_n^{asym}(j\omega_1, \cdots j\omega_n) \quad (2.6)$$

3. Identification of nonlinear systems: the NARX model

The generalised frequency response functions described above form a non-parametric black box description for a wide class of nonlinear systems. Unfortunately direct estimation of these functions by non-parametric means, such as multidimensional FFT or correlation techniques, is computationally burdensome, and results in inordinately large system descriptions [Billings 1980].

If the system under analysis can be described equally well by some compact parametric model form, then another approach would be to perform identification using this model, and then derive its frequency response analytically from the estimated coefficients.

One such model is given by the NARMAX representation [Leontaritis,Billings 1985]

$$y(t) = F[y(t-1), \cdots, y(t-k_y), u(t-1), \cdots, u(t-k_u), \zeta(t-1), \cdots, \zeta(t-k_\zeta)] \qquad (3.1)$$

where $F[\cdot]$ is some discrete time nonlinear function of lagged input signals $u(t-k_u)$, outputs $y(t-k_y)$, and noise $\zeta(t-k_\zeta)$, with t used to enumerate the sampling intervals, and k the lags. Algorithms for detecting the model structure, estimating the parameters, and validating these models are now well developed, so physical systems are readily identified from real plant data.

Once this identification process is complete, the Moving Average noise terms (which were included to ensure unbiased estimation), may be discarded, yielding a deterministic 'NARX' model containing input and output terms only. The polynomial structural form of the NARX model may be described by,

$$y(t) = \sum_{m=1}^{M} y_m(t) \qquad (3.2)$$

where $y_m(t)$, the 'NARX m-th order output' of the system is given by,

$$y_m(t) = \sum_{p=0}^{m} \sum_{k_1, k_{p+q}=1}^{K} c_{p,q}(k_1, \cdots, k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \qquad (3.3)$$

Each term is seen to contain a p-th order factor in $y(t-k_i)$ and a q-th order factor in $u(t-k_i)$ (such that $p + q = m$), and each is multiplied by a coefficient $c_{p,q}(k_1, \cdots, k_{p+q})$, while the multiple summation over the k_i, ($k_i = 1, \cdots, K$), generates all the possible permutations of lags which might appear in these terms. Thus for example a specific instance of the NARX model such as,

$$y(t) = a_0 y(t-1) + a_1 u(t-1) + a_2 y(t-2)u(t-1) + a_3 u(t-1)^2 \tag{3.4}$$

may be obtained from the general form (3.2),(3.3) with

$$c_{1,0}(1) = a_0; \quad c_{0,1}(1) = a_1; \tag{3.5}$$

$$c_{1,1}(2,1) = a_2; \quad c_{0,2}(1,1) = a_3; \quad else \;\; c_{p,q}(\cdot) = 0;$$

The NARX model has a comparatively modest parameter set because it encodes information from past outputs as well as past inputs, and it is this recursive property which fundamentally distinguishes it from the Volterra model. However an algorithm has been developed to derive the Volterra transfer functions from the coefficients of a general NARX model, [Peyton Jones, Billings 1989]. The algorithm, and the probing method on which it is based, are briefly outlined in Section 4, while in latter sections these results are extended to models having integro-differential form.

4. Computing the generalised frequency response: The probing method

Consider an harmonic input consisting of the sum of R exponentials,

$$u(t) = \sum_{r=1}^{R} e^{j\omega_r t} \tag{4.1}$$

which is applied both to the Volterra model, and to some other model S. The outputs from each of these models may then be equated, thereby relating the (unknown) n-th order frequency response functions, $H_n(\cdot)$, which parameterise the Volterra model, to the (known) parameters θ which describe the model S. Thus,

$$y(t;\, H,\omega_r) = y(t;\, \theta,y,\omega_r) \tag{4.2}$$

where the notation $y(t;\, H,\omega_r)$ is used to denote an expression for the output $y(t)$ as a functional expansion of all the Volterra parameters $H_n(\cdot)$, and the harmonic input. Similarly $y(t;\, \theta,y,\omega_r)$ implies that the model S, parameterised by θ, is a functional of both output and input, and is therefore quite general.

The left hand side of (4.2) may be found by applying the input (4.1) to (2.1),(2.2), and integrating to yield,

$$y(t;\, H,\omega_r) = \sum_{n=1}^{N} \sum_{r_1,r_n=1}^{R} H_n(j\omega_{r_1}, \cdots j\omega_{r_n})\; e^{j(\omega_{r_1}+ \cdots +\omega_{r_n})t} \tag{4.3}$$

$$= \sum_{n=1}^{N} \sum_{\substack{\textit{all combinations} \\ \textit{of R frequencies} \\ \textit{taken n at a time}}} \sum_{\substack{\textit{all permutations} \\ \textit{of } \omega_{r_1} \cdots \omega_{r_n}}} H_n(j\omega_{r_1}, \cdots j\omega_{r_n})\; e^{j(\omega_{r_1}+ \cdots +\omega_{r_n})t}$$

For the case $R = n$, the latter sum of permutations may be recognised as $n!$ times the symmetric n-th order transfer function, and this appears as the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$. Equating such coefficients across (4.2) gives the probing method result [Bedrosian and Rice 1971],

$$H_n^{sym}(j\omega_1,\cdots j\omega_n) = \frac{1}{n!} \times \left[coefficient\ of\ e^{j(\omega_1+\cdots+\omega_n)t}\ in\ y(t;\ \theta,H,\omega_r) \right] \quad (4.4)$$

which may be abbreviated as,

$$H_n^{sym}(j\omega_1,\cdots,j\omega_n) = \frac{1}{n!}\ C_n\ [\ y(t;\ \theta,H,\omega_r)\ ] \quad (4.5)$$

Note that the output terms in $y(t;\ \theta,y,\omega_r)$ must be eliminated by substituting their corresponding Volterra expansion, as indicated by the new notation $y(t;\ \theta,H,\omega_r)$. For this reason (4.5) does not necessarily possess a solution, since any coefficient in $y(t;\ \theta,H,\omega_r)$ is in general a functional of all Volterra parameters $H_i(\cdot)$. The equation can however be solved recursively from $H_1(\cdot)$ providing the model S is of polynomial form.

4.1. Probing the NARX model

The probing method may be applied specifically to the NARX model, whose coefficients $c_{p,q}(\cdot)$ then correspond to the parameters θ in equation (4.5). By equating coefficients of $e^{j(\omega_1+\cdots+\omega_n)t}$ in the harmonic expansion of the NARX model, $y(t;\ c_{p,q}(\cdot),H,\omega_r)$, the following result is obtained,

$$\left[1 - \sum_{k_1=1}^{K} c_{1,0}(k_1)\ e^{-j(\omega_1+\cdots+\omega_n)k_1} \right] H_n^{asym}(j\omega_1,\cdots,j\omega_n) =$$

$$\sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)\ e^{-j(\omega_1 k_1+\cdots+\omega_n k_n)}$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_n=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{n-q+1}+\cdots+\omega_{p+q}k_{p+q})}\ H_{n-q,p}(j\omega_1,\cdots j\omega_{n-q})$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\cdots,k_p)\ H_{n,p}(j\omega_1,\cdots j\omega_n) \quad (4.1.1)$$

where $H_{n,p}(j\omega_1,\cdots j\omega_n)$ is used to denote the contribution to the n-th order frequency response function that is generated by the p-th degree of nonlinearity in the (recursive) output. It is perhaps not surprising therefore that $H_{n,p}(\cdot)$ itself may be expressed recursively (for $n{\ge}p$) as, (4.1.2)

$$H^{asym}_{n,p}(j\omega_1,\cdots,j\omega_n) = \sum_{i=1}^{n-p+1} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\, e^{-j(\omega_1+\cdots+\omega_i)k_p}$$

Note that the recursion finishes with p=1, and that $H_{n,1}(j\omega_1,\cdots,j\omega_n)$ has the property,

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n(j\omega_1,\cdots,j\omega_n)\, e^{-j(\omega_1+\cdots+\omega_n)k_1} \tag{4.1.3}$$

5. Modelling of nonlinear systems: Nonlinear integro-differential equation models

The NARMAX model of Section 3 was introduced as a practical means of identifying black box nonlinear systems. The system's frequency response may then be derived analytically as detailed in the preceeding section. In some situations however there may be considerable knowledge about the processes within the system, and a physical analysis or modelling approach may have been adopted. Such analysis is generally performed in continuous time, resulting in nonlinear integro-differential equation (NIDE) models. A polynomial form for a wide class of these models may be represented as,

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{l_1,l_{p+q}=-L}^{L} c_{p,q}(l_1,\cdots,l_{p+q}) \prod_{i=1}^{p} D^{l_i}y(t) \prod_{i=p+1}^{p+q} D^{l_i}u(t) = 0 \tag{5.1}$$

where $p+q=m$, and where the operator D is defined by,

$$D^l x(t) = \begin{cases} \dfrac{d^l x(t)}{dt^l} & l \geq 0 \\ \displaystyle\int_{-\infty}^{t}\cdots\int_{-\infty}^{\tau_2} x(\tau_1)d\tau_1\cdots d\tau_{|l|} & l < 0 \end{cases} \tag{5.2}$$

Note that the lower limits of integration in (5.2) may be raised to zero for causal systems where $x(\tau)=0\ \forall\tau<0$.

For (5.1) to represent a valid input/output map however, there must exist at least one non-zero linear output term, and this will become apparent in the final result (6.4.2). For the present a suitable device is to assume the coefficient $c_{1,0}(0) \neq 0$, and rearrange (5.1) to give,

$$-c_{1,0}(0)\, y(t) = \sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{\substack{l_1,l_{p+q}=-L \\ except\ c_{1,0}(0)}}^{L} c_{p,q}(l_1,\cdots,l_{p+q}) \prod_{i=1}^{p} D^{l_i}y(t) \prod_{i=p+1}^{p+q} D^{l_i}u(t) \tag{5.3}$$

Thus for example the expansion of (5.3) up to 2nd order gives,

$$- c_{1,0}(0)\, y(t) = \sum_{\substack{l_1=-L \\ except\ c_{1,0}(0)}}^{L} c_{1,0}(l_1)\, D^{l_1}y(t) +$$

$$\sum_{l_1=-L}^{L} c_{0,1}(l_1)\, D^{l_1}u(t)$$

$$+ \sum_{l_1=1}^{L} \sum_{l_2=-L}^{L} c_{2,0}(l_1,l_2)\, D^{l_1}y(t)\, D^{l_2}y(t) +$$

$$\sum_{l_1=-L}^{L} \sum_{l_2=-L}^{L} c_{1,1}(l_1,l_2)\, D^{l_1}y(t)\, D^{l_2}u(t) +$$

$$\sum_{l_1=-L}^{L} \sum_{l_2=-L}^{L} c_{0,2}(l_1,l_2)\, D^{l_1}u(t)\, D^{l_2}u(t) \tag{5.4}$$

Alternatively a specific instance of the NIDE model such as the well known Duffings equation,

$$D^2y(t) + 2\zeta\omega_n Dy(t) + \omega_n^2 y(t) + \omega_n^2 \varepsilon y(t)^3 - u(t)=0 \tag{5.5}$$

may be obtained from the general form (5.1) with

$$c_{1,0}(2) = 1.0; \qquad c_{1,0}(1) = 2\zeta\omega_n; \qquad c_{1,0}(0) = \omega_n^2; \qquad (5.6)$$

$$c_{3,0}(0) = \omega_n^2\varepsilon; \qquad c_{0,1}(0) = 1.0; \qquad else\ \ c_{p,q}(\cdot) = 0; \qquad (5.6)$$

where ζ is the damping ratio, and ω_n the natural frequency of the Duffings oscillator.

Once again the advantage of the NIDE model is its relatively small parameter set, and its widespread usage in physical system modelling. The remaining task is now to map these physical parameters to obtain the Volterra frequency response characteristics of the system under analysis, and this is achieved by applying the probing method.

6. Probing the NIDE model

6.1. Pure Input Nonlinearity

The simplest sub-class of NDE to consider corresponds to p=0 in expression (5.3), which yields,

$$- c_{1,0}(0)\, y(t) = \sum_{m=q=1}^{M} \sum_{l_1,l_q=-L}^{L} c_{0,q}(l_1, \cdots, l_q) \prod_{i=1}^{q} D^{l_i}u(t) \tag{6.1.1}$$

Applying the D operator to the harmonic input (4.1) gives,

$$D^{l_i} \sum_{r=1}^{R} e^{j\omega_r t} = \sum_{r=1}^{R} (j\omega_r)^{l_i} e^{j\omega_r t} \tag{6.1.2}$$

so that the harmonic expansion of (6.1.1) becomes,

$$- c_{1,0}(0)\, y(t) = \sum_{m=q=1}^{M} \sum_{l_1,l_q=-L}^{L} c_{0,q}(l_1, \cdots ,l_q) \prod_{i=1}^{q} \sum_{r=1}^{R} (j\omega_r)^{l_i} e^{j\omega_r t} \tag{6.1.3}$$

$$= \sum_{m=q=1}^{M} \sum_{l_1,l_q=-L}^{L} c_{0,q}(l_1, \cdots ,l_q) \sum_{r_1,r_n=1}^{R} \prod_{i=1}^{q} (j\omega_r)^{l_i} e^{j\omega_r t}$$

By the probing method, the n-th order frequency response function from pure input terms is given as the coefficient of $e^{j(\omega_1+ \cdots +\omega_n)t}$ in (6.1.3), with $R=q=n$. Thus

$$- c_{1,0}(0)\, H_{n_u}^{asym}(j\omega_1, \cdots j\omega_n) = \sum_{l_1,l_n=1}^{\infty} c_{0,n}(l_1, \cdots ,l_n)\, (j\omega_1)^{l_1} \cdots (j\omega_n)^{l_n} \tag{6.1.4}$$

and a pure n-th order term in $u(t)$ is seen to contribute *only* to the n-th order kernel, or frequency response of the system.

6.2. Pure Output Nonlinearity

A much more complicated sub-class of NDE contains only pure output nonlinearity terms, and corresponds to $q=0$ in expression (5.3). This yields,

$$- c_{1,0}(0)\, y(t) = \sum_{\substack{m=p=1 \\ except\ c_{1,0}(0)}}^{M} \sum_{l_1,l_p=-L}^{L} c_{p,0}(l_1, \cdots ,l_p) \prod_{i=1}^{p} D^{l_i} y(t) \tag{6.2.1}$$

which again is of polynomial form, only this time in terms of the output $y(t)$. A Volterra model, and its frequency response, are however functionals of the input, so the n-th order frequency response is obtained by applying the probing method. A harmonic expansion of (6.2.1) is obtained by substituting for each output term using the corresponding Volterra model expansion of (4.3),

$$D^{l_i} y(t) = \sum_{\gamma=1}^{N} \alpha^{\gamma} \sum_{r_1,r_\gamma=1}^{R} H_\gamma(j\omega_{r_1}, \cdots j\omega_{r_\gamma})\, D^{l_i} e^{j(\omega_{r_1}+ \cdots +\omega_{r_\gamma})t} \tag{6.2.2}$$

where γ is being used to enumerate the various order terms, and where the dummy variable $\alpha=1$ is included to keep track of all terms homogenous to degree γ.

Pure output nonlinearities may therefore be expressed as,

$$\prod_{i=1}^{p} D^{l_i}y(t) = \prod_{i=1}^{p} \sum_{\gamma=1}^{N} \alpha^{\gamma} \sum_{r_1,r_\gamma=1}^{R} H_\gamma(j\omega_{r_1}, \cdots j\omega_{r_\gamma})\ D^{l_i} e^{j(\omega_{r_1}+\cdots+\omega_{r_\gamma})t} \qquad (6.2.3)$$

$$= \sum_{\gamma_1,\gamma_p=1}^{N} \alpha^{\gamma_1+\cdots+\gamma_p} \prod_{i=1}^{p} \sum_{r_1,r_\gamma=1}^{R} H_{\gamma_i}(j\omega_{r_1}, \cdots j\omega_{r_{\gamma_i}})\ D^{l_i} e^{j(\omega_{r_1}+\cdots+\omega_{r_{\gamma_i}})t}$$

By inspecting the power of the dummy variable α, (6.2.3) is seen to contain terms from order p up to Np. The leftmost summation may therefore be subdivided into terms of like order n, giving,

$$\prod_{i=1}^{p} D^{l_i}y(t) = \sum_{n=p}^{Np} \alpha^{n} \sum_{\substack{\gamma_1,\gamma_p=1 \\ \mid \sum\gamma_i=n}}^{n-p+1} \prod_{i=1}^{p} \sum_{r_1,r_\gamma=1}^{R} H_{\gamma_i}(j\omega_{r_1}, \cdots j\omega_{r_{\gamma_i}})\ D^{l_i} e^{j(\omega_{r_1}+\cdots+\omega_{r_{\gamma_i}})t} \qquad (6.2.4)$$

where the constraint that $\sum\gamma_i=n$ also lowers the limit N to $n-p+1$.

Now by the probing method, the contribution to the n-th order frequency response function that is generated by a p-th degree of nonlinearity in $y(t)$, is given by the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$ in (6.2.4), with $R=n$; that is,

$$n!\ H^{sym}_{n,p}(\cdot) = C_n\left[\sum_{\substack{\gamma_1,\gamma_p=1 \\ \mid \sum\gamma_i=n}}^{n-p+1} \prod_{i=1}^{p} \sum_{r_1,r_\gamma=1}^{n} H_{\gamma_i}(j\omega_{r_1}, \cdots j\omega_{r_{\gamma_i}})\ D^{l_i} e^{j(\omega_{r_1}+\cdots+\omega_{r_{\gamma_i}})t} \right] \qquad (6.2.5)$$

Each permutation described by the multiple summation over r is an ordered set, (denoted W_{γ_i}), of γ_i frequencies, taken from the n input harmonics. Applying the product would then generate terms with arguments given by the union of W_{γ_i} , $i=1..p$. As expected there are n such arguments (since $\sum\gamma_i = n$), but these will contain repeated frequencies unless the sets W_{γ_i}, and their elements are disjoint, i.e. unless $W_{\gamma_a} \cap W_{\gamma_b} = 0\ \forall\ a \neq b$, and $\omega_{r_a} \neq \omega_{r_b}\ \forall\ a \neq b$.

The simplest construction of such disjoint sets W_{γ_i} may be obtained by choosing $W_{\gamma_i} = \{\omega_{X+1} \cdots \omega_{X+\gamma_i}\}$ where $X = \sum\gamma_x$, $x = 1..i-1$. The union of such sets then comprises the n *different* frequencies $\{\omega_1 \cdots \omega_n\}$. Thus, (6.2.6)

$$n!\ H^{sym}_{n,p}(\cdot) = C_n\left[\sum_{\substack{\gamma_1,\gamma_p=1 \\ \mid \sum\gamma_i=n}}^{n-p+1} \sum_{\substack{r_1,r_n=1 \\ r_a\neq r_b\ \forall\ a\neq b}}^{n} \prod_{i=1}^{p} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots j\omega_{r_{X+\gamma_i}})\ D^{l_i} e^{j(\omega_{r_{X+1}}+\cdots+\omega_{r_{X+\gamma_i}})t} \right]$$

where X is given as above, and where the multiple summation over r now generates all other constructions of disjoint sets by permutation. However given,

$$n!\ H^{sym}_{n,p}(j\omega_1, \cdots j\omega_n) = \sum_{\substack{r_1,r_n=1 \\ r_a \neq r_b\ \forall\ a \neq b}}^{n} H^{asym}_{n,p}(j\omega_1, \cdots j\omega_n) \tag{6.2.7}$$

then it is more simple to write,

$$H^{asym}_{n,p}(\cdot) = C_n \left[\sum_{\substack{\gamma_1,\gamma_p=1 \\ | \sum\gamma_i=n}}^{n-p+1} \prod_{i=1}^{p} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots j\omega_{r_{X+\gamma_i}})\ D^{l_i} e^{j(\omega_{r_{X+1}}+ \cdots +\omega_{r_{X+\gamma_i}})t} \right] \tag{6.2.8}$$

Applying the D operator, and then extracting the time exponential term from within the product, gives $H^{asym}_{n,p}(\cdot)$ as,

$$H^{asym}_{n,p}(\cdot) = \sum_{\substack{\gamma_1,\gamma_p=1 \\ | \sum\gamma_i=n}}^{n-p+1} \prod_{i=1}^{p} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots j\omega_{r_{X+\gamma_i}})\ (j\omega_{r_{X+1}}+ \cdots +j\omega_{r_{X+\gamma_i}})^{l_i} \tag{6.2.9}$$

Equation (6.2.9) may be cast in recursive form by expanding the last term of the product. This yields,

$$H^{asym}_{n,p}(\cdot) = \sum_{\gamma_p=1}^{n-p+1} H_{\gamma_p}(j\omega_{n-\gamma_p+1}, \cdots j\omega_n)\ (j\omega_{n-\gamma_p+1}+ \cdots +j\omega_n)^{l_p} \tag{6.2.10}$$

$$\times \sum_{\substack{\gamma_1,\gamma_{p-1}=1 \\ | \sum\gamma_i=n-\gamma_p}}^{(n-\gamma_p)-(p-1)+1} \prod_{i=1}^{p-1} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots j\omega_{r_{X+\gamma_i}})\ (j\omega_{r_{X+1}}+ \cdots +j\omega_{r_{X+\gamma_i}})^{l_i}$$

$$= \sum_{\gamma_p=1}^{n-p+1} H_{\gamma_p}(j\omega_{n-\gamma_p+1}, \cdots j\omega_n)\ (j\omega_{n-\gamma_p+1}+ \cdots +j\omega_n)^{l_p}\ H^{asym}_{n-\gamma_p,p-1}(j\omega_1, \cdots j\omega_{n-\gamma_p})$$

Equation (6.2.10) may be written more convieniently using new subscripts, and a different (aysmmetric) permutation of frequencies, as,

$$H^{asym}_{n,p}(\cdot) = \sum_{i=1}^{n-p+1} H_i(j\omega_1, \cdots j\omega_i) H_{n-i,p-1}(j\omega_{i+1}, \cdots j\omega_n)\ (j\omega_1+ \cdots +j\omega_i)^{l_p} \tag{6.2.11}$$

Note that the recursion finishes with p=1, and that $H_{n,1}(j\omega_1, \cdots j\omega_n)$ has the property,

$$H_{n,1}(j\omega_1, \cdots j\omega_n) = H_n(j\omega_1, \cdots j\omega_n)\ (j\omega_1+ \cdots +j\omega_n)^{l_1} \tag{6.2.12}$$

Whilst equation (6.2.11) gives the contribution to the n-th order frequency response function generated by a p-th nonlinearity in $y(t)$, it is seen from (6.2.1) that the output is composed of many such terms. Their combined contribution to the n-th order frequency response function is therefore,

$$- c_{1,0}(0)\, H^{asym}_{n_y}(j\omega_1, \cdots, j\omega_n) = \sum_{p=1}^{n} \sum_{\substack{l_1,l_p=-L \\ except\ c_{1,0}(0)}}^{L} c_{p,0}(k_1, \cdots, k_p)\ H_{n,p}(j\omega_1, \cdots j\omega_n) \quad (6.2.13)$$

where the uppermost limit on the first summation has been reduced from M to n since $H_{n,p}(\cdot) = 0$ for $p > n$. This merely reiterates that the p-th nonlinearity in $y(t)$ cannot contribute to kernels less than p, although it will contribute (recursively) to all higher kernels.

Finally note that the linear terms in the output, corresponding to p=1, may be collected and brought over to the left side of equation (6.2.13), thereby defining the poles of the n-th order transfer function. Thus (6.2.13) may be rewritten as,

$$- \left[\sum_{l_1=-L}^{L} c_{1,0}(l_1)\ (j\omega_1 + \cdots + j\omega_n)^{l_1} \right] H^{asym}_{n_y}(j\omega_1, \cdots, j\omega_n) = \quad (6.2.14)$$

$$\sum_{p=2}^{n} \sum_{l_1,l_p=-L}^{L} c_{p,0}(l_1, \cdots, l_p)\ H_{n,p}(j\omega_1, \cdots j\omega_p)$$

6.3. Pure cross product nonlinearity

The largest sub-class of NARX, which contains only pure cross product terms, corresponds to $p{\neq}0, q{\neq}0$ in equation (3.3), giving,

$$- c_{1,0}(0)\ y(t) = \sum_{m=1}^{M} \sum_{p=1}^{m-1} \sum_{l_1,l_{p+q}=-L}^{L} c_{p,q}(l_1, \cdots, l_{p+q}) \prod_{i=1}^{p} D^{l_i}y(t) \prod_{i=p+1}^{p+q} D^{l_i}u(t) \quad (6.3.1)$$

This structure suggests that the n-th order frequency response function is obtained by multiplying the pure q-th order response function in $u(t)$ with the pure (and recursive) ($N{-}q$)-th order response function in $y(t)$, within the major summator. Fortunately the multiplication is a relatively simple operation, for although the response of an output term contains many (recursive) components, the response of an input term contains only one, and there are no awkward cross terms.

For notational reasons however, it is convenient to rewrite (6.3.1) with q, the order of the input nonlinearity, as the major summator; substituting $p = m - q$, and reversing the direction of summation, then gives,

$$- c_{1,0}(0)\, y(t) \;=\; \sum_{m=1}^{M} \sum_{q=1}^{m-1} \sum_{l_1, l_{p+q}=-L}^{L} c_{p,q}(l_1, \cdots, l_{p+q}) \prod_{i=1}^{m-q} D^{l_i} y(t) \prod_{i=m-q+1}^{p+q} D^{l_i} u(t) \tag{6.3.2}$$

The corresponding frequency response, (for $n>1$), is then,

$$- c_{1,0}(0)\, H^{asym}_{n_{uy}}(j\omega_1, \cdots j\omega_n) \;= \tag{6.3.3}$$

$$\sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{l_1, l_n=-L}^{L} c_{p,q}(l_1, \cdots, l_{p+q})\; (j\omega_{n-q+1})^{l_{n-q+1}} \cdots (j\omega_{p+q})^{l_{p+q}}\, H_{n-q,p}(j\omega_1, \cdots j\omega_{n-q})$$

where the upper limits on the summations have been lowered as before (since $H_{n_{uy}}(\cdot) = 0$ for $m = p+q > n$), and where $H_{n-q,p}(\cdot)$ is generated by the recursive relation (6.2.11). This somewhat heuristic approach may be validated by applying the probing method directly as in Section 6.2.

6.4. Combining NIDE sub-classes

The total frequency response of the NIDE model may now be found by summing the contributions from the various sub-classes,

$$H_n(j\omega_1, \cdots j\omega_n) \;=\; H_{n_u}(\cdot) \;+\; H_{n_{uy}}(\cdot) \;+\; H_{n_y}(\cdot) \tag{6.4.1}$$

Substituting (6.1.2),(6.2.14) and (6.3.3) in (6.4.1) gives the total frequency response,

$$- \left[\sum_{l_1=-L}^{L} c_{1,0}(l_1)\, (j\omega_1 + \cdots + j\omega_n)^{l_1} \right] H_n(j\omega_1, \cdots j\omega_n) \;=$$

$$\sum_{l_1, l_n=-L}^{L} c_{0,n}(l_1, \cdots, l_n)\, (j\omega_1)^{l_1} \cdots (j\omega_n)^{l_n}$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{l_1, l_n=-L}^{L} c_{p,q}(l_1, \cdots, l_{p+q})\; (j\omega_{n-q+1})^{l_{n-q+1}} \cdots (j\omega_{p+q})^{l_{p+q}}\, H_{n-q,p}(j\omega_1, \cdots j\omega_{n-q})$$

$$+ \sum_{p=2}^{n} \sum_{l_1, l_p=-L}^{L} c_{p,0}(l_1, \cdots, l_p)\; H_{n,p}(j\omega_1, \cdots j\omega_n) \tag{6.4.2}$$

together with the recursive relation (6.2.11). Note that (6.4.2) gives the asymmetric Volterra transfer function, although it is a simple matter to obtain unique symmetric values by applying (2.6).

7. Examples

7.1. Duffings equation

Consider then the example of Duffings oscillator given in equation (5.5),

$$D^2y(t) + 2\zeta\omega_n Dy(t) + \omega_n^2 y(t) + \omega_n^2 \varepsilon y(t)^3 - u(t) = 0$$

where the cubic output nonlinearity derives physically from the cubic stiffness of a spring. However the majority of terms are linear, and as expected the first order frequency response obtained from (6.4.2) is,

$$H_1(j\omega) = \frac{1}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \tag{7.1.1}$$

This response has been plotted in Figure 1, with the values $\zeta = 0.01$, $\omega_n = 45\pi$, and $\varepsilon = 3.0$, giving a resonant peak at a frequency of 22.5Hz.

The cubic term of (5.6) however generates frequency response functions for orders 3 and higher. In the third order case, the absence of other nonlinear terms means that the response from (6.4.2) is simply,

$$H_3(j\omega_1, j\omega_2, j\omega_3) = \omega_n^2 \, \varepsilon \times \frac{H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)}{(j\omega_1+j\omega_2+j\omega_3)^2 + 2\zeta\omega_n(j\omega_1+j\omega_2+j\omega_3) + \omega_n^2} \tag{7.1.2}$$

and this is plotted in Figure 2, for the same values given above, (and with $f_1=f_2$). The plot contains a number of resonant peaks and ridges, which can also be seen on the contour plot, Figure 3. The origin of these features may may be understood by inspection of (7.1.2). The ridges are generated whenever one of the factors $H_1(\cdot)$ in the numerator of (7.1.2) is excited at the linear resonant frequency, namely ±22.5Hz, and the peaks occur when this is true for all three factors, (see dotted lines on Figure 3).

Notice also that the poles of $H_3(\cdot)$, given by the denominator of (7.1.2), have the same form as the linear characteristic equation, and so these are excited whenever $\omega_1+\omega_2+\omega_3 = \omega_n$, (see solid line on Figure 3). It is for this reason that the resonant peak at $H_3(j\omega_n, -j\omega_n, j\omega_n)$ is larger than the others.

The example therefore demonstrates how the features of higher order frequency responses may be related to parameters (such as the damping ratio and natural frequency of $H_1(\cdot)$) in the original oscillator equation. In this way equation (6.4.2) may not only be useful in computing higher order frequency response functions, but may also aid designers in shaping nonlinear frequency characteristics through the parameters under their control.

7.2. Van-der-Pol model

As a second example consider the Van-der-Pol model,

$$D^2y(t) + 2\zeta\omega_n\ (y(t)^2 - 1)\ Dy(t) + \omega_n^2 y(t) - u(t) \ = \ 0 \qquad (7.2.1)$$

In this case the nonlinearity occurs in the damping term, so that for small displacements the damping is negative (self-exciting), and for large displacements the damping is positive (limiting). The linear terms however are very similar to those of the previous example, and yield the first order response,

$$H_1(j\omega) \ = \ \frac{1}{(j\omega)^2 - 2\zeta\omega_n(j\omega) + \omega_n^2} \qquad (7.2.2)$$

This response has been plotted in Figure 4, with the same values for ζ, ω_n as above, and it is seen that the changed sign of the damping term merely reverses the phase characteristic, (compare with Figure 1).

The nonlinear damping term, though cubic as before, contains a factor in the differential of $y(t)$, and gives the third order response function (from (6.4.2), symmeterised by (2.6)),

$$H_3(j\omega_1,j\omega_2,j\omega_3) \ = \ \frac{2\zeta\omega_n}{6} \times \frac{(j\omega_1+j\omega_2+j\omega_3)\ H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)}{(j\omega_1+j\omega_2+j\omega_3)^2 + 2\zeta\omega_n(j\omega_1+j\omega_2+j\omega_3) + \omega_n^2} \qquad (7.2.3)$$

which is shown together with its contour plots in Figures 5,6. These still exhibit the same peaks and ridges as the third order Duffings equation response, (compare with Figures 2,3), and the same discussion applies.

The additional feature of the third order Van-der-Pol response is however the deep gorge running across the magnitude plot. This is caused by the zero $(j\omega_1+j\omega_2+j\omega_3)$ in $H_3(\cdot)$ which was introduced by the differential factor in the cubic nonlinearity. The zero is excited whenever $\omega_1+\omega_2+\omega_3 = 0$, and there is consequently no response at points where this condition is satisfied.

Finally notice from (7.2.3),(7.1.2) that the Van-der-Pol and Duffings third order response also differ in their scaling factors. The former is dependent on the product of the damping factor and natural frequency of the system, whereas the latter depends on the spring stiffness ε, and the square of the natural frequency. This is in agreement with the original equations (7.2.1),(5.5).

8. Extension to time-lagged nonlinear integro-differential models

Whilst the NIDE model of Section 5 is sufficient for a wide variety of modelling needs, it cannot adequately describe systems which have time delays. Such systems can be described however by combining the time-lagged structure of the NARMAX model, with the integro-differential NIDE model. This results in the 'Time-lagged Nonlinear Integro-Differential Equation' (TNIDE) model, (8.1)

$$- c_{1,0}(0)\, y(t) = \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{\substack{l_1,l_{p+q}=-L \\ except\ c_{1,0}(0)}}^{L} \sum_{k_1,k_{p+q}} c_{p,q}(l_1k_1, \cdots ,l_{p+q}k_{p+q}) \prod_{i=1}^{p} D^{l_i}y(t-k_i) \prod_{i=p+1}^{p+q} D^{l_i}u(t-k_i)$$

where in this case the lags $K > k_i \geq 0$ are defined as a finite set of positive real numbers. In practice many physical systems have only one dominant lag, and equation (7.1) does not contain an inordinate number of terms.

The n-th order frequency response of the TNIDE model follows directly from those already derived for the NARMAX and NIDE forms; the lags merely introduce a (time independent) factor which is not affected by time differentiation/integration. The desired response is therefore,

$$\left[\sum_{l_1=-L}^{L} \sum_{k_1} c_{1,0}(l_1k_1)\, e^{-j(\omega_1+ \cdots +\omega_n)k_1} (j\omega_1+ \cdots +j\omega_n)^{l_1} \right] H_n(j\omega_1, \cdots ,j\omega_n) =$$

$$\sum_{l_1,l_n=-L}^{L} \sum_{k_1,k_n} c_{0,n}(l_1k_1, \cdots ,l_nk_n) \prod_{i=1}^{n} (j\omega_i)^{l_i}\, e^{-(j\omega_i)k_i}$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{l_1,l_n=-L}^{L} c_{p,q}(l_1, \cdots ,l_{p+q})\ H_{n-q,p}(j\omega_1, \cdots ,j\omega_{n-q}) \prod_{i=n-q+1}^{p+q} (j\omega_i)^{l_i}\, e^{-(j\omega_i)k_i}$$

$$+ \sum_{p=2}^{n} \sum_{l_1,l_p=-L}^{L} c_{p,0}(l_1, \cdots ,l_p)\ H_{n,p}(j\omega_1, \cdots ,j\omega_n) \qquad (8.2)$$

together with the modified recursive relationship, (8.3)

$$H_{n,p}^{asym}(\cdot) = \sum_{i=1}^{n-p+1} H_i(j\omega_1, \cdots ,j\omega_i) H_{n-i,p-1}(j\omega_{i+1}, \cdots ,j\omega_n)\, (j\omega_1+ \cdots +j\omega_i)^{l_p}\, e^{-j(\omega_1+ \cdots +\omega_i)k_p}$$

9. Laplace and Z-transform results

The results of previous sections have been presented entirely in the multi-dimensional frequency domain. It is a simple matter to cast these into the corresponding Laplace or Z-transform representation, and for completeness such equivalent results have been included below.

Replacing $e^{j\omega_n}$ by z_n in (4.1.1), (4.1.2) gives the Z-transform of the NARX model as,

$$\left[1 - \sum_{k_1=1}^{K} c_{1,0}(k_1)\,(z_1 \cdots z_n)^{-k_1}\right] H_n^{asym}(z_1, \cdots, z_n) =$$

$$\sum_{k_1,k_n=1}^{K} c_{0,n}(k_1, \cdots, k_n)\, z_1^{-k_1} \cdots z_n^{-k_n}$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1,k_n=1}^{K} c_{p,q}(k_1, \cdots, k_{p+q})\, z_{n-q+1}^{-k_{n-q+1}} \cdots z_{p+q}^{-k_{p+q}}\, H_{n-q,p}(z_1, \cdots, z_{n-q})$$

$$+ \sum_{p=2}^{n} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p)\, H_{n,p}(z_1, \cdots, z_n) \tag{9.1}$$

where,

$$H_{n,p}^{asym}(z_1, \cdots, z_n) = \sum_{i=1}^{n-p+1} H_i(z_1, \cdots, z_i) H_{n-i,p-1}(z_{i+1}, \cdots, z_n)\,(z_1 \cdots z_i)^{-k_p} \tag{9.2}$$

Likewise, by replacing $j\omega_n$ by s_n, the relations (6.4.2),(6.2.11) for the NIDE model become,

$$-\left[\sum_{l_1=-L}^{L} c_{1,0}(l_1)\,(s_1 + \cdots + s_n)^{l_1}\right] H_n(s_1, \cdots, s_n) =$$

$$\sum_{l_1,l_n=-L}^{L} c_{0,n}(l_1, \cdots, l_n)\,(s_1)^{l_1} \cdots (s_n)^{l_n}$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{l_1,l_n=-L}^{L} c_{p,q}(l_1, \cdots, l_{p+q})\,(s_{n-q+1})^{l_{n-q+1}} \cdots (s_{p+q})^{l_{p+q}}\, H_{n-q,p}(s_1, \cdots, s_{n-q})$$

$$+ \sum_{p=2}^{n} \sum_{l_1,l_p=-L}^{L} c_{p,0}(l_1, \cdots, l_p)\, H_{n,p}(s_1, \cdots, s_n) \tag{9.3}$$

where,

$$H_{n,p}^{asym}(s_1, \cdots, s_n) = \sum_{i=1}^{n-p+1} H_i(s_1, \cdots, s_i) H_{n-i,p-1}(s_{i+1}, \cdots, s_n)\,(s_1 + \cdots + s_i)^{l_p} \tag{9.4}$$

The Laplace transform representation for the TNIDE model of Section 8 is very similar to (9.3),(9.4), but with exponential terms of the form $e^{-s_i k_i}$ accounting for the delays.

10. Conclusion

An analytical relationship which expresses the generalised frequency response functions in terms of the coefficients in nonlinear integro-differential equations has been derived. This allows the generalised frequency response functions of any order to be calculated in a manner which avoids most of the disadvantages of prevoius methods. Application of the results to the Duffing and Van der Pol equations have been included to demonstrate how the new approach provides insight into the relationship between physical parameters in the differential equation model and the higher order frequency response functions.

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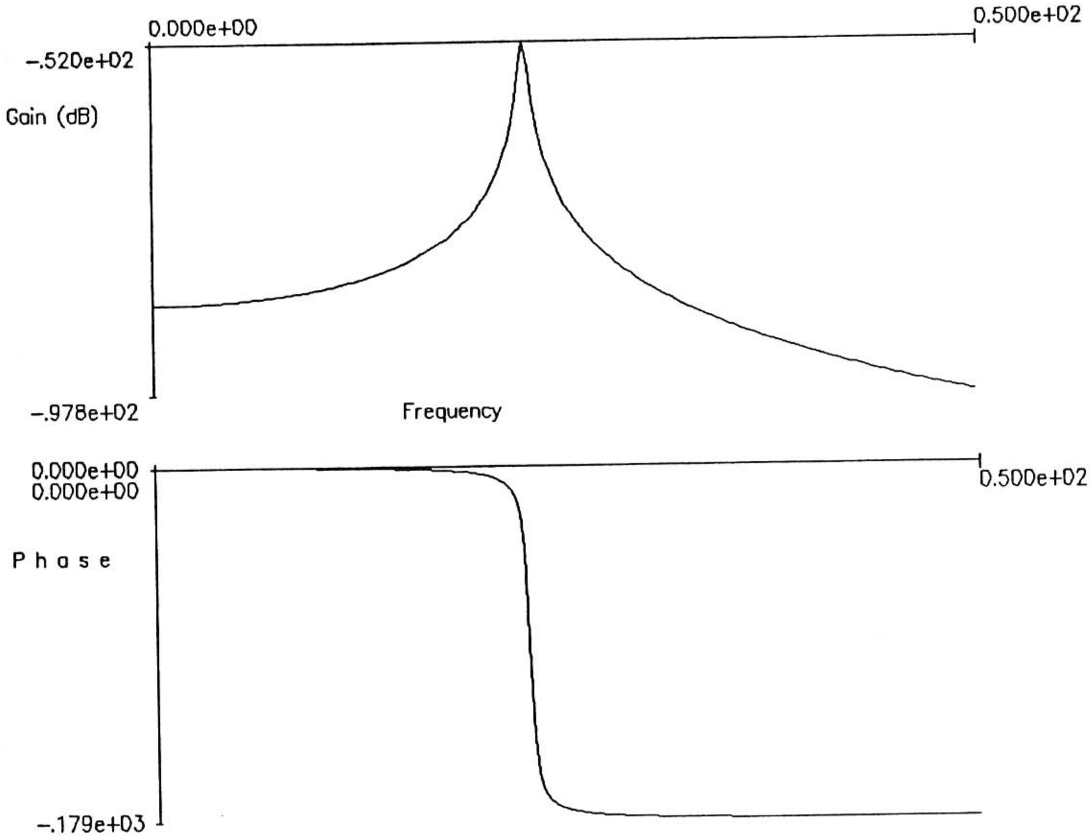

Figure 1
1st order Duffings frequency response function

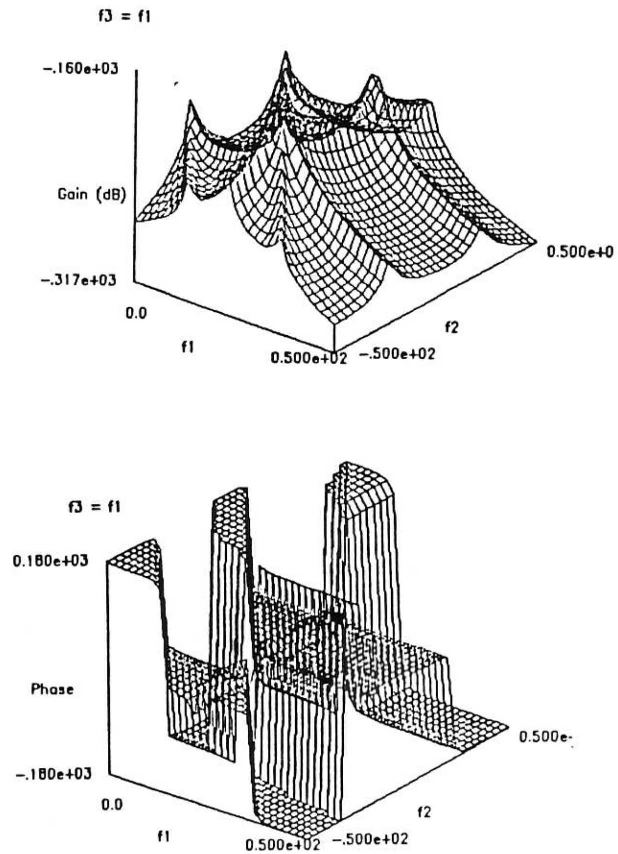

Figure 2
3rd order Duffings frequency response function ($f_3=f_1$)

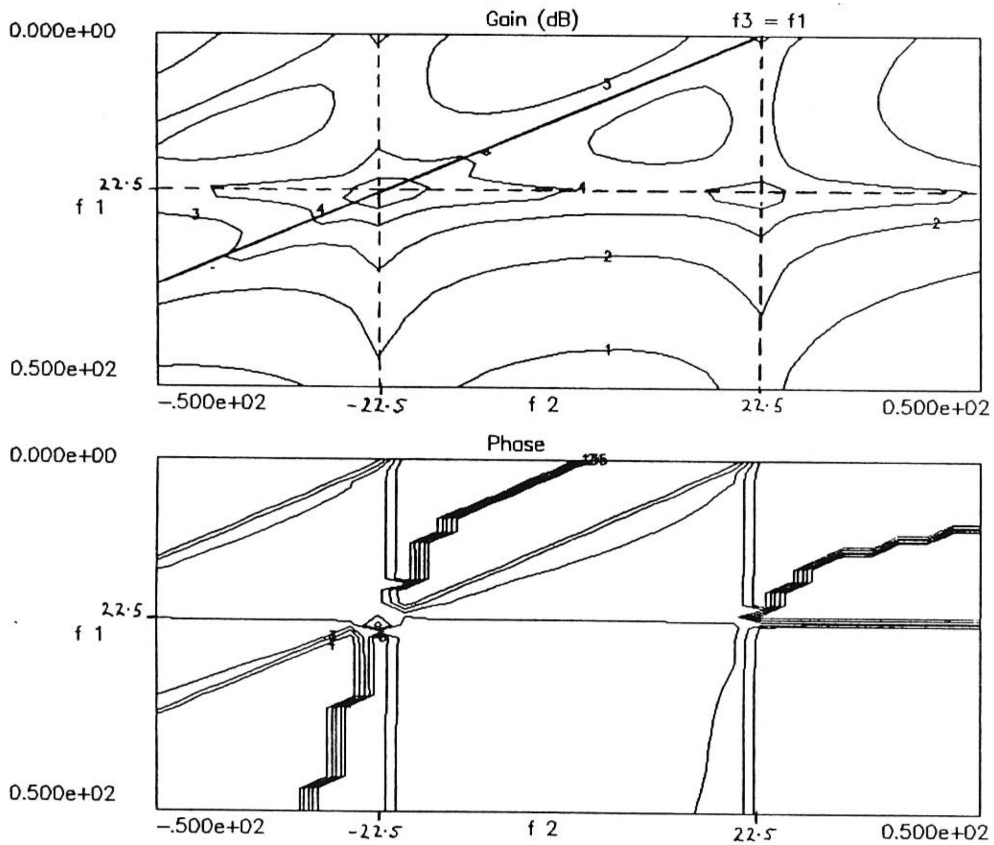

Figure 3
3rd order Duffings frequency response contour plot

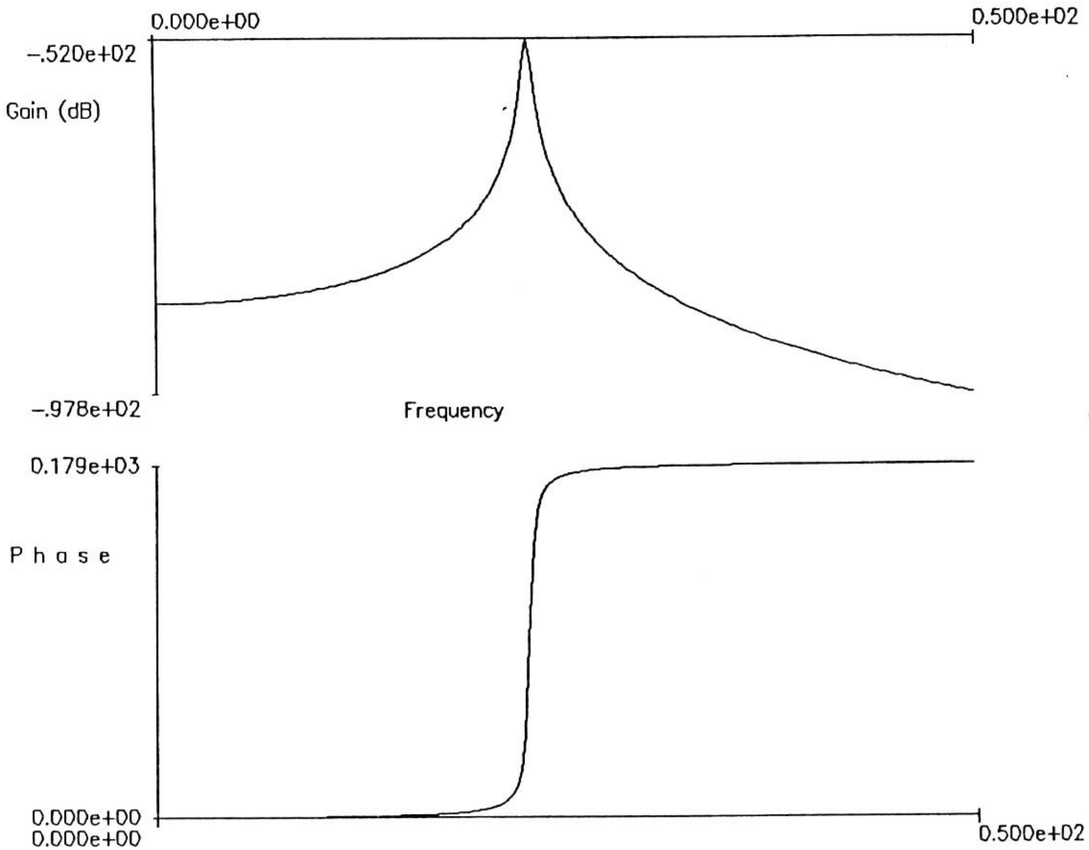

Figure 4
1st order Van-der-Pol frequency response function

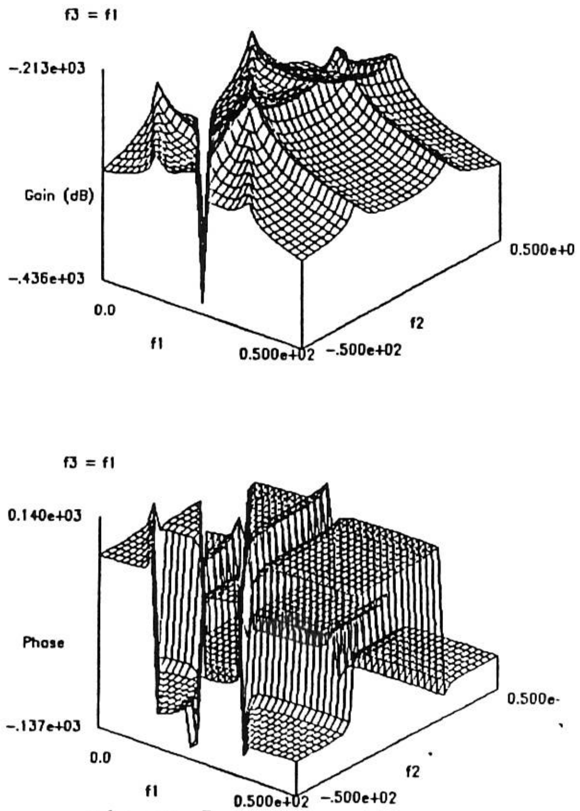

Figure 5
3rd order Van-der-Pol frequency response function ($f_3=f_1$)

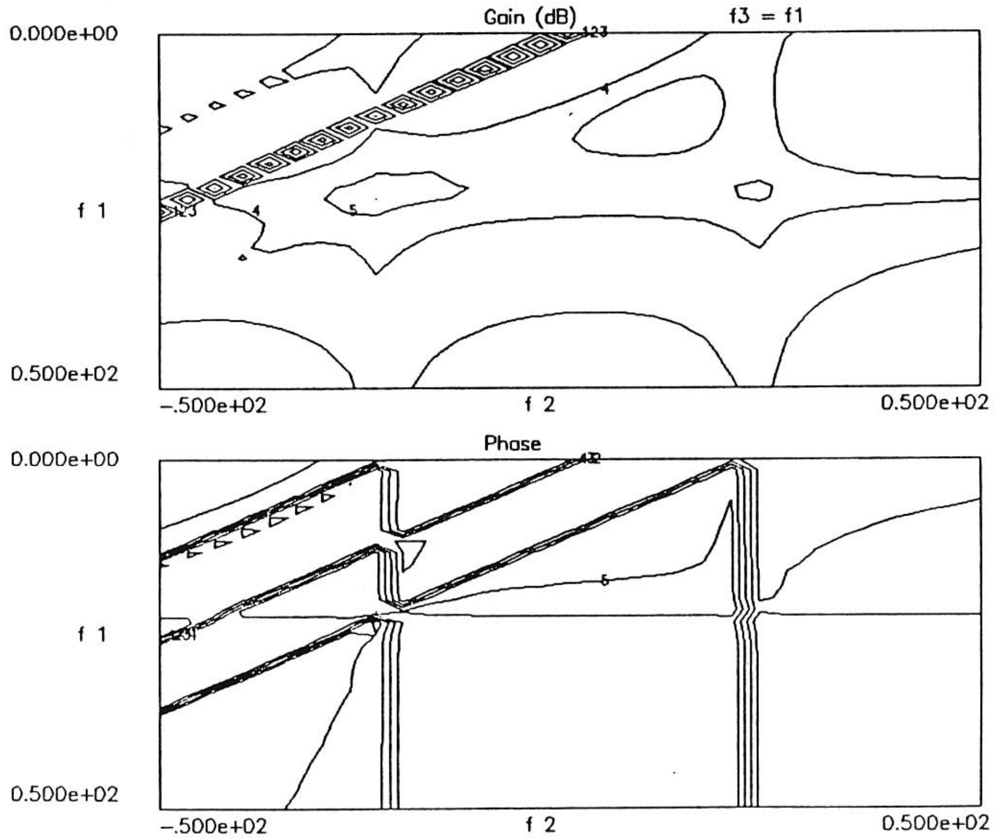

Figure 6
3rd order Van-der-Pol frequency response contour plot.