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Nonlinear Systems and Kolmogorov's Representation Theorem

by

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Abstract

A new representation for discrete dynamical systems is presented by applying Kolmogorov's representation theorem to the system functions .

Keywords : Kolmogorov's Representation Theorem , Separable Dynamical Systems .

1 Introduction

In this paper we shall consider the application of Kolmogorov's representation theorem ([1]) to discrete dynamical systems. The theorem states that any continuous function of n variables may be written in terms of functions of one variable. The result has been refined by a number of authors - see , for example, [2] and [3]. Here it will be convenient , however , to use the theorem in its original form , although the later versions of the result could be used in the same way.

In section 2 we shall show that any discrete system defined by continuous functions may be written in a separable form in which the defining equations consist of sums of functions of single variables. The stability of such systems will then be discussed in section 3 , by representing them in a 'quasi-linear 'form. Systems with controls are then considered in section 4 and a similar separable representation to that of section 2 is obtained. Finally the stability results of section 3 are applied to this representation in section 5.

2 Representation of Discrete Systems

In this section we shall consider a discrete nonlinear system of the form

$$x(k+1) - x(k) = f(x(k)) \tag{2.1}$$

for some function $f : R^n \longrightarrow R^n$, where $x(k) \in R^n$ for each n . First recall the famous theorem of Kolmogorov relating a general continuous function of n

variables to functions of one variable .

Theorem 2.1

For each $n \geq 2$ there exist continuous real functions $\psi^{pq}(x)$, defined on $E^1 = [0,1]$, such that every continuous real function $f(x_1, \ldots, x_n)$ defined on the n-cube E^n , is representable in the form

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^{n} \psi^{pq}(x_p) \right] .$$

□

We emphasize here that the functions ψ^{pq} are *independent* of f ;only χ_q depends on the particular function f under consideration . The functions ψ^{pq} are strictly monotonically increasing and have values in E^1 .

Corollary 2.2

If $n \geq 2$ and $f(y_1, \ldots, y_n)$ is a continuous function defined on the n-cube $[-a, a] \times \ldots \times [-a, a]$, then we can write

$$f(y_1, \ldots, y_n) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^{n} \psi^{pq} \left(\frac{y_p + a}{2a} \right) \right] .$$

where $\psi^{pq}(x)$ is defined on E^1 .

Proof

Put

$$x_i = \frac{y_i + a}{2a}$$

in theorem 2.1 . □

It follows that , if the solutions to (2.1) are bounded we can assume that $x(k+1)$

is in E^n for each k by scaling the variables $x(k)$; this will be done in the following discussion . Consider the p^{th} component of equation (2.1) , namely ,

$$x_p(k+1) = x_p(k) + f_p(x_1(k), x_2(k), \ldots, x_n(k)).$$

Then we have

$$\psi^{pq}(x_p(k+1)) = \psi^{pq}(x_p(k) + f_p(x_1(k), x_2(k), \ldots, x_n(k))) \tag{2.2}$$

Since the right hand side is continuous we can write

$$\psi^{pq}(x_p(k) + f_p(x_1(k), x_2(k), \ldots, x_n(k))) = \sum_{q'=1}^{2n+1} \chi_{q'}^{pq} \left[\sum_{r=1}^{n} \psi^{rq'}(x_r(k)) \right] \tag{2.3}$$

by theorem 2.1 , for some functions $\chi_{q'}^{pq}, 1 \leq p \leq n, 1 \leq q, q' \leq 2n+1$. Next introduce the new variables

$$\begin{aligned}
y_{1q'} &= \psi^{1q'}(x_1) \\
y_{2q'} &= \psi^{2q'}(x_2) \\
\ldots & \ldots \ldots \\
y_{n-1q'} &= \psi^{n-1q'}(x_{n-1}) \\
y_{nq'} &= \psi^{1q'}(x_1) + \psi^{2q'}(x_2) + \ldots + \psi^{nq'}(x_n), 1 \leq q' \leq 2n+1
\end{aligned}$$

Then , from (2.2) and (2.3) we have

$$\begin{aligned}
y_{1q}(k+1) &= \sum_{q'=1}^{2n+1} \chi_{q'}^{1q} \left[y_{nq'}(k) \right] \\
y_{2q}(k+1) &= \sum_{q'=1}^{2n+1} \chi_{q'}^{2q} \left[y_{nq'}(k) \right] \\
\ldots & \ldots \ldots
\end{aligned}$$

$$
\begin{aligned}
y_{nq}(k+1) &= \sum_{p=1}^{n}\sum_{q'=1}^{2n+1}\chi_{q'}^{pq}\left[\sum_{r=1}^{n}\psi^{rq'}(x_r(k))\right] \\
&= \sum_{p=1}^{n}\sum_{q'=1}^{2n+1}\chi_{q'}^{pq}\left[y_{nq'}(k)\right] \\
&= \sum_{q'=1}^{2n+1}\sum_{p=1}^{n}\chi_{q'}^{pq}\left[y_{nq'}(k)\right]
\end{aligned}
$$

Define the $(2n+1)^2$ functions $\chi_{q'}^{q}$ by

$$
\chi_{q'}^{q} = \sum_{p=1}^{n}\chi_{q'}^{pq}.
$$

Then we have

$$
y_{nq}(k+1) = \sum_{q'=1}^{2n+1}\chi_{q'}^{q}(y_{nq'}(k)),
$$

and we have proved

Theorem 2.3

A general discrete system of the form (2.1) may be transformed into the system

$$
z_q(k+1) = \sum_{q'=1}^{2n+1}\gamma_{q'}^{q}(z_{q'}(k)), 1 \le q \le 2n+1
$$

where

$$
z_{q'} = \psi^{1q'}(x_1) + \psi^{2q'}(x_2) + \ldots + \psi^{nq'}(x_n)
$$

for some functions $\chi_{q'}^{q}$. Moreover , we have

$$
\psi^{pq}(x_1) = \sum_{q'=1}^{2n+1}\chi_{q'}^{pq}\left[z_{q'}(k)\right], 1 \le p \le n-1
$$

where

$$
\gamma_{q'}^{q} = \sum_{p=1}^{n}\chi_{q'}^{pq}.
$$

By the strict monotonicity of the $\psi' s$ we can therefore evaluate x in terms of z. □

3 Stability of Separable Discrete Systems

The main point of theorem 2.3 is that we can write any discrete system in the following separable form :

$$\begin{aligned} x_1(k+1) &= f_{11}(x_1(k)) + f_{12}(x_2(k)) + \ldots + f_{1n}(x_n(k)) \\ \ldots & \ldots\ldots \\ x_n(k+1) &= f_{n1}(x_1(k)) + f_{n2}(x_2(k)) + \ldots + f_{nn}(x_n(k)) \end{aligned} \tag{3.1}$$

for some odd integer n . It is therefore desirable to study the systems of the form (3.1) in more detail. In this section we shall consider the stability theory of such systems. Note first that we can write this system in the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \ldots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} f_{11}(x_1(k) & f_{12}(x_2(k) & \ldots & f_{1n}(x_n(k) \\ f_{21}(x_1(k) & f_{22}(x_2(k) & \ldots & f_{2n}(x_n(k) \\ \ldots & \ldots & \ldots & \ldots \\ f_{n1}(x_1(k) & f_{n2}(x_2(k) & \ldots & f_{nn}(x_n(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \ldots \\ x_n(k) \end{bmatrix}$$

or

$$x(k+1) = A(x(k))x(k), \tag{3.2}$$

where

$$A(x(k)) = (f_{ij}(x_j)/x_j), x(k) = (x_1(k), \ldots, x_n(k))^T \tag{3.3}$$

The first result approximates $A(x(k))$ by a constant matrix A :

Lemma 3.1

Suppose that there exists a matrix A such that $A(x(k))$, given by (3.3) , satisfies

$$\| A(x) - A \| \leq K \| A \|, \forall x \in R^n$$

for some $K \geq 0$. Then the solution $x(k)$ of (3.2) is bounded by

$$\| x(k) \| \leq ((1+K) \| A \|)^k \| x(0) \| .$$

Proof

This is a standard approximation result ; however , since it is usually stated for continuous-time systems , we shall prove it here for completeness . Thus , from (3.2) we have

$$x(k+1) = Ax(k) + (A(x(k)) - A)x(k)$$

so that

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i}(A(x(i)) - A)x(i).$$

(This is the discrete variation of constants formula.) Hence

$$\| x(k) \| \leq \| A \|^k \| x(0) \| + \sum_{i=0}^{k-1} \| A \|^{k-1-i} \| A(x(i)) - A \| \| x(i) \|$$

or

$$\alpha(k) \leq \| x(0) \| + \sum_{i=0}^{k-1} K\alpha(i)$$

where

$$\alpha(k) = \| A \|^{-k} \| x(k) \|$$

Hence ,

$$\alpha(k) = (1+K)^k \| x(0) \|$$

(by the discrete Gronwall inequality) and so

$$\| x(k) \| \leq ((1+K) \| A \|^k \| x(0) \| .$$

□

Corollary 3.2

Under the hypotheses of lemma 3.1 , if A is stable and satisfies

$$\| A \| \leq 1 - \epsilon$$

for some $\epsilon > 0$, then the system (3.2) is stable if $\frac{\epsilon}{1-\epsilon} > K$, i.e. if

$$\| A(x) - A \| < 1 - \| A \| . \tag{3.4}$$

□

Remark

Note that (3.4) implies that $\| A(x) \| < 1$ for all $x \in R^n$. If $A(x)$ is continuous in x then this implies stability in the following simple way . We have

$$\| x(k+1) \| \leq \| A(x(k)) \| \| x(k) \| . \tag{3.5}$$

Since $\| A(x) \|$ is continuous and $\| A(x) \| < 1$ we have

$$\| A(x) \| \leq \alpha(\delta) < 1 \quad , \quad x \in B_\delta = \{x : \| x \| \leq \delta\}$$

for some $\alpha(\delta)$ by compactness of B_δ . Now , by (3.5) , $\| x(k) \|$ is decreasing so that if $\| x(0) \| = \delta_1$, then

$$\| A(x(k)) \| \leq \alpha(\delta_1) < 1$$

for all k , so that by (3.5) again

$$\begin{aligned} \| x(k+1) \| &\leq \alpha(\delta_1) \| x(k) \| \\ &\leq (\alpha(\delta_1))^k \| x(0) \| \\ &\longrightarrow 0 \end{aligned}$$

as $k \longrightarrow \infty$. Thus lemma 3.1 is unnecessary in this case . However , if $A(x)$ is not continuous then the inequality $\| A(x) \| < 1$ for all x is *not* sufficient for (asymptotic) stability ; we require $\| A(x) \| \leq \alpha < 1$ uniformly in x . □

Let $r(C)$ denote the spectral radius of the matrix C ; then

$$r(C) \leq \| C \|$$

for any (induced) norm $\| . \|$ on the set of matrices , while

$$r(C) = \| C \|_{S(C)}$$

where $\| . \|_{S(C)}$ is the spectral norm (with respect to C). Choosing the norm in corollary 3.2 to be $\| . \|_{S(A)}$ we obtain

Corollary 3.3

Under the hypotheses of lemma 3.1 , if A is stable and satifies

$$\| A \|_{S(A)} \leq 1 - \epsilon$$

for some $\epsilon > 0$, then the system (3.2) is stable if

$$r(A(x) - A) < 1 - r(A).$$

Proof

By corollary 3.2 we have

$$\begin{aligned} r(A(x) - A) &\leq \| A(x) - A \|_{S(A)} \\ &< 1 - \| A \|_{S(A)} \\ &= 1 - r(A). \end{aligned}$$

□

Theorem 3.4

Suppose the matrix function $A(x) = (f_{ij}(x_j)/x_j)$ satisfies the following condition : Each function f_{ij} has the form

$$f_{ij}(x_j) = a_{ij}x_j + g_{ij}(x_j)$$

and the matrices

$$A = (a_{ij}), G(x) = (g_{ij}(x_j)/x_j)$$

satisfy the spectral inequality

$$r(G(x)) < 1 - r(A)$$

for all $x \in R^n$. Then the system (3.2) is stable . □

(This follows immediately from corollary 3.3.) Note that g_{ij} is not assumed to be continuous .

4 Systems with Control

Consider next the nonlinear control system

$$x(k+1) = x(k) + f(x(k), u(k)) \quad , \tag{4.1}$$

where

$$x \in R^n \quad , \quad u \in R^m.$$

Then , as in theorem 2.1 , we can write

$$f(x,u) = \sum_{q=1}^{2(n+m)+1} \chi_q \left[\sum_{p=1}^{n} \psi^{pq}(x_p) + \sum_{r=1}^{m} \psi^{r+n,q}(u_r) \right]$$

for any continuous function $f : R^{n+m} \longrightarrow R$ where the function $\psi^{pq}, 1 \leq p \leq n+m, 1 \leq q \leq 2(n+m)+1$ are independent of f . Hence , as in section 2 , we have

$$\begin{aligned} \psi^{pq}(x_p(k+1)) &= \psi^{pq}\left(x_p(k) + f_p(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k))\right) \\ &= \sum_{q'=1}^{2(n+m)+1} \chi_{q'}^{pq} \left[\sum_{r=1}^{n} \psi^{rq'}(x_r) + \sum_{r=1}^{m} \psi^{n+r,q'}(u_r) \right] \end{aligned}$$

for some functions $\chi_{q'}^{pq}$. $(1 \leq p \leq n+m, 1 \leq q \leq 2(n+m)+1.)$ As before , define

$$\begin{aligned} y_{1q} &= \psi^{1q}(x_1) \\ \ldots & \ldots \quad \ldots \\ y_{n-1q} &= \psi^{n-1q}(x_{n-1}) \\ y_{nq} &= \psi^{1q}(x_1) + \ldots + \psi^{nq}(x_n) = z_q \end{aligned}$$

and

$$v_q = \sum_{r=1}^{m} \psi^{n+r,q}(u_r).$$

Then we have

$$z_q(k+1) = \sum_{q'=1}^{2(n+m)+1} \gamma_{q'}^{q}(z_{q'}(k) + v_{q'})$$

where

$$\gamma_{q'}^{q} = \sum_{p=1}^{n} \chi_{q'}^{pq}.$$

Note , however , that the controls v_q are not independent . In fact , v can be chosen only from the nonlinear subset of $R^{2(n+m)+1}$ which is the image of the function $g : R^m \longrightarrow R^{2(n+m)+1}$ given by

$$g_q(u) = \sum_{r=1}^{m} \psi^{n+r,q}(u_r). \tag{4.2}$$

We have therefore proved the control analogue of theorem 2.3 , namely :

Theorem 4.1

A general discrete control system of the form (4.1) (where f is continuous) may be written in the form

$$z_q(k+1) = \sum_{q'=1}^{2(n+m)+1} \gamma_{q'}^{q}(z_{q'}(k) + v_{q'})$$

where

$$z_q = \psi^{1q}(x_1) + \ldots + \psi^{nq}(x_n), 1 \leq q \leq 2(n+m)+1$$

for some functions $\gamma_{q'}^{q}$. The v control space is a (nonlinear) subset of $R^{2(n+m)+1}$ given by the image of the function g as in (4.2) . □

5 Application to Control of Nonlinear Systems

From theorem 4.1 we can write any discrete control system (with continuous dynamics) in the form

$$\begin{aligned} x_1(k+1) &= f_{11}(x_1(k)+v_1(k))+\ldots+f_{1N}(x_N(k)+v_N(k)) \\ \ldots & \ldots\ldots \\ x_N(k+1) &= f_{N1}(x_1(k)+v_1(k))+\ldots+f_{NN}(x_N(k)+v_N(k)) \end{aligned} \tag{5.1}$$

for some odd integer N where $v(k) \in \Gamma \subseteq R^N$ for some subset Γ . As before we can express this system in a 'pseudo-linear 'form

$$x(k+1) = A(x(k)+v(k))x(k),$$

where

$$[A(x(k)+v(k))]_{ij} = f_{ij}(x_j(k)+v_j(k))/x_j(k).$$

In order to use the theory of section 3 to obtain a stabilizing controller , we must determine the values of $v(k)$ and $x(k)$ for which $A(x(k)+v(k))$ is 'close' to some constant matrix A . For fixed $x \in R^n$ let $G(x)$ be the subset of those elements v of Γ for which

$$\| A(x+v) \| < 1.$$

(Of course , $G(x)$ may be empty .) Moreover , let $X \subseteq R^n$ be defined by

$$X = \{x \in R^n : G(x) \neq \emptyset\}.$$

Finally , we shall call a subset $S \subseteq X$ *invariant* (under the dynamics of (5.1)) if , for each $x \in X$ there exists $v = v(x) \in G(x)$ such that

$$f(x+v) \triangleq (f_{11}(x_1+v_1)+\ldots+f_{1N}(x_N+v_N),\ldots, \tag{5.2}$$

$$f_{N1}(x_1+v_1)+\ldots+f_{NN}(x_N+v_N)) \in S.$$

Let

$$X_I \;=\; \cup\{S \subseteq X \;:\; S \; invariant\},$$

i.e X_I is the largest invariant set in X . Then we have

Theorem 5.1

If $X_I \neq \emptyset$, $0 \in X_I$ and there exists a stable matrix A such that for each $x \in X_I$, and for each $v \in G(x)$ we have

$$\| A(x+v) - A \| < 1 - \| A \| \tag{5.3}$$

on the set X_I then the system (5.1) is stabilizable .

Proof

This follows from corollary 3.2 by choosing $v \in G(x)$ so that $f(x+v) \in X_I$ for any $x \in X_I$, since the solution is then guaranteed to belong to X_I for all k . □

Remark

This result also holds if X_I is replaced by any invariant set S for which $0 \in S$.□

Corollary 5.2

If $X = R^n$ and there exists a stable matrix A such that (5.2) holds , then the system (5.1) is stabilizable . □

6 Conclusions

A new representation for nonlinear systems has been introduced by applying Kolmogorov's representation theorem to the right hand sides of the system equations . It has been shown that a difference equation (without control) can be written in a separable form while a control system has a similar representation with the controls being restricted to a certain subset of the space .

Some stabilizability results for such separable systems have been proved. Because of the nature of the functions in Kolmogorov's theorem the implementation of the ideas contained in this paper must be based on numerical procedures . These will be examined in a future paper .

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