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Fast Sampling And Stability Of Nonlinear Sampled-Data Systems -- Part I: Existence Theorems

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Part I: Existence Theorems

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Abstract: The objective of this paper is to investigate how the choice of sampling interval is related to the stability of a class of nonlinear sampled-data systems, and in particular how fast sampling may stabilise a sampled-data system when the underlying continuous system is known to be stable. In this the first part of the paper two fast sampling theorems are derived for a class of nonlinear sampled-data systems and it is shown that provided the underlying continuous system is stable, there exists a maximum sampling interval such that when the system is sampled below this interval it will remain stable. In the second part of the paper a special class of nonlinear sampled-data systems are studied and an analytical relationship between sampling rates and the domains of attraction of the system is derived.

1. Introduction

Due to the rapid development in computer technology, many systems need to be studied as sampled-data systems. Although it is a well known fact that a sampled-data system can become unstable if the sampling rate is not properly chosen, analytical studies to obtain a relationship between sampling rate and the domain of attraction of a general nonlinear sampled-data system have not been carried out. This is due to the fact that most studies on sampled-data systems to date have not taken sampling rate as a parameter in the modelling of sampled-data systems. For instance, any modelling involving the z-domain transfer function of the system assumes a known fixed sampling rate. When a sampled-data system is modelled approximately by a set of difference equations using numerical methods, any relationship obtained on the sampling rate and domain of attraction based on the approximate discrete modelling will be valid for that model. However its validity on the original sampled-data system will always be in doubt due to the errors associated with approximate modelling [1].

In this paper, the concept of hybrid modelling is introduced, that is the behaviour of the sampled-data system during the k^{th} sampling interval ($0<k<\infty$) is modelled by a set of nonlinear differential equations valid during the time length $h_k=t_k-t_{k-1}$. As k varies the initial conditions of the differential equations form a sequence which is a sampled-data sequence of the system, and it is assumed that the complete time range of interest is covered as $k\rightarrow\infty$. This hybrid model describes a nonlinear sampled-data system exactly without attempting to digitise it, and it takes the k^{th} sampling period h_k as a parameter of the model. An important consequence of such modelling is that h_k is allowed to vary with k, hence allowing variable sampling rate in sampled-data systems.

Based on such modelling, section 2 below studies a class of nonlinear sampled-data systems whose underlying continuous system is stable on some domain about an equilibrium point of interest. Intuitively, the sampled-data system would be expected to be stable provided it is sampled fast enough. Two theorems will be derived to show that there exists a finite time interval h^* independent of any sampled datum such that provided the sampled-data system is sampled fast enough ($h_k\leq h^*$, k=1,2,...), it will be stable or asymptotically stable on some domain about the equilibrium point of interest. Section 3 relates the stability conditions given by those theorems in section 2 to the Lyapunov function matrix of the linearised underlying continuous system, making the theorems easier to apply in practice. Conclusions are given in section 4. A detailed study to actually obtain a relationship between the sampling rate and the estimated domain of attraction of a special class of sampled-data systems will be given in part II of this paper.

2. Fast sampling theorems

The sampled-data system to be studied here has the general mathematical model

$$\dot{x} = f(x, x_k), \qquad x(t_k)=x_k, \quad t \in [t_k, t_{k+1}[, \quad k \geq 0 \tag{2.1}$$

where $x \in \mathbf{R}^n$; the mapping f: $\mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a nonlinear mapping in general with $f(0,0)=0$; the $(k+1)^{th}$ sampling period is $h_{k+1} = t_{k+1} - t_k$, and $t_k \rightarrow \infty$ as $k \rightarrow \infty$. The origin is assumed to be the equilibrium point of interest without loss of generality. The differential equation holds only during each sampling period within which x_k (k=0,1,2,...) is constant. In the following analysis, in order to avoid referring to any particular sequence, x_s will be used to denote the sampled datum at some sampling instant t_s.

The aim of this stability analysis is to search for conditions to be satisfied by system (2.1) so that it will have the required stability properties. Several notations are first introduced.

Define the mapping V: $\mathbf{R}^n \rightarrow \mathbf{R}_+$ as the quadratic function

$$V = x^T P\, x, \qquad P = P^T > 0 \quad (P \text{ real}) \tag{2.2}$$

Thus, V is positive definite. Define the P-matrix norm as

$$\|x\|_P = (x^T P\, x)^{1/2} = V(x)^{1/2} \tag{2.3}$$

so that X=($\|.\|_P$, $\mathbf{R}^n$) represents a Banach space. Let D, A_{δ_0} and B_{δ_0} be three compact regions in X defined by

$$D = \{ x \,/\, V(x) \le d \} \tag{2.4}$$

$$A_{\delta_0} = \{ x \,/\, \delta_0 \le V(x) \le d \} \tag{2.5}$$

$$B_{\delta_0} = \{ x \,/\, V(x) \le \delta_0 \} \tag{2.6}$$

respectively, as shown in Fig.1, with $\delta_0<d$.

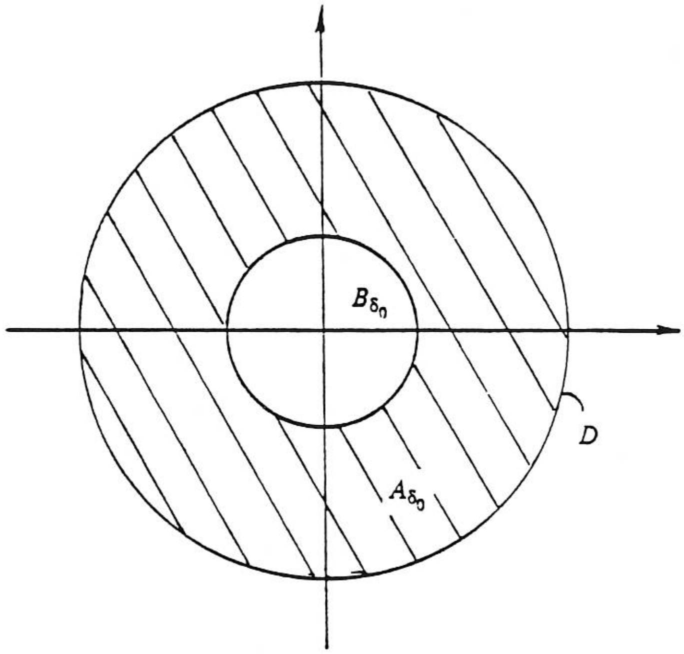

Fig. 1

With the above notations and definitions, the stability analysis of the sampled-data system will be carried out in three stages. Initially lemma 2.1 will show that if the mapping $V(x_s)$ decays exponentially $\forall x_s \in A_{\delta_0}$, then there exists a finite time period *independent of* x_s during which V still decays exponentially, bounded by a smaller decay constant. This implies that the motion $x(t;x_s,t_s)$ of the system converges towards a neighbourhood of the origin. Then (lemma 2.2), it will show that for any sampling sequence $\{x_k, k=0,1,2,...\}$ sampled at intervals less than the above period, there exists a finite $k>0$, where k is dependent on the initial condition x_0, such that x_k falls into the region B_{δ_0} provided x_{k-1} lies still in A_{δ_0}. Finally (lemma 2.3), it will show that for all $x_s \in B_{\delta_0}$, there exists a finite time period *independent of* x_s such that the motion $x(t;x_s,t_s)$ is bounded in D. Hence, the system (2.1) is stable in the region

D in the sense that every motion of the system starting from it is bounded in it.

Lemma 2.1. Assume that the nonlinear mapping f in system (2.1) is continuous on the domain $D \times A_{\delta_0}$. Let the mapping x: $(x_s,t_s,t) \rightarrow x(t;x_s,t_s)$ denote a motion of the dynamical system (2.1) for all $x_s \in A_{\delta_0}$, i.e., $x(t;x_s,t_s)$ exists uniquely and is continuous with respect to its arguments. If the system (2.1) satisfies the inequality

$$\dot{V}(x_s) \leq -\xi\, V(x_s)\,, \qquad \forall x_s \in A_{\delta_0}\,, \quad \xi > 0, \tag{2.7}$$

then there exists a finite sampling period $0 < h_a^* < \infty$ such that

$$\dot{V}(x(t,x_s,t_s)) < -\frac{\xi}{2}\, V(x(t,x_s,t_s))\,, \qquad \forall x_s \in A_{\delta_0}\,, \quad t \in [t_s, t_s + h_a^*[\,. \tag{2.8}$$

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Essentially, this lemma suggests that

(a) at the sampling instant t_s, if $V(x_s)$ decays exponentially at a rate bounded by ξ for all $x_s \in A_{\delta_0}$, then $V(x(t;x_s,t_s))$ will keep decay exponentially at a rate bounded by $\frac{\xi}{2}$ provided t is sufficiently small;

(b) the finite time period for the above to be true is independent of x_s for all $x_s \in A_{\delta_0}$.

The proof below first shows that $\forall x_s \in A_{\delta_0}$, the 'distance' between x_s and the subsequent motion $x(t;x_s,t_s)$ can be made as small as required by taking a sufficiently small time period independent of x_s. It then proves that for all $x_s \in A_{\delta_0}$, provided the 'distance' between x_s and $x(t;x_s,t_s)$ is small enough, the inequality (2.7) implies the inequality (2.8). This 'distance' is in terms of the P-matrix norm and is closely related to the quadratic mapping V.

Proof. Without loss of generality, assume $t_s=0$ so that $x(t;x_s,t_s)$ can be written simply as $x(t,x_s)$. As $x(t,x_s)$ is continuous with respect to its arguments, for all $t \in [0,T_1]=I_1$ where T_1 is arbitrarily large and for all $x_s \in A_{\delta_0}$, $x(t,x_s)$ is continuous on the compact domain $I_1 \times A_{\delta_0}$, and therefore $x(t,x_s)$ is *uniformly continuous* on $I_1 \times A_{\delta_0}$. Thus for each

$\delta>0$, there exist $\varepsilon_1>0$ and $\varepsilon_2>0$ such that

$$\|x(t_1, x_{s1}) - x(t_2, x_{s2})\|_P < \delta \tag{2.9}$$

provided $t_1,t_2\in I_1$, $x_{s1},x_{s2}\in A_{\delta_0}$, and $|t_1-t_2|<\varepsilon_1$, $\|x_{s1}-x_{s2}\|_P<\varepsilon_2$. By setting $t_1=t$, $t_2=0$, $x_{s1}=x_{s2}=x_s$ and noting that $x(0,x_s) = x_s$, (2.9) becomes

$$\|x(t, x_s) - x_s\|_P < \delta \,, \qquad \forall x_s\in A_{\delta_0} \tag{2.10}$$

provided $|t|<\varepsilon_1$, or $t\in[0,\varepsilon_1[$ since only the positive time is considered. In other words, provided the time period t is small enough, the motion $x(t,x_s)$ can be made close enough to its initial condition x_s $\forall x_s\in A_{\delta_0}$.

Now, let the above δ be such that $\delta\leq\delta_a<\delta_0^{1/2}$ where δ_0 is defined in (2.6); let the corresponding $\varepsilon_1\leq h_a^*$. Define $\gamma = \delta_0^{1/2} - \delta_a$ so that γ is positive finite. Then, because

$$\|x_s\|_P \leq \|x_s - x\|_P + \|x\|_P \tag{2.11}$$

this yields, for $t\in[0,h_a^*[$,

$$\begin{aligned}\|x\|_P &\geq \|x_s\|_P - \|x_s - x\|_P \\ &\geq \delta_0^{1/2} - \delta_a \\ &= \gamma\end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned}\|x\|_P &\leq \|x - x_s\|_P + \|x_s\|_P \\ &\leq \delta_a + d^{1/2}\,, \qquad x_s\in A_{\delta_0}\end{aligned} \tag{2.13}$$

where d is given by (2.4). Hence, provided $t\in[0,h_a^*[$ and $x_s\in A_{\delta_0}$, the motion trajectory $x(t,x_s)$ of the system is bounded in a region D_e defined by

$$D_e = \{\, x \;/\; \gamma \leq \|x\|_p \leq \delta_a + d^{1/2} \,\} \tag{2.14}$$

Note that D_e extends the domain A_{δ_0} on both its lower and upper bounds. The next step is to prove that (2.7) implies (2.8). By noting that

$$\begin{aligned}\dot V &= \dot x^T P x + x^T P \dot x = 2x^T P \dot x \\ &= 2x^T P f(x, x_s)\end{aligned} \tag{2.15}$$

the inequality (2.7) can be written as

$$2x_s^T P f(x_s, x_s) \le -\xi\, x_s^T P x_s\,, \qquad x_s \in A_{\delta_0}$$

or, because $x_s \in A_{\delta_0}$, which does not include the origin, the above inequality can be written as

$$\frac{2x_s^T P f(x_s, x_s)}{x_s^T P x_s} \le -\xi\,, \qquad x_s \in A_{\delta_0}\,. \tag{2.16}$$

Similarly, the inequality (2.8) can be written as

$$\frac{2x^T P f(x, x_s)}{x^T P x} \le -\frac{\xi}{2}\,, \qquad (x, x_s) \in D_e \times A_{\delta_0}\,. \tag{2.17}$$

Thus, define a mapping $L\colon D_e \times A_{\delta_0} \to \mathbf{R}$ as

$$L(x, x_s) = \frac{2x^T P\, f(x, x_s)}{x^T P\, x} \tag{2.18}$$

Now in lemma 2.1, f is assumed to be continuous on $D \times A_{\delta_0}$. In the following proof, it will initially be assumed that f is continuous on $D_e \times A_{\delta_0}$ so that the motion $x(t, x_s)$ is allowed to lie anywhere in D_e. Later analysis will then show that $x(t, x_s)$ actually lies only in D and the assumption on f made in lemma 2.1 is thus restored. Hence, assume f is continuous on the compact region $D_e \times A_{\delta_0}$. L is then continuous on $D_e \times A_{\delta_0}$ and thus L is uniformly continuous on $D_e \times A_{\delta_0}$. Thus, for each $\varepsilon > 0$, there exist $\delta > 0$ and $\delta_s > 0$ such that

$$|L(x_1, x_{s1}) - L(x_2, x_{s2})| < \varepsilon \tag{2.19}$$

whenever $x_1, x_2 \in D_e$, $x_{s1}, x_{s2} \in A_{\delta_0}$ and $\|x_1 - x_2\|_P < \delta$, $\|x_{s1} - x_{s2}\|_P < \delta_s$. Now set $x_1 = x$, $x_2 = x_{1s} = x_{2s} = x_s$ and note that

$$L(x_s, x_s) \le -\xi, \qquad x_s \in A_{\delta_0}, \tag{2.20}$$

If ε is set at $\varepsilon = \frac{\xi}{2}$ and the corresponding $\delta = \delta'$, $\delta_s = \delta_s'$, then,

$$\begin{aligned} L(x, x_s) &\le |\, L(x, x_s) - L(x_s, x_s)\,| + L(x_s, x_s) \\ &< \frac{\xi}{2} - \xi \\ &= -\frac{\xi}{2} \end{aligned} \tag{2.21}$$

whenever

$$\|x - x_s\|_P < \delta', \qquad (x,x_s)\in D_\epsilon\times A_{\delta_0} \tag{2.22}$$

As it is always possible to let h_a^* defined above be small enough so that δ_a satisfies

$$\delta_a \le \delta' \tag{2.23}$$

if x is a point on the solution $x(t,x_s)$ for some t, then provided $x_s\in A_{\delta_0}$ and $t\in[0,\, h_a^*[$, the distance between x and x_s is bounded by $\|x-x_s\| < \delta_a \le \delta'$. Hence (2.21) is satisfied if $t\in[0,\, h_a^*[$ where h_a^* is independent of x_s. Substituting into (2.21) the definitions of L and V, it is easy to see that

$$\dot V(x(t,x_s)) \;<\; -\frac{\xi}{2}\, V(x(t,x_s))\,, \qquad x_s\in A_{\delta_0}, \quad t\in[0,\, h_a^*[\,.$$

Now, return to the original assumption that f is continuous on $D\times A_{\delta_0}$. Because the above inequality yields

$$V(x(t,x_s)) \;<\; e^{-\frac{\xi}{2}t}\, V(x_s)$$

it is clear that $\|x(t,x_s)\|_P$ can never exceed $\|x_s\|_P$ provided $t\in[0,h_a^*[$, and $x_s\in A_{\delta_0}$. Consequently $\|x(t,x_s)\|_P$ is actually bounded by

$$\gamma \;\le\; \|x(t,x_s)\|_P \;\le\; d^{1/2}$$

and therefore $x(t,x_s)$ lies always within the domain D. Thus f only needs to be continuous on $D\times A_{\delta_0}$ for lemma 2.1 to be true.

Q.E.D.

The above proof has shown the necessity of introducing the domain A_{δ_0}: to ensure the continuity of L. The assumption that f is continuous on $D\times A_{\delta_0}$ allows f to possess some 'odd' characteristics near the origin for $x_s\in B_{\delta_0}$. For instance, f can be discontinuous or a sector bounded function with respect to x_s and the behaviour of the system in the vicinity of the origin may not be 'nice'.

The next lemma concerns the discrete property of the sampled-data system under the assumption of lemma 2.1. It is quite clear from the proof above that if $x_{k-1} \in A_{\delta_0}$, then $\|x_k\|_P < \|x_{k-1}\|_P$ provided the sampling interval is sufficiently small, i.e., x_k is 'closer' to the origin. But as soon as x_k enters B_{δ_0}, lemma 2.1 no longer holds. Evidently, for every $x_0 \in A_{\delta_0}$, there exists a sampling sequence $\{x_k\}$ which converges to the domain B_{δ_0}.

Lemma 2.2. Assume lemma 2.1 holds and let x_0 represent the initial condition of the sampled-data system (2.1). Then, there exists a finite time duration $T_{k(x_0)} > 0$ such that the sampled datum $x_k = x(T, x_0)$ of the solution sequence $\{x_k\}$ of the system lies in B_{δ_0} for all $x_0 \in A_{\delta_0}$.

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Proof. From lemma 2.1, inequality (2.7) implies the existence of an h_a^* such that (2.8) holds. Then, for any $x_0 \in A_{\delta_0}$,

$$V(x(t, x_0)) < e^{-\frac{\xi}{2} t} V(x_0) , \qquad t \in [0, h_a^*[, \tag{2.24}$$

and in particular,

$$V(x_1) < e^{-\frac{\xi}{2} h_1} V(x_0) , \qquad h_1 < h_a^* \tag{2.25}$$

where $x_1 = x(h_1)$. Similarly,

$$\begin{aligned} V(x_2) &< e^{-\frac{\xi}{2} h_2} V(x_1) \\ &< e^{-\frac{\xi}{2} (h_1+h_2)} V(x_0) \end{aligned} \tag{2.26}$$

By induction,

$$V(x_j) < e^{-\frac{\xi}{2} (h_1+h_2+\ldots+h_j)} V(x_0) , \qquad 0 < h_i < h_a^* \tag{2.27}$$

provided x_{j-1} is still within A_{δ_0}. It is clear from (2.27) that x_j is closer to the region B_{δ_0} than x_0. Now, suppose during the k^{th} sampling period, $x(t, x_{k-1})$ crosses the boundary $V(x) = \delta_0$ and enters the region B_{δ_0}, then, (2.27) says that x_k stays within B_{δ_0} provided $x_{k-1} \in A_{\delta_0}$ and $h_k < h_a^*$. Therefore, there exists a $T_{k(x_0)}$ dependent on x_0 where

$$T_{k(x_0)} = h_1 + h_2 + \cdots + h_k, \qquad 0<h_i<h_a^* \tag{2.28}$$

such that x_k lies within B_{δ_0}.

Q.E.D.

After the sampling sequence $\{x_k\}$ enters the region B_{δ_0}, because the behaviour of the system in B_{δ_0} can be irrational, the sequence may or may not stay in B_{δ_0} for $t>T$. The following lemma states a rather mild requirement on f which will ensure the stability of the sampled-data system on the domain D.

Lemma 2.3. If the mapping f is such that $x(t;x_s,t_s)$ exists uniquely for all $x_s \in B_{\delta_0}$ and is continuous with respect to its arguments, then there exists a finite sampling period $0 < h_b^* \leq \infty$ such that for all $x_s \in B_{\delta_0}$ and $t \in [0, h_b^*[$, $x(t;x_s,t_s)$ is bounded in D.

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Proof. Again assume $t_s=0$ without loss of generality and write $x(t;x_s,t_s)$ as $x(t,x_s)$. As $x(t,x_s)$ is unique and continuous with respect to its arguments, for all $t \in [0,T_1]=I_1$ where T_1 is arbitrarily large and for all $x_s \in B_{\delta_0}$, $x(t,x_s)$ is continuous on the compact domain $I_1 \times B_{\delta_0}$, and hence $x(t,x_s)$ is uniformly continuous on $I_1 \times B_{\delta_0}$. Therefore, for each $\delta>0$, there exist $\varepsilon_1>0$ and $\varepsilon_2>0$ such that

$$||x(t_1,x_{s1})-x(t_2,x_{s2})||_P < \delta \tag{2.29}$$

provided $t_1,t_2 \in I_1$, $x_{s1},x_{s2} \in B_{\delta_0}$, and $|t_1-t_2|<\varepsilon_1$, $||x_{s1}-x_{s2}||_P<\varepsilon_2$. By letting $t_1=t$, $t_2=0$, $x_{s1}=x_{s2}=x_s$ and noting that $x(0,x_s)=x_s$, (2.29) becomes

$$||x(t,x_s)-x_s||_P < \delta , \qquad x_s \in B_{\delta_0} \tag{2.30}$$

provided $|t|<\varepsilon_1$, or $t \in [0,\varepsilon_1[$ since only positive time is concerned. Therefore, provided $x_s \in B_{\delta_0}$ and $t \in [0,\varepsilon_1[$, $x(t,x_s)$ is bounded by

$$\begin{aligned} ||x(t,x_s)||_P &\leq ||x(t,x_s)-x_s||_P + ||x_s||_P \\ &< \delta + \delta_0^{1/2} \end{aligned} \tag{2.31}$$

If δ is set to be

$$\delta = \delta_b \leq d^{1/2} - \delta_0^{1/2} \tag{2.32}$$

then, there exists a corresponding $\varepsilon_1 = h_b^* < \infty$ such that

$$\|x(t,x_s)\|_P < d^{1/2}, \qquad x_s \in B_{\delta_0} \tag{2.33}$$

provided $t \in [0, h_b^*[$. Thus, $x(t,x_s)$ is bounded in D.

Q.E.D.

The above three lemmas easily lead to the following theorem which concerns the stability of the sampled-data system (2.1).

Theorem 2.4 (Bounded Stability). Consider a sampled-data system whose mathematical model is represented by eq.(2.1). Let V be the quadratic mapping defined in eq.(2.2) and let D, A_{δ_0} and B_{δ_0} be defined as in eq.s(2.4), (2.5) and (2.6) respectively.

If

(a) $f(x, x_s)$ is continuous on the domain $D \times A_{\delta_0}$;

(b) f is such that for all $x_s \in D$, $x(t; x_s, t_s)$ exists uniquely and is continuous with respect to its arguments;

(c) $2x^T P f(x,x) \leq -\xi x^T P x$, $\quad \forall x \in A_{\delta_0}$, $\quad \xi > 0$;

then, there exists an $h^* > 0$ such that for every initial condition $x(t_0) = x_0 \in D$, the motion $x(t; x_0, t_0)$ of the sampled-data system stays in D for all $t > 0$ provided $0 < t_{k+1} - t_k < h^*$.

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Proof. The proof simply follows from lemmas 4.1, 4.2 and 4.3, and h^* is chosen such that

$$h^* = \min(h_a^*, h_b^*) \tag{2.34}$$

Q.E.D.

To further extend the study of stability of the sampled-data system (2.1) to that of asymptotic stability, an additional restriction will have to be made on the system, this is given in the following theorem.

Theorem 2.5 (Asymptotic Stability). Let the assumptions (a)-(c) of theorem 2.4 hold for the sampled-data system (2.1). Assume $t_0=0$. If there exists an $h_c^*>0$ such that

$$(d) \quad \lim_{t\to\infty} x(t, x_0) = 0, \quad \forall x_0 \in B_{\delta_0}, \quad 0<t_{k+1}-t_k<h_c^*,$$

then, there exists an $h^*>0$ such that $x(t, x_0)\to 0$ for all $x_0\in D$.

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Proof. From lemmas 2.1 and 2.2, it is clear that $\forall x_0\in A_{\delta_0}$, there exists a $T_{k(x_0)}$ such that $x_k\in B_{\delta_0}$ provided $x_{k-1}\in A_{\delta_0}$ and $0<t_{k+1}-t_k<h_a^*$. Because the sampled-data system is time-invariant, this x_k can also be treated as an initial condition of the system. Hence, if the condition (d) is fulfilled, then $x(t, x_0)\to 0$ as $t\to\infty$ for all $x_0\in D$ and for $0<t_{k+1}-t_k<h^*$ where $h^* = min(\, h_a^*, h_c^* \,)$.

Q.E.D.

Note that the condition (d) implies that system (2.1) in the vicinity of the origin is asymptotically stable for $0<h_k<h_c^*$, $k\geq 0$.

It has been observed that both theorems 2.4 and 2.5 state only conditions for the *existence* of a maximum sampling period ensuring the stability of the system. It does not provide a method of calculating h^* on the domain D on which relevant stability conditions are satisfied. This does not, however, mean that the above theorems are of only theoretical value. Suppose an estimated DOA of a nonlinear sampled-data system has already been established by some analytical method or by trial-and-error, the above theorems may be used to check if this estimated DOA can be further increased with respect to some sampling period by treating this estimated

DOA as the domain B_{δ_0}. It is possible that a larger domain exists on which (a)-(c) (or (a)-(d)) are satisfied and the chosen sampling period still gives stability (or asymptotic stability).

3. The choice of the quadratic function V

An observation made on the above theorems is that in defining the quadratic mapping V, the choice of the positive definite matrix P seemed quite arbitrary, i.e., no specific relationship between P and the characteristics of the sampled-data system (2.1) were required. Consequently, it is possible that for some choice of P, the condition (c) of theorem 2.4 cannot be satisfied, i.e., a finite domain A_{δ_0} can not be found, whereas for the same system with some other choice of P, (c) can be fulfilled. This indicates that the choice of P can directly affect the existence of the sampling period h^*! Faced with such arbitrariness, the next theorem investigates how, under certain conditions, the matrix P can be related to the characteristics of the sampled-data system so that a systematic method of determining the required matrix P becomes available. Perhaps the obvious choice is to try to relate P to the linearised system of the sampled-data system.

Theorem 3.1. Let

$$\dot{x} = f(x, x), \qquad x(t_0)=x_0 \tag{3.1}$$

be the underlying continuous system of the sampled-data system (2.1). Let V be the quadratic mapping defined by eq.(2.2). If $f(x,x)$ is continuous with respect to its arguments and for every x_0 in a small neighbourhood of the origin, $x(t;x_0,t_0)$ exists uniquely and is also continuous with respect to its arguments, then along $x(t;x_0,t_0)$, the inequality

$$\dot{V}(x) \leq -\xi V(x), \qquad \xi>0 \tag{3.2}$$

will be satisfied if and only if the linearised system of (3.1) is asymptotically stable and P must take the form of a Lyapunov function matrix of the linearised system.

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Note that the above inequality (3.2) is equivalent to condition (c) of theorem 2.4. Basically, this lemma says the necessary and sufficient condition for (3.2) to be satisfied is to take P as a Lyapunov function of the linearised system of $\dot{x}=f(x,x)$.

Proof. Without loss of generality, assume $t_0=0$ and write $x(t,x_0)=x(t;x_0,t_0)$. As $x(t,x_0)$ is continuous with respect to its arguments, it can be made as close to the origin as required by making x_0 close enough to the origin and t close enough to 0.

To prove that the satisfaction of the inequality (3.2) implies the asymptotic stability of the linearised system of (3.1), write

$$f(x, x) = Ax + g(x) \tag{3.3}$$

where A is an $(n \times n)$ matrix and

$$\lim_{||x|| \to 0} \frac{||g(x)||}{||x||} = 0 \tag{3.4}$$

i.e., $g(x)$ contains at least second order nonlinearity and is continuous. By noting that

$$\dot{V}(x) = 2x^T P\, f(x, x) = 2x^T P(Ax + g(x))$$

the inequality (3.2) becomes

$$x^T(A^T P + PA)x + 2x^T P g(x) \;\leq\; -\xi\, x^T P x\,, \qquad \xi > 0 \tag{3.5}$$

satisfied for x in the vicinity of the origin. As x is close enough to the origin, the nonlinear term is negligible (from (3.4)) compared with the linear term. Hence, (3.5) becomes

$$x^T(A^T P + PA)x \;\leq\; -\xi\, x^T P x\,, \qquad \xi > 0 \tag{3.6}$$

valid for x in a small enough neighbourhood of the origin. Now, introduce a matrix Q where

$$A^T P + P\,A = -\,Q$$

so that (3.6) becomes

$$x^TQx \ \geq \ \xi \, x^TPx \,, \qquad \xi>0 \tag{3.7}$$

i.e., the inequality (3.2) is equivalent to the above inequality. As P is a positive definite matrix, (3.7) will hold only if Q is a positive definite matrix. Thus P is a Lyapunov function matrix of the stable linear system

$$\dot{x} = Ax, \qquad x(0)=x_0$$

To show that the asymptotic stability of the linearised system of (3.1) implies inequality (3.2), let $f(x, x)$ be as defined in (3.3), assume A is a stable matrix, Q is a positive definite matrix and the matrix P satisfies the equation

$$A^TP + P\,A = -\,Q$$

Then P is a Lyapunov function matrix of the linear system $\dot{x}=Ax$, $x(0)=x_0$. Thus it is always possible to find a sufficiently small $\varepsilon>0$ such that

$$x^TQx \ \geq \ \varepsilon \, x^TPx$$

or

$$x^T(A^TP+PA)x \ \leq \ -\,\varepsilon \, x^TPx \tag{3.8}$$

Now, due to the property of $g(x)$ given in eq.(3.4), provided x is sufficiently small, it is always possible to find a small enough $\xi>0$ such that

$$\varepsilon \ \geq \ \xi + \frac{2x^TP\,g(x)}{x^TPx}$$

and inequality (3.8) thus becomes

$$x^T(A^TP+PA)x + 2x^TPg(x) \ \leq \ -\,\xi \, x^TPx \,, \qquad \xi>0$$

Hence inequality (3.2) is satisfied for some $\xi>0$.

Q.E.D.

Observe that although in theorem 3.1, inequality (3.2) was assumed to hold only in the vicinity of the origin, it is possible to obtain a finite domain on which it would be satisfied by computing inequality (3.5), and then defining the domain A_{δ_0} inside the above domain. Thus theorem 3.1 provides a convenient method of ensuring that condition (c) of theorem 2.4 will be met if the sampled-data system has an underlying continuous system whose linearised system is asymptotically stable. In such cases, the matrix P is a Lyapunov function matrix of the linearised system.

4. Conclusions

This the first part of the paper has focussed upon finding restrictions under which there would exist a finite time period h^* sampled below which system (2.1) would be stable (theorem 2.4) or asymptotically stable (theorem 2.5) on a certain domain about the origin. The restrictions on such sampled-data systems were weak enough to include a wide range of nonlinear sampled-data systems, for example, systems which represent nonlinear continuous plants, or systems with static non-linear elements in their feedback loops, or systems possessing some 'odd' behaviour in the vicinity of their equilibrium points, etc..

The theoretical evidence of the existence of such an h^* is significant in that it encourages the practice of searching for such an h^* once a sampled-data system of the type (2.1) is found to satisfy various conditions given in the theorems. Even if such an h^* may be theoretically difficult to obtain, the theorems given above make it possible to conclude that a sampled-data system satisfying certain conditions can be stabilised by fast enough sampling rates. Note that both theorems 2.4 and 2.5 provide sufficient, but not necessary, conditions for stability.

The second part of this paper will concentrate on finding, analytically, a relationship between the sampling rate and an estimated domain of attraction of a particular class of sampled-data systems. It will show that such relationship can be rather simple.

References

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