[redacted]

Fast Sampling And Stability Of Nonlinear Sampled-Data Systems -- Part II: Sampling Rate Estimations

Y. Zheng#, D. H. Owens*, S. A. Billings#

Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield, S1 3JD, England

* Department of Mathematics,
University of Strathclyde,
26, Richmond Street,
Glasgow G1 1XH, Scotland

University of Sheffield
Research Report No. 341
August, 1988

Fast Sampling And Stability Of Nonlinear Sampled-Data Systems -- Part II: Sampling Rate Estimations

Y. Zheng

Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, England

D. H. Owens

Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, Scotland

S. A. Billings

Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, England

Keywords: *Fast sampling, nonlinear sampled-data systems, stability, sampling intervals, domains of attraction.*

Abstract: This paper uses a time-domain approach to study the effect of fast sampling on the stability of a class of nonlinear sampled-data systems which have the property that their underlying continuous systems are stable on a certain domain. In this the second part of the paper a relationship between sampling rate and an estimated domain of attraction of the system is obtained by extending the analysis in Part I [1]. This relationship indicates that a faster sampling rate can increase the estimated domain of attraction of the sampled-data system, a phenomenon well known in practice. Hence this study is a significant step towards understanding the effect of sampling on the behaviour of a sampled-data system. It may also help in designing a sampled-data system in the best interests of economic instrumentation and system performance.

1. Introduction

This part of the paper attempts to find a relationship between sampling rate and some estimated domain of attraction (DOA) of a class of sampled-data systems. The class of sampled-data systems which is studied is assumed to have a general block diagram structure shown in Fig.1.

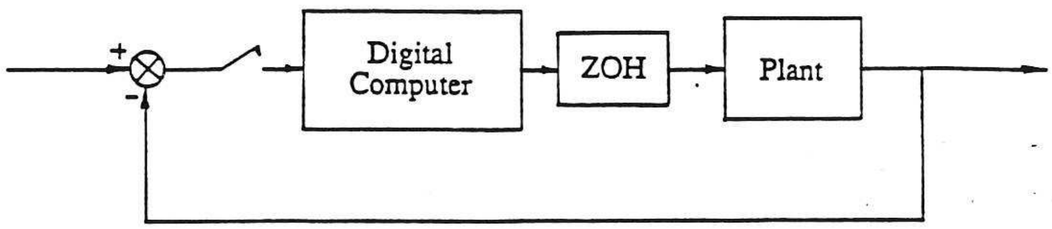

Fig. 1

It is assumed that the plant can be described by the model

$$\dot{x}(t) = Ax(t) + F(x(t)) + Bu(t)$$
$$y(t) = Cx(t)\ , \qquad x(0)=x_0 \tag{1.1}$$

where $y \in \mathbf{R}^m,\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^l$; A, B and C are the characteristic, input and output matrices of the plant of dimension $(n \times n)$, $(n \times l)$ and $(m \times n)$ respectively; $t_0=0$ without loss of generality; the nonlinearity of the plant $F(x)$ is assumed a polynomial in x of degree at most N, written as

$$F(x) = \sum_{j=2}^{N} F_j(x)\ , \qquad F_j(\alpha x) = \alpha^j F_j(x), \quad \forall \alpha \in \mathbf{R}, \quad \forall x \in \mathbf{R}^n \tag{1.2}$$

Assume that the linearised model of the plant is both controllable and observable. The plant is to be controlled by a digital computer, and due to the ZOH device, the plant inputs $u(t)$ are piece-wise constant signals which can be written as

$$u(t) = u_k \qquad t \in [t_k,\ t_{k+1}[, \qquad k \geq 0 \tag{1.3}$$

Note that constant sampling rate is *not* assumed. In the following, two types of digital controllers are to be used to control the plant. One is a simple proportional controller, the other is a proportional-plus-integral controller. Both types are widely used in practice.

2. Proportional control

In proportional control, the digital computer takes the control strategy

$$u_k = K(-y_k) = -Ky_k \tag{2.1}$$

where K is a constant matrix of dimension $(l{\times}m)$. Thus the state-space equation of the feedback autonomous sampled-data system of Fig.1 can be written as

$$\begin{aligned}\dot{x} &= Ax + F(x) + Gx_k \\ &= f(x, x_k), \qquad x(t_k)=x_k, \quad t\in[t_k,t_{k+1}[, \quad k\geq 0\end{aligned} \tag{2.2}$$

where

$$G = -BKC \tag{2.3}$$

Clearly in this case, the mapping f: $\mathbf{R}^n\times\mathbf{R}^n \rightarrow \mathbf{R}^n$ is polynomial in x but linear in x_k. $f(x,x_k)$ is thus continuous on $\mathbf{R}^n\times\mathbf{R}^n$.

The problem to be solved is the following. Assume the sampled-data system satisfies certain conditions so that the underlying continuous system is asymptotically stable on a certain domain about the origin. Find a range of allowable sampling rates so that the sampled-data system can be stabilised on the same domain.

One condition the system is assumed to satisfy is that the underlying continuous system of the sampled-data system, written as

$$\dot{x} = Ax + F(x) + Gx \tag{2.4}$$

has an asymptotically stable linearised system

$$\dot{x} = (A+G)x \tag{2.5}$$

This assumption can be fulfilled by designing the controller matrix K to result a stable matrix $(A+G)$. With the controller thus designed, the following study takes two steps. First, an estimated DOA of the continuous system is obtained by finding a Lyapunov function of the system over the domain. Then by imposing upon the system the condition $\|x(t;x_s,t_s)\|\leq\varepsilon\|x_s\|$ where $0<\varepsilon\leq 1$ and x_s lies in the above domain, a range of allowable t is obtained. In particular, a relationship between the maximum of such t and the boundary of the estimated DOA is given.

2.1 The DOA estimation

The continuous system (2.4) is now written in the form:

$$\dot{x} = (A+G)x + F(x)\,, \qquad x(0)=x_0$$
$$= f(x,x) \tag{2.6}$$

and due to the choice of the controller matrix K, the above system is asymptotically stable about the origin, i.e., $(A+G)$ is stable in the continuous sense. Hence there exists a unique matrix P which is the solution of the equation

$$(A+G)^T P + P(A+G) = -I_n \tag{2.7}$$

where I_n is the $(n{\times}n)$ identity matrix and thus P is a positive definite matrix. Let the quadratic mapping V and the P-matrix norm be defined as

$$V(x) = x^T P x\,, \qquad (P=P^T>0) \qquad \|x\|_P = \sqrt{x^T P x} = V(x)^{1/2} \tag{2.8}$$

Let the Banach space $X=(\|.\|_P, \mathbf{R}^n)$ be the state space of the system (2.6). Then along the motion of the system,

$$\dot{V}(x) = \dot{x}^T P\, x + x^T P\, \dot{x} = 2\, x^T P\, f(x,x)$$
$$= x^T[\, (A+G)^T P + P(A+G)\,]x + 2x^T P\, F(x)$$
$$\leq -\, x^T x + 2x^T P \sum_{j=2}^{N} F_j(x) \tag{2.9}$$

From the inequalities 1, 2 and 3 of Appendix I, inequality (2.9) becomes

$$\dot{V}(x) \leq -\frac{1}{\bar{\sigma}(P)}\, V(x) + 2\sum_{j=2}^{N} M_j\, \|x\|_P^{j+1}$$
$$= -\left(\frac{1}{\bar{\sigma}(P)} - 2\sum_{j=2}^{N} M_j\, V(x)^{\frac{j-1}{2}} \right) V(x) \tag{2.10}$$

where $\bar{\sigma}(P)$ is the largest eigenvalue of the matrix P and

$$M_j = \sup_{\|y\|_P=1} \|F_j(y)\|_P \tag{2.11}$$

Note that both $\bar{\sigma}(P)$ and M_j are positive constants independent of x. As $x \rightarrow 0$, $V(x) \rightarrow 0$, and the expression in the bracket of (2.10) tends to $\frac{1}{\bar{\sigma}(P)}$. This implies that in the vicinity of the origin, $-\dot{V}(x)$ is positive definite. Thus from Lyapunov's theorem, V

is a Lyapunov function of the system and the origin is asymptotically stable.

Now suppose that for some small enough $\xi>0$, there exists a $d_\xi>0$ and a domain

$$D_\xi = \{ x \,/\, V(x) \le d_\xi \} \tag{2.12}$$

such that along the motion trajectory of the system,

$$\dot V(x) \le -\xi V(x) , \qquad \forall x \in D_\xi \tag{2.13}$$

By denoting $V(x_0)=V_0$, this yields

$$V(x) \le e^{-\xi t} V_0 , \qquad x_0 \in D_\xi \tag{2.14}$$

which clearly indicates that provided $x_0 \in D_\xi$, $V(x)$ tends to 0 as $t \to \infty$, or $x \to 0$ as $t \to \infty$. Therefore, the domain D_ξ, if it exists, is an estimated DOA of the continuous system (2.6). The objective therefore will be to search for conditions that, once satisfied by the system, will enable the domain D_ξ to be found.

Comparing (2.10) with (2.13), it is clear that to obtain D_ξ, it is only necessary that

$$\xi \le \frac{1}{\bar\sigma(P)} - 2\sum_{j=2}^{N} M_j V(x)^{\frac{j-1}{2}} , \qquad \forall x \in D_\xi \tag{2.15}$$

or,

$$2\sum_{j=2}^{N} M_j V(x)^{\frac{j-1}{2}} \le \frac{1}{\bar\sigma(P)} - \xi , \qquad \forall x \in D_\xi \tag{2.16}$$

Hence, provided $0<\xi<\frac{1}{\bar\sigma(P)}$, there exists a $d_\xi>0$ given by

$$2\sum_{j=2}^{N} M_j d_\xi^{\frac{j-1}{2}} = \frac{1}{\bar\sigma(P)} - \xi \tag{2.17}$$

such that $\dot V \le -\xi V$ will be satisfied on D_ξ.

The above analysis is concluded by the following lemma.

Lemma 2.1. Consider the nonlinear continuous system (2.6). Assume the matrix $(A+G)$ is stable, i.e. the real parts of all its eigenvalues are strictly negative. Let the matrix P be defined as in (2.7) and the quadratic mapping V and the P-matrix norm be defined as in (2.8). There exists a domain D_ξ of the form

$$D_\xi = \{ x \,/\, V(x) \le d_\xi \}$$

where d_ξ is given by

$$2 \sum_{j=2}^{N} M_j \, d_\xi^{\frac{j-1}{2}} = \frac{1}{\bar{\sigma}(P)} - \xi \,, \qquad 0<\xi<\frac{1}{\bar{\sigma}(P)}$$

such that system (2.6) is asymptotically stable on D_ξ. M_j is defined by (2.11) and $\bar{\sigma}(P)$ is the largest eigenvalue of P.

o

Remarks.

(a) Note that the condition $\dot{V} \le -\xi V$ imposed on the mapping V is much more restrictive than the condition $\dot{V} \le 0$ given by the La Salle's theorem. In fact, V is required to decay exponentially over D_ξ.

(b) The estimated DOA D_ξ is compact and varies with ξ which restricts the decay rate of V on D_ξ. From (2.17), for a sharper decay rate, i.e., a bigger ξ, the corresponding d_ξ is decreased, hence D_ξ is reduced but remains finite as long as $\xi<\frac{1}{\bar{\sigma}(P)}$. This basically says that the estimated DOA becomes smaller if the decay rate imposed on V becomes bigger, as one expects.

Now, as the continuous system (2.6) is the limiting case of the sampled-data system (2.2) when $h_k \to 0$ for all k, it is reasonable to ask the following question: will the sampled-data system (2.2) be stabilised on the domain D_ξ estimated above by fast enough sampling rates? i.e., by letting h_k close enough to zero? This is to be investigated in the next subsection.

2.2 The relationship between sampling periods and D_ξ

The sampled-data system (2.2) can be written in the following form:

$$\dot{x} = (A+G)x + F(x) + G(x_k - x) \,, \qquad x(t_k)=x_k, \quad t\in[t_k, t_{k+1}[, \quad k\ge 0 \qquad (2.18)$$

Comparing it with the continuous system $\dot{x}=(A+G)x+F(x)$, it is obvious that the term $G(x_k-x)$ is due to sampling: if $h_k \to 0$, then $x_k \to x$ and hence the term $G(x_k-x) \to 0$. Lemma 2.1 has given an estimated DOA D_ξ for the continuous system (2.6). The following analysis will prove that the sampled-data system can also be stabilised on D_ξ provided h_k is small enough for all k.

Lemma 2.2. Let V be the quadratic function as defined in eq.(2.8). Let x_s denote a sampled datum at some sampling instant t_s. Let $x(t;x_s,t_s)$ be the subsequent motion trajectory of the sampled-data system (2.18).

If there exist $h^*>0$, $d>0$ and a connected region $D=\{ x \,/\, V(x) \le d \}$ such that

$$V(x(t;x_s,t_s)) \le e^{-rt}\, V(x_s), \qquad \forall x_s \in D, \qquad \forall t \in [t_s,\, t_s+h^*[\tag{2.19}$$

for some $r \ge 0$, then the sampled-data system has solutions from any initial condition $x_0 \in D$ and every solution is bounded in D provided $0 < t_{k+1}-t_k \le h^*$, $k \ge 0$.

If (2.19) holds for $r>0$, then every solution of the system converges to the origin provided $x_0 \in D$, $0 < t_{k+1}-t_k \le h^*$, $k \ge 0$.

o

Proof. Consider the sampled-data system (2.2) during the first sampling period where $x(0)=x_0$, $t \in [0, h_1[$. If (2.19) holds, then, provided $h_1 \le h^*$ and $x_0 \in D$,

$$V(x(t,x_0)) \le e^{-rt}\, V(x_0), \qquad t \in [0, h_1[$$

and in particular at the first sampling instant,

$$V(x_1) \le e^{-rh_1}\, V(x_0)$$

which implies that x_1 lies in D. Now, because the sampled-data system is time-invariant, by induction,

$$V(x_k) \le e^{-r(h_1+h_2+\ldots+h_k)}\, V(x_0), \qquad \forall x_0 \in D, \quad 0<h_i \le h^*, \quad 1 \le i \le k$$

Thus, for $r>0$, $V(x_k) \to 0$ as $k \to \infty$. Therefore, the sampled-data system is asymptotically stable on D provided $0<t_{k+1}-t_k \le h^*$ for all k. If $r=0$ however, then $V(x_k) \le V(x_0)$ for all

$k \geq 0$ and the system is stable on D.

Q.E.D.

In lemma 2.2, there is no suggestion that the connected region D, if exists, is the domain D_ξ given by lemma 2.1. However, in the following procedure of searching for an h^* which is related to the domain D, the relationship between D and D_ξ will become apparent. Now, the inequality (2.19) can also be written as

$$\|x(t;x_s,t_s)\|_P \leq e^{-\frac{\eta}{2}} \|x_s\|_P , \qquad \forall x_s \in D, \quad \forall t \in [t_s,\ t_s+h^*[\tag{2.20}$$

In order for this inequality to be satisfied, the idea is to initially find an upper bound on $\|x(t;x_s,t_s)\|_P$ in terms of the parameters of the system. If there exists an h^* and a domain D such that this upper bound is also bounded by the right-hand of the inequality (2.20), then (2.20) will be automatically satisfied.

Without loss of generality, let $t_s=0$ so that $x(t;x_s,t_s) \equiv x(t,x_s)$, or for simplicity in notation, $x(t)$ will be used to represent the motion trajectory of the system without causing misunderstanding. Then from the model of the sampled-data system,

$$x(t) = e^{(A+G)t}x_s + \int_0^t e^{(A+G)(t-\tau)} [\ F(x(\tau)) + G(x_s - x(\tau))\ ]\ d\tau \tag{2.21}$$

Take the P-matrix norms on both sides, make use of the inequalities 2 and 4 provided in Appendix I, define the error term

$$e(t) = x_s - x(t) \tag{2.22}$$

and

$$\eta_t = \sup_{0 \leq t' \leq t} \|\ e(t')\ \|_P \tag{2.23}$$

Note that η_t is nondecreasing with t, and that for $0 \leq t' \leq t$,

$$\begin{aligned} \|x(t')\|_P &= \|x(t') - x_s + x_s\|_P \\ &\leq \eta_t + \|x_s\|_P \end{aligned} \tag{2.24}$$

thus (2.21) becomes

$$\|x(t)\|_P \le e^{-\frac{t}{2\bar{\sigma}(P)}}\|x_s\|_P + 2\bar{\sigma}(P)(1-e^{-\frac{t}{2\bar{\sigma}(P)}})\,[\,\sum_{j=2}^{N} M_j(\|x_s\|_P + \eta_t)^j + \|G\|_P\,\eta_t\,]$$

$$= U(t, \|x_s\|_P, \eta_t) \tag{2.25}$$

where U is an upper bound of $\|x(t)\|_P$ defined by

$$U(t, \|x_s\|_P, \eta_t) = e^{-\frac{t}{2\bar{\sigma}(P)}}\|x_s\|_P + 2\bar{\sigma}(P)\,(1-e^{-\frac{t}{2\bar{\sigma}(P)}})\,[\,\sum_{j=2}^{N} M_j(\|x_s\|_P + \eta_t)^j + \|G\|_P\,\eta_t\,] \tag{2.26}$$

Therefore, condition (2.20) is further restricted to yield the following corollary.

Corollary 2.3. Consider the sampled-data system (2.18). If there exist $h^*>0$, $d>0$, and a domain D of the form $D = \{\, x \,/\, V(x) \le d \,\}$ such that

$$U(t, \|x_s\|_P, \eta_t) \;\le\; e^{-\frac{rt}{2}}\|x_s\|_P, \qquad r\ge 0, \qquad t\in [t_s, t_s+h^*[\tag{2.27}$$

for all $x_s \in D$, then the sampled-data system is stable in D provided $0 < t_{k+1}-t_k \le h^*$ for all $k\ge 0$.

o

Note that although the above corollary sets a more restrictive condition, it is now possible to find a range of time periods and a corresponding range of $\|x_s\|_P$ that satisfy (2.27), and hence guarantee the stability of the sampled-data system. In what follows, the cases of stability ($r=0$) and asymptotic stability ($r>0$) are to be discussed separately. Without loss of generality, assume $t_s=0$.

2.2.1. Stability conditions ($r=0$).

In this case, the inequality (2.27) becomes

$$U(t, \|x_s\|_P, \eta_t) \le \|x_s\|_P\,, \qquad x_s\in D, \quad t\in [0,h^*[\tag{2.28}$$

By substituting (2.26) for U into the above and after a little manipulation, it becomes

$$2\bar{\sigma}(P)[\sum_{j=2}^{N} M_j\,(\|x_s\|_P+\eta_t)^j + \|G\|_P\,\eta_t] \le \|x_s\|_P\,, \qquad x_s\in D,\ t\in[0,h^*[\tag{2.29}$$

Note that when $\|x_s\|_P=0$, $x_s=0$ which is the equilibrium point. Excluding this equilibrium point, define

$$\lambda_t = \frac{\eta_t}{\|x_s\|_P}\,, \qquad x_s\neq 0 \tag{2.30}$$

so that (2.29) can be written as

$$2\bar{\sigma}(P)[\sum_{j=2}^{N} M_j\,(1+\lambda_t)^j\,\|x_s\|_P^{j-1} + \|G\|_P\,\lambda_t] \le 1\,, \qquad x_s\in D|0,\ \ t\in[0,h^*[\tag{2.31}$$

Note that λ_t is, like η_t, nondecreasing with t. Further, define

$$\phi(\lambda_t,\|x_s\|_P) = 2\bar{\sigma}(P)\,[\sum_{j=2}^{N} M_j\,(1+\lambda_t)^j\,\|x_s\|_P^{j-1} + \|G\|_P\,\lambda_t\,] \tag{2.32}$$

so that (2.31) becomes

$$\phi(\lambda_t,\|x_s\|_P)\ \le\ 1, \qquad x_s\in D|0,\ \ t\in[0,h^*[\tag{2.33}$$

The question of stability has now become:

Corollary 2.4. For the sampled-data system (2.2), if there exist $h^*>0$, $d>0$, and a domain D of the form $D=\{\,x\,/\,V(x)\le d\,\}$ such that $\forall x_s\in D|0$, the inequality (2.32) holds for all $\lambda_t\in[0,\lambda^*]$, where

$$\lambda^* = \sup_{t\in[0,h^*[}\ \lambda_t \tag{2.34}$$

then the system is stable on D with $0<t_{k+1}-t_k\le h^*$. Furthermore, if $x_s=0$, then $x(t)=0$ for all $t>0$.

o

To start searching for such h^* and d, the properties of the mapping ϕ is first investigated. It is noted that (a) $\phi(0,0)=0$; (b) from its definition, ϕ is a polynomial in both of its arguments. Thus ϕ is continuous on **R**×**R** and increases monotonically with its arguments. Hence ϕ is uniformly continuous on some compact domain Ω in

R×**R** containing the origin. In other words, ϕ can be made as small as required by making λ_r and $\|x_s\|_P$ sufficiently small. Let Ω denote the domain on which $\phi \le 1$. Because ϕ increases monotonically, the boundary of Ω is found by setting $\phi=1$, or, there exists a $\lambda^*>0$ for each $d>0$ such that

$$\phi(\lambda^*, d^{1/2}) = 1 \tag{2.35}$$

and $\phi \le 1$ holds for all $\lambda_r \in [0, \lambda^*]$, $0 \le \|x_s\|_P \le d^{1/2}$. Note from (2.35) that λ^* and d are inversely related to each other. The maximum of d is obtained when $\lambda^*=0$ in which case

$$\phi(0, d^{1/2}) = 2\bar{\sigma}(P) \sum_{j=2}^{N} M_j \, d^{\frac{j-1}{2}} = 1 \tag{2.36}$$

As λ^* increases, d decreases and the maximum of λ^* is reached when $d=0$ in which case

$$\phi(\lambda^*, 0) = 2\bar{\sigma}(P) \, \|G\|_P \, \lambda^* = 1 \tag{2.37}$$

The compact domain Ω thus can be found by, say, solving (2.35) for λ^* corresponding to each d.

From the above, the domain D would be maximum if d is maximum. Now, note that (2.36) can be written as

$$2 \sum_{j=2}^{N} M_j \, d^{\frac{j-1}{2}} = \frac{1}{\bar{\sigma}(P)}$$

Comparing it with (2.17), it is clear that d_ξ is less than the maximum possible d, or that D_ξ is contained in the maximum possible D. This implies that at some $d = d_\xi$, there exists a $\lambda^*>0$ such that $\phi(\lambda^*, d_\xi)<1$. Now, as the underlying continuous system associated with the sampled-data system is asymptotically stable in D_ξ, it is much more interesting to study the relationship between D_ξ and the sampling periods that stabilise the sampled-data system. Therefore, the domain D is replaced by D_ξ from now on.

Having found the relationship between λ^* and d_ξ, the next step is to relate λ^*

with h^* in terms of the parameters of the sampled-data system in the hope that a relationship between h^* and d_ξ will eventually be found.

From eq.s(2.21) and (2.22), an expression for the term $e(t)$ is found to be

$$e(t) = (I-e^{(A+G)t})\,x_s - \int_0^t e^{(A+G)(t-\tau)}\,[\,F(x(\tau))+G(x_s-x(\tau))\,]d\tau \tag{2.38}$$

Again, take the P-matrix norms on both sides, use the inequalities 2 and 5 in Appendix I and use the definitions of η_t and λ_t respectively, the above equation yields

$$\lambda_t \le 2\bar{\sigma}(P)(1-e^{\overline{\frac{t}{2\bar{\sigma}(P)}}})\,[||A+G||_P + \sum_{j=2}^{N} M_j(1+\lambda_t)^j||x_s||_P^{j-1} + ||G||_P\lambda_t] \tag{2.39}$$

for all $x_s \in \mathbf{R}^n \,|\, 0$, $t \ge 0$. Now, define

$$\psi(t,\lambda_t,||x_s||_P) = 2\bar{\sigma}(P)\,(1-e^{\overline{\frac{t}{2\bar{\sigma}(P)}}})\,[||A+G||_P + \sum_{j=2}^{N} M_j\,(1+\lambda_t)^j||x_s||_P^{j-1} + ||G||_P\lambda_t] \tag{2.40}$$

so that ψ is an upper bound of λ_t at all time $t\ge0$ for all $x_s \in \mathbf{R}^n \,|\, 0$. Note that unfortunately this upper bound of λ_t is itself a function of λ_t! But all is not lost. The following lemma shows that provided t is sufficiently small, there indeed exists a finite range of λ_t such that the inequality (2.39) can be satisfied.

Lemma 2.5. Consider the mapping ψ defined by (2.40). At $||x_s||_P \ne 0$, there exists a $\hat{\lambda}_t > 0$ such that provided t is small enough,

$$\lambda_t < \psi(t,\,\lambda_t,\,||x_s||_P)\,, \qquad \lambda_t \in [0,\,\hat{\lambda}_t[\tag{2.41}$$

and at $\lambda_t = \hat{\lambda}_t$,

$$\hat{\lambda}_t = \psi(t,\,\hat{\lambda}_t,\,||x_s||_P) \tag{2.42}$$

o

Proof. Define

$$\psi_s(t,\,\lambda_t) = \psi(t,\,\lambda_t,\,||x_s||_P)$$

Then, ψ_s is continuous in both of its arguments and that ψ_s increases monotonically with respect to its arguments, with $\psi_s(0,0)=0$. Also, $\partial\psi_s/\partial\lambda_t$ is continuous and is also a monotonically increasing function of its arguments with $\partial\psi_s(0,0)/\partial\lambda_t=0$. Appendix II provides a proof for the above lemma.

Q.E.D.

By keeping $\|x_s\|_P$ fixed and taking t as a parameter, a family of curves ψ_s vs. λ_t can be drawn shown in Fig.2, in which $t_1<t_2<t_3$, x_s is nonzero. The straight line represents $\psi_s = \lambda_t$.

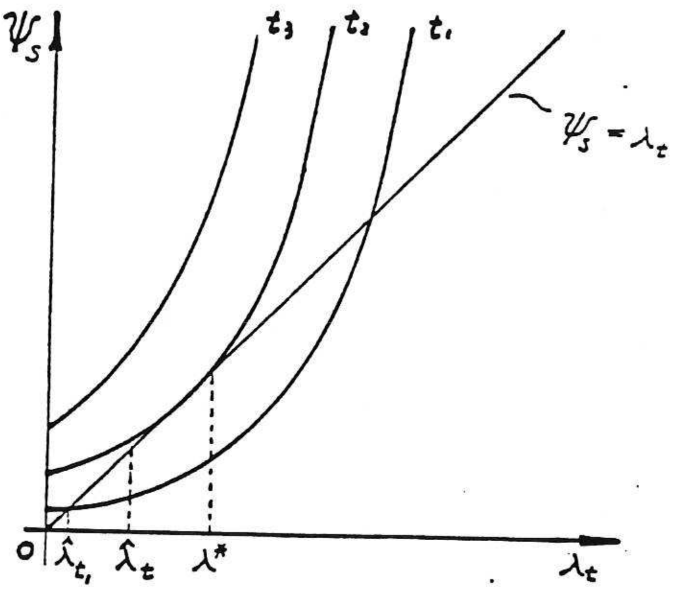

Fig. 2

With the help of this diagram, it can be seen that provided the time t is sufficiently small, there always exists a $\hat\lambda_t$ dependent on t and a range $\lambda_t \in [0, \hat\lambda_t]$ within which the straight line is bounded by the curve. This implies the inequality (2.39), and is essentially what lemma 2.5 states.

Now, consider the inequality (2.39) on the domain D_ξ. As discussed before, there exists a $\lambda^*>0$ such that the equality $\phi(\lambda^*,d_\xi)=1$ holds. Now, if this λ^* is such that

$$\lambda^* \geq \psi(t, \lambda^*, d_\xi^{1/2}) \tag{2.43}$$

for some $t>0$, then there exists a $\hat{\lambda}_t \leq \lambda^*$ given by

$$\hat{\lambda}_t = \psi(t, \hat{\lambda}_t, d_\xi^{1/2})$$

such that for all $\lambda_t \in [0, \hat{\lambda}_t]$ and $x_s \in D_\xi | 0$,

$$\lambda_t \leq \psi(t, \lambda_t, d_\xi^{1/2})$$

From the definition of ψ, the requirement (2.43) can be written as

$$\lambda^* \geq 2\overline{\sigma}(P)(1 - e^{-\frac{t}{2\overline{\sigma}(P)}}) \, [\|A+G\|_P + \sum_{j=2}^{N} M_j \, (1+\lambda^*)^j \, d_\xi^{\frac{j-1}{2}} + \|G\|_P \, \lambda^*] \tag{2.44}$$

Substituting the eq.(2.43) into the above yields

$$\lambda^* \;\geq\; (1 - e^{-\frac{t}{2\overline{\sigma}(P)}}) \, (2\,\overline{\sigma}(P) \, \|A+G\|_P + 1) \tag{2.45}$$

The maximum of such a t, denoted by h^*, which ensures the above inequality is obtained from

$$\lambda^* \;=\; (1 - e^{-\frac{h^*}{2\overline{\sigma}(P)}}) \, (2\,\overline{\sigma}(P) \, \|A+G\|_P + 1) \tag{2.46}$$

Note that the case

$$\lambda^* \;\geq\; 2\,\overline{\sigma}(P) \, \|A+G\|_P + 1 \tag{2.47}$$

corresponds to the situation when, from Fig.2, the whole family of curves ψ_s for $t \geq 0$ intersect with the straight line $\psi_s = \lambda_t$. This might be possible if both d_ξ and the controller gain are chosen sufficiently small to give a sufficiently large λ^*. (This will be discussed in greater detail later.) If so, the sampled-data system would be stable on D_ξ at all sampling rates, or the maximum possible h^* is $+\infty$. A perhaps much more general situation is when the reverse of the inequality (2.47) holds. If so,

$$h^* = -\,2\,\overline{\sigma}(P) \, ln(\,1 - \frac{\lambda^*}{2\overline{\sigma}(P)\|A+G\|_P + 1}\,)$$

In this final expression for h^*, it is worth noting that λ^* is not explicitly related to d_ξ. In general, λ^* needs to be computed from $\phi(\lambda^*, d_\xi) = 1$. Thus h^* is also not explicitly related to the size of the estimated DOA D_ξ. Before the effects of various parameters on the estimations of h^* and d_ξ are discussed, the above result is stated as a stability criterion below for easy reference later.

Theorem 2.6. (Stability Criterion)

Consider the sampled-data system given by (2.18) where the matrix $(A+G)$ is stable, i.e., the real parts of its eigenvalues are negative, $F(x)$ is a polynomial in x, the $(k+1)^{th}$ sampling period is denoted by $h_{k+1}=t_{k+1}-t_k$. The quadratic mapping V and the P-matrix norm are defined as in eq.(2.8).

On the compact domain D_ξ of the form

$$D_\xi = \{ x \,/\, V(x) \le d_\xi \} \tag{2.48}$$

where d_ξ is evaluated from

$$2 \sum_{j=2}^{N} M_j \, d_\xi^{\frac{j-1}{2}} = \frac{1}{\bar{\sigma}(P)} - \xi, \qquad 0 < \xi < \frac{1}{\bar{\sigma}(P)} , \tag{2.49}$$

if the real positive quantity λ^*, given by

$$2\,\bar{\sigma}(P) \, [\, \sum_{j=2}^{N} M_j (1+\lambda^*)^j \, d_\xi^{\frac{j-1}{2}} + \|G\|_P \, \lambda^* \,] = 1 \tag{2.50}$$

is such that

(a) $\lambda^* \ge 2\bar{\sigma}(P)\|A+G\|_P+1$, then there exists an h^* where

$$h^* = +\infty \; ;$$

or,

(b) $\lambda^* < 2\bar{\sigma}(P)\|A+G\|_P+1$, then there exists an h^* given by

$$h^* = -\,2\,\bar{\sigma}(P)\; ln(\, 1 - \frac{\lambda^*}{2\,\bar{\sigma}(P)\; \|A+G\|_P + 1} \,) \tag{2.51}$$

and for all $h_k \le h^*$, $k>0$, the sampled-data system has solutions on D_ξ.

o

Remarks. From the above theorem, there are two factors affecting the evaluation of the domain D_ξ and the maximum sampling interval h^*. One is the parameter ξ, the other is the controller gain K. The effect of ξ is quite obvious. By choosing ξ closer to zero, d_ξ is increased, resulting in a larger domain D_ξ. However from (2.50), an increase in d_ξ decreases λ^* and hence h^* is decreased according to (2.51). Consequently, the variation of ξ is such that on a larger estimated DOA D_ξ (by decreasing ξ), the maximum sampling interval h^*, which stabilises the sampled-data system, will be reduced. The effect of the controller gain K is reflected in the matrix $G=-BKC$ and is more complicated. Because P is given by

$$(A+G)^T P + P\,(A+G) = -I_n$$

from which it can be shown that

$$2(\|A\|_P + \|G\|_P)\bar{\sigma}(P) \;\geq\; 2\|A+G\|_P\bar{\sigma}(P) \;\geq\; 1$$

a sufficiently small $\bar{\sigma}(P)$ implies that $\|A+G\|_P$ or $\|G\|_P$ must be sufficiently high. This requires higher eigenvalues of G. Furthermore, the size of the sphere

$$x^T P\, x = 1$$

tends to increase as $\bar{\sigma}(P)$ decreases, which implies that M_j as defined in (2.11) tends to increase as $\bar{\sigma}(P)$ decreases. Hence, an increase in the controller gain in terms of higher eigenvalues of G has the tendency of decreasing $\bar{\sigma}(P)$, but increasing M_j at a rate dependent on j. It is therefore difficult to conclude from (2.49) how d_ξ varies with the controller gain. If the order of the nonlinearity is sufficiently high, then M_j may increase rapidly with a slight decrease in $\bar{\sigma}(P)$, resulting in a decrease in the size of the domain D_ξ. From (2.50), λ^* also has the tendency of decreasing with a larger $\|G\|_P$, and h^* decreases with a decrease in λ^* as well as in $\bar{\sigma}(P)$. Hence too high a controller gain tends to destabilise the system, as is the case in classical control theory for linear feedback systems.

Example 2.1. Consider a nonlinear scalar system

$$\dot{x} = -x + \frac{1}{8}x^3 - x_k, \qquad x(t_k)=x_k, \quad t\in[t_k, t_{k+1}[, \quad k\geq 0$$

From theorem 2.6, the estimated relationship between h^* and $|x|$ is found, shown in Fig.3. To compare the estimated h^* with the actual value, consider on the domain

$$D = \{\, x \,/\, |x| \leq 1 \,\}$$

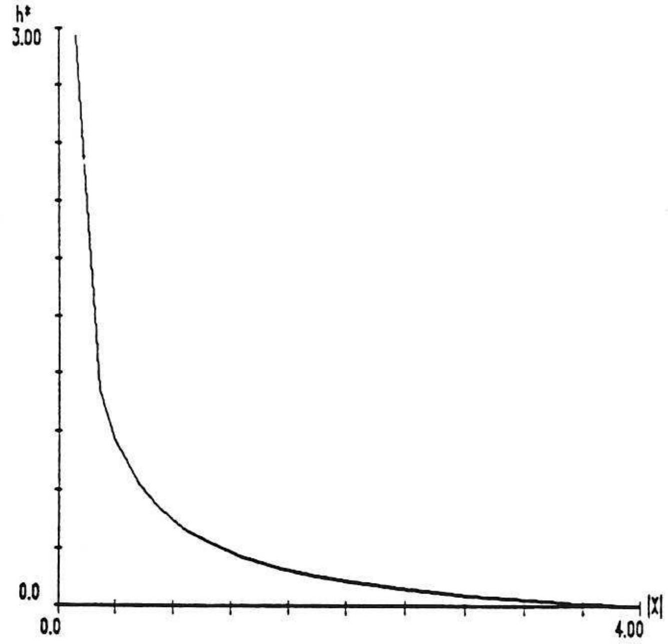

Fig. 3

The estimated h^* *is found to be*

$$h^* = 0.35$$

It can be found by solving the differential equation that the actual h^* *is*

$$h^* = 3.29$$

Hence the estimated value is considerably conservative due to the fact that in estimating h^**, no negative sign was taken into account. All terms used their supremum of norms during any sampling interval.*

Example 2.2. Consider a second order nonlinear plant with the mathematical model

$$\ddot{y} + 3\dot{y} + y - 0.1y^3 = u, \qquad y(0)=y_0,\ \dot{y}(0)=\dot{y}_0$$

Assume that the feedback proportional controller used is

$$u_k = 2y_k$$

Then the state equation for the feedback sampled-data system can be written as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1x_\alpha^3 \end{bmatrix}, \qquad x(t_k)=x_k, \quad t\in[t_k, t_{k+1}[, \quad k\geq 0$$

where the state vector $x=(x_\alpha, x_\beta)^T$. *The estimated relationship between* d_ξ *and* h^* *are shown in Fig.4.*

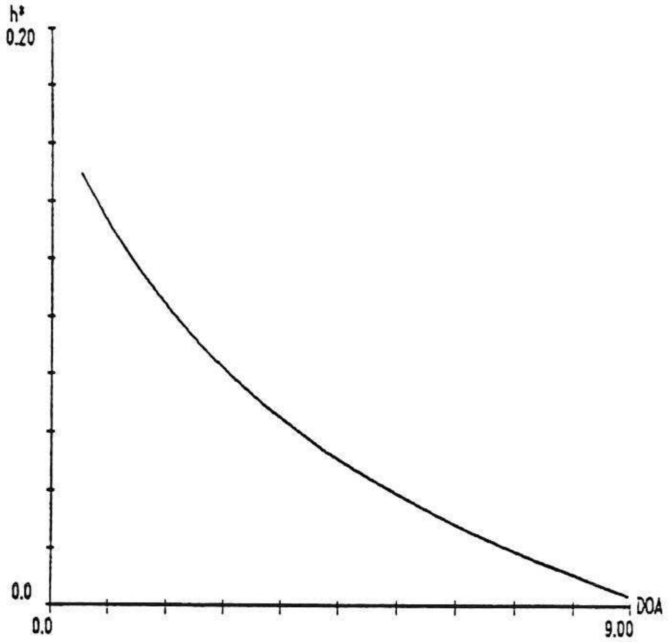

Fig. 4

Note that there is no easy way of finding the actual relationship between h^* *and the corresponding DOA for such a nonlinear multivariable system. Numerical method could be used, but it would involve considerable amount of computation.*

The above examples showed that theorem 2.6 could provide useful information on how fast a sampling rate would be in order to stabilise a sampled-data system on certain domain, even though results thus obtained could be quite conservative. For nonlinear multivariable sampled-data systems, it was generaly difficult to estimate a range of sampling rates which stabilise the corresponding system on a certain domain. Usually the sampled-data system would first be approximated by a discrete model and then numerical methods would have to be used to search for the maximum sampling rate. Note that the parameters of the discrete system are related to the sampling rate. One application of theorem 2.6 is that it can be taken as the first and, most importantly, reliable estimation of the allowable sampling rates for multivariable nonlinear sampled-data systems.

As mentioned before, theorem 2.6 does not guarantee the asymptotic stability of the sampled-data system. For instance, there may exist some limit cycles in D_ξ. The asymptotic stability requires $r>0$.

2.2.2. Asymptotic stability conditions ($r>0$).

In this case, the basic procedure of finding an h^* to stabilise the sampled-data system on the domain D_ξ is the same as in the previous case. But the mathematics becomes more involved due to the fact that the inequality (2.27) cannot be simplified. Thus by taking D_ξ as the domain D and after a simple manipulation, (2.27) becomes

$$2\bar{\sigma}(P)(1-e^{-\frac{t}{2\bar{\sigma}(P)}})\,[\sum_{j=2}^{N} M_j(1+\lambda_t)^j\,||x_s||_P^{j-1} + ||G||_P\,\lambda_t] \le e^{-\frac{rt}{2}} - e^{-\frac{t}{2\bar{\sigma}(P)}} \qquad (2.52)$$

First note that the above inequality would hold for some $t>0$ only if $r<\frac{1}{\bar{\sigma}(P)}$. Now, substituting the mapping ϕ defined in (2.32) into the above yields

$$\phi(\lambda_t, ||x_s||_P) \le \frac{e^{-\frac{rt}{2}} - e^{-\frac{t}{2\bar{\sigma}(P)}}}{1-e^{-\frac{t}{2\bar{\sigma}(P)}}}, \qquad x_s \in D_\xi|0$$

$$= E(t) \qquad (2.53)$$

where

$$E(t) = \frac{e^{-\frac{rt}{2}} - e^{-\frac{t}{2\bar{\sigma}(P)}}}{1-e^{-\frac{t}{2\bar{\sigma}(P)}}} \qquad (2.54)$$

Obviously, $E(t)=1$ if $r=0$, the situation discussed previously. Now, provided $0<r<\frac{1}{\bar{\sigma}(P)}$, $E(t)$ is a monotonically decreasing function of t, with

$$\lim_{t\to 0} E(t) = 1 - r\,\bar{\sigma}(P), \qquad \lim_{t\to\infty} E(t) = 0 \qquad (2.55)$$

The asymptotic stability of the sampled-data system on the domain D_ξ requires the existence of some finite $t>0$ and a corresponding $\lambda_t>0$ such that

$$\phi(\lambda_t, d_\xi^{1/2}) \le E(t) \tag{2.56}$$

The following lemma shows that provided r is chosen properly, this requirement can be satisfied for some $t>0$ and a range within which λ_t may lie.

Lemma 2.7. If r is chosen such that

$$r \le (1-\varepsilon)\,\xi\,, \qquad 0<\varepsilon<1 \tag{2.57}$$

then, there exists some $t>0$ and a corresponding $\bar{\lambda}_t\ge 0$ such that (2.56) is satisfied for all $\lambda_t \in [0, \bar{\lambda}_t]$.

o

Proof. As is known, ϕ is continuous and increases monotonically with λ_t, and

$$\phi(0, d_\xi^{1/2}) = 2\,\bar{\sigma}(P) \sum_{j=2}^{N} M_j\, d_\xi^{\frac{j-1}{2}}$$

$$= 1 - \xi\,\bar{\sigma}(P) \qquad \textit{(from (2.17))}$$

On the other hand, $E(t)$ decreases monotonically with t with the two end conditions given by (2.55). If r is chosen according to (2.57), then,

$$\lim_{t\to 0} E(t) = E(0) = 1 - r\bar{\sigma}(P) \ge 1 - (1-\varepsilon)\xi\bar{\sigma}(P)$$
$$= 1 - \xi\bar{\sigma}(P) + \varepsilon\xi\bar{\sigma}(P)$$
$$= \phi(0, d_\xi^{1/2}) + \varepsilon\xi\bar{\sigma}(P)$$

i.e., $E(0)$ is greater than $\phi(0, d_\xi^{1/2})$. Now, as $E(t)$ is continuous in t, it is continuous on the compact interval $t\in[0,T_1]=I_1$ where T_1 can be arbitrarily large. Hence, E is uniformly continuous on I_1. For each $\varepsilon'>0$ therefore, there exists $\delta'>0$ such that

$$|E(t_1) - E(t_2)| < \varepsilon'$$

whenever $t_1, t_2 \in I_1$, and $|t_1 - t_2| < \delta'$. Since E also decreases monotonically, by setting $t_1 = t > 0$, $t_2 = 0$, the above inequality becomes

$$E(0) - E(t) < \varepsilon' \tag{2.58}$$

whenever $t \in I_1$ and $t < \delta'$. In other words, $E(t)$ can be made as close to $E(0)$ as possible by making t small enough. On the other hand, the mapping ϕ is continuous with λ_t and λ_t is continuous with t, i.e., ϕ is a composite function of t and it is also continuous in t. Thus ϕ is continuous on the compact interval $t \in I_1$ and it is thus uniformly continuous on I_1. Hence for each $\varepsilon''>0$, there exists $\delta''>0$ such that

$$|\phi(\lambda_{t1}, d_\xi^{1/2}) - \phi(\lambda_{t2}, d_\xi^{1/2})| < \varepsilon''$$

whenever $t_1, t_2 \in I_1$, and $|t_1 - t_2| < \delta''$ (note that λ_t is also uniformly continuous on I_1 and hence the quantity $|\lambda_{t1} - \lambda_{t2}|$ can be made as small as required). By noting that ϕ monotonically increases with λ_t and λ_t is non-decreasing with t, let $t_1 = t > 0$, $t_2 = 0$, so that $\lambda_{t1} = \lambda_t$ and $\lambda_{t2} = 0$ and

$$\phi(\lambda_t, d_\xi^{1/2}) - \phi(0, d_\xi^{1/2}) < \varepsilon'' \tag{2.59}$$

whenever $t \in I_1$ and $t < \delta''$. Hence from (2.58) and (2.59), if $t = min\{\delta', \delta''\}$, then

$$\begin{aligned} \phi(\lambda_t, d_\xi^{1/2}) &< \phi(0, d_\xi^{1/2}) + \varepsilon'' \\ &\leq E(0) - \varepsilon\xi\bar{\sigma}(P) + \varepsilon'' \\ &< E(t) + \varepsilon' + \varepsilon'' - \varepsilon\xi\bar{\sigma}(P) \end{aligned}$$

By choosing δ' and δ'' such that

$$\varepsilon' + \varepsilon'' = \varepsilon\xi\bar{\sigma}(P)$$

it is clear that

$$\phi(\lambda_t, d_\xi^{1/2}) < E(t)$$

As the relationship between λ_t and t is not known, it is only possible to say that because ϕ increases monotonically with λ_t, λ_t lies in the region $[0, \bar{\lambda}_t[$ where $\bar{\lambda}_t$ is given by

$$\phi(\bar{\lambda}_t, d_\xi^{1/2}) = E(t) \tag{2.60}$$

Q.E.D.

Now, in order to find how t is related to d_ξ, it is crucial to find how t is related to λ_t. Such a relationship has in fact already been stated in lemma 2.5. Thus if $\bar{\lambda}_t$ is such that

$$\bar{\lambda}_\tau \geq \psi(t, \bar{\lambda}_t, d_\xi^{1/2}) \tag{2.61}$$

then there exists a $\hat{\lambda}_\tau \leq \bar{\lambda}_t$ such that (2.39) holds for all $\lambda_\tau \in [0, \hat{\lambda}_t]$, $x_s \in D_\xi|0$. Substituting the definition of ψ into the above yields

$$\bar{\lambda}_\tau \geq 2\bar{\sigma}(P)\,(1-e^{-\frac{t}{2\bar{\sigma}(P)}})\,[\|A+G\|_P + \sum_{j=2}^{N} M_j(1+\bar{\lambda}_t)^j\, d_\xi^{\frac{j-1}{2}} + \|G\|_P\bar{\lambda}_t] \tag{2.62}$$

and from (2.60), after rearranging, (2.62) becomes

$$\bar{\lambda}_\tau \geq (1-e^{-\frac{t}{2\bar{\sigma}(P)}})\,(2\bar{\sigma}(P)\,\|A+G\|_P + 1) - (1-e^{-\frac{rt}{2}}) \tag{2.63}$$

Because $r \leq (1-\varepsilon)\xi < \dfrac{1}{\bar{\sigma}(P)}$, it can be shown easily that the right-hand expression of the above inequality increases monotonically with t. Thus the maximum of such a t, denoted by h^*, is given when

$$\lambda^* = (1-e^{-\frac{h^*}{2\bar{\sigma}(P)}})\,(2\bar{\sigma}(P)\,\|A+G\|_P + 1) - (1-e^{-\frac{rh^*}{2}})$$

where λ^* also satisfies

$$\phi(\lambda^*, d_\xi^{1/2}) = E(h^*)$$

Note that in this case, no explicit expression for h^* in terms of λ^* and the system parameters is available. The above analysis results in the following stability criterion.

Theorem 2.8 (Asymptotic stability).

Consider the sampled-data system (2.18) having the properties as described in theorem 2.6. The mappings ϕ and E are defined in (2.32) and (2.54) respectively. The parameter r lies in the range given by (2.57).

On the compact domain D_ξ as defined before by (2.12), the sampled-data system is asymptotically stable provided $0<t_{k+1}-t_k \leq h^*$ for all $k>0$ where h^* is found by solving simultaneously the two equations

$$\lambda^* = (1-e^{-\frac{h^*}{2\bar{\sigma}(P)}})\,(2\bar{\sigma}(P)\,\|A+G\|_P + 1) - (1-e^{-\frac{rh^*}{2}}) \tag{2.64}$$

$$\phi(\lambda^*, d_\xi^{1/2}) = E(h^*) \qquad (2.65)$$

to eliminate λ^*.

o

Note from (2.64) that λ^* increases monotonically with h^*, converging to $2\bar{\sigma}(P)\,\|A+G\|_P$ as $h^*\rightarrow+\infty$. It can also be seen from (2.65) that for each h^*, there exists a unique real positive λ^* which decreases monotonically to zero with increasing h^*. Hence it is concluded that the above two equations always intersect uniquely in the region $h^*\geq 0$ and $\lambda^*\geq 0$.

Remarks. The effect of the parameter ξ and the controller gain K on the estimates of D_ξ and h^* are as discussed in 2.2.1. The additional parameter that affects the evaluation of h^* is the decay parameter r imposed on V. (Note r has no effect on D_ξ.) One would expect that as r is decreased, the restriction on V is relaxed and h^* might be increased. This is, however, not apparent from the above stability criterion. As r is decreased, the value of $E(h^*)$ is increased $\forall h^*\geq 0$. For the same h^*, the value of λ^* is thus increased (eq.(2.65)). However from (2.64), a decrease in r implies that the rate of increase of the right-hand expression with respect to h^* is also increased. It is therefore not possible to conclude if a decrease in r would increase the value of h^*.

3. Proportional-plus-integral control

The sampled-data system to be studied here has the same structure as that shown in Fig.1. The plant is assumed to have the same mathematical model given by eq.(1.1). However, instead of proportional control, the digital controller takes the proportional-plus-integral $(P+I)$ control strategy, i.e., the controller signal u consists of two parts: the proportional control action denoted by u_p, and the integral control action denoted by u_i. Thus, u can be written as

$$u_k = u_{pk} + u_{ik}, \qquad k\geq 0 \qquad (3.1)$$

As in the previous section, the proportional control signal u_p is such that

$$u_{pk} = K_p\,(-\,y_k) = -\,K_p\,C\,x_k, \qquad k\geq 0 \tag{3.2}$$

where K_p is the proportional controller gain matrix of dimension $(l\times m)$.

The action of a digital integrator is very similar to that of a continuous integrator where the control action at any time t is proportional to the area swept by the signal at the input during the time period $[0,t]$. But the difference is that for a digital integrator, the area depends on the functions to be assumed between any two adjacent data at the input of the integrator. Therefore, this assumed function determines the structure of the digital integrator. For instance, if the integral action is such that

$$u_{ik+1} = -\,\hat{K}_i\,(y_0 h_1 + y_1 h_2 + \cdots + y_k h_{k+1}), \qquad k\geq 0 \tag{3.3}$$

where $\hat{K}_i$ is the integrator parameter matrix of dimension $(l\times m)$, and h_i is the i^{th} time interval, then the function between any two adjacent data is assumed to be a zero-order hold function carrying the amplitude of the previous datum. For notation simplicity, let u_i be denoted by v. Then, (3.3) has the closed form

$$v_{k+1} = v_k - h_{k+1}\,\hat{K}_i\,y_k \tag{3.4}$$

or, because $y_k = C\,x_k$,

$$v_{k+1} = v_k + h_{k+1}\,K_i\,x_k \tag{3.5}$$

where

$$K_i = -\,\hat{K}_i\,C \tag{3.6}$$

However, with such an integrator, the present input datum x_{k+1} is not taken into account. A modified version of it would be

$$v_{k+1} = v_k + h_{k+1}\,(\Delta K_i x_k + (I-\Delta)K_i x_{k+1}), \qquad k\geq 0 \tag{3.7}$$

where the additional matrix Δ can be chosen as any matrix having the property

$$0 \leq \|\Delta\| \leq 1 \tag{3.8}$$

Thus, the action of the integrator (3.7) varies with Δ, and the integrator given by (3.5) is a special case when $\Delta = I$. Hence the $(P+I)$-controlled sampled-data system can be written in general as

$$\dot{x}(t) = Ax(t) + F(x(t)) + Gx_k + Bv_k \,, \tag{3.9a}$$

$$v_{k+1} = v_k + h_{k+1}\,(\Delta K_i x_k + (I-\Delta)K_i x_{k+1}) \,, \tag{3.9b}$$

$$x(t_k)=x_k, \quad t\in[t_k,\, t_{k+1}[, \quad k\geq 0$$

where $G=-BK_pC$ and K_i is given by (3.6). The stability of such a feedback system will obviously be affected by Δ, which should be chosen to provide a less conservative stability criterion.

3.1 $\Delta = I$, h_k variable.

Here, the integrator takes the form (3.5). It is not difficult to see that this integrator has a control action which exactly coincides with the continuous integrator

$$\dot{w}(t) = K_i\, x_k, \qquad w(t_k)=w_k, \qquad t\in[t_k,\, t_{k+1}[, \quad k\geq 0 \tag{3.10}$$

at all sampling instants t_k , with $v_0=w_0$. In other words,

$$v_k = w_k, \qquad \forall k\geq 0 \tag{3.11}$$

Thus, (3.9b) can be replaced by (3.10) and the feedback sampled-data system can be written as

$$\begin{bmatrix}\dot{x}(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix} + \begin{bmatrix}F(x(t))\\ 0\end{bmatrix} + \begin{bmatrix}G & B\\ K_i & 0\end{bmatrix}\begin{bmatrix}x_k\\ w_k\end{bmatrix}, \tag{3.12}$$

$$x(t_k)=x_k, \quad w(t_k)=w_k, \quad t\in[t_k,\, t_{k+1}[, \quad k\geq 0$$

Writing $z=(x,w)^T$ and comparing the above system with the proportionally controlled system (2.2), it is clear that they have exactly the same form. Define

$$A_c = \begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}, \qquad G_c = \begin{bmatrix}G & B\\ K_i & 0\end{bmatrix} \tag{3.13}$$

so that

$$A_c + G_c = \begin{bmatrix}A+G & B\\ K_i & 0\end{bmatrix} \tag{3.14}$$

Assume the proportional controller K_p is chosen such that $A+G$ is a stable matrix in the continuous sense and K_i is chosen (through $\hat{K}_i$) such that A_c+G_c is also stable in the continuous sense. Then both theorems 2.6 and 2.8 are applicable to the stability and asymptotic stability of this class of $(P+I)$-controlled sampled-data systems. Note that the matrix $(A+G)$ is required to be stable so that in the absence of the integral action, the system can still be stabilised by fast sampling.

3.2 $0 \le \|\Delta\| \le 1,\ h_k = h,\ \forall k \ge 0.$

This case corresponds to a *fixed* sampling rate. This is because the analysis below is already quite involved technically and it would be better to study the simplest case in order to illustrate the crucial points. The integrator (3.7) thus becomes

$$v_{k+1} = v_k + h\ (\Delta K_i x_k + (I-\Delta)K_i x_{k+1})\ , \qquad k \ge 0 \tag{3.15}$$

whose expansion in terms of $\{x_k\}$ is

$$v_{k+1} = h\ K_i\ (x_0+x_1+...+x_k) + h(I-\Delta)K_i x_{k+1}\ , \qquad k \ge 0 \tag{3.16}$$

By introducing the dummy variable w where w satisfies (3.10) but with a fixed sampling rate, i.e.

$$w_{k+1} = h\ K_i\ (x_0+x_1+...+x_k) \tag{3.17}$$

at the $(k+1)^{th}$ sampling instant, then v_k can be written in terms of w_k and x_k as

$$v_k = w_k + h\ (I-\Delta)\ K_i x_k \tag{3.18}$$

and the feedback sampled-data system with this integrator can be written as

$$\dot{x}(t) = Ax(t) + F(x(t)) + Gx_k + Bv_k \tag{3.19a}$$

$$\dot{w}(t) = K_i x_k \tag{3.19b}$$

$$v_k = w_k + h(I-\Delta)K_i x_k, \qquad x(t_k)=x_k,\quad w(t_k)=w_k,\quad t\in[t_k,\ t_{k+1}[,\ k\ge 0 \tag{3.19c}$$

The purpose of introducing the dummy variable w is to build a mathematical model in the similar fashion to the previous model (3.12) in the hope that the stability analysis may be carried out in the similar fashion too. Comparing (3.19a) and (3.19b) with the model (3.12), it can be seen that the only difference between them is: w_k in (3.12) is now replaced by v_k in (3.19a). The two systems have the same underlying continuous system, and from (3.19c),

$$\begin{bmatrix} v_k \\ 0 \end{bmatrix} = \begin{bmatrix} h(I-\Delta)K_i & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \tag{3.20}$$

The analysis below is based on the following observation. Suppose that as $k \rightarrow \infty$, $\|(x_s, w_s)\| \rightarrow 0$. This implies from (3.20) that $\|(v_k, 0)\| \rightarrow 0$, or $\|v_k\| \rightarrow 0$ as $k \rightarrow \infty$. Therefore, it may be possible that by restricting the behaviour of the dummy state vector (x,w), the behaviour of the actual state vector (x_k, v_k) of the feedback sampled-data system can be controlled.

As before, the first step in the study is to find a domain on which the underlying continuous system of (3.19) is stable. Let K_p and $\hat{K}_i$ be chosen such that A_c+G_c is stable. Then from lemma 2.1, there exists a domain D_ξ on which the underlying continuous system of (3.19) is asymptotically stable. Denote $z=(x,w)^T$ so that the domain D_ξ is given by the connected region

$$\{ \ z: \ 2 \sum_{j=2}^{N} M_j V(z)^{\frac{j-1}{2}} \ \leq \ \frac{1}{\bar{\sigma}(P)} - \xi \ \} \tag{3.21}$$

of the form

$$D_\xi = \{ \ z \ / \ V(z) \leq d_\xi \ \} \tag{3.22}$$

where d_ξ is evaluated from

$$2 \sum_{j=2}^{N} M_j \ d_\xi^{\frac{j-1}{2}} = \frac{1}{\bar{\sigma}(P)} - \xi \tag{3.23}$$

Here, P is determined by

$$(A_c+G_c)^T P + P \ (A_c+G_c) = - I_{n+l}$$

where A_c and G_c are as defined in eq.(3.13). M_j is defined as

$$M_j = \sup_{\|z\|_P=1} \|F_j(x,w)\|_P \tag{3.24}$$

where

$$F_j(x,w) = \begin{bmatrix} F_j(x) \\ 0 \end{bmatrix} \tag{3.25}$$

Now denote x_s, w_s and v_s as the sampled data for $x(t)$, $w(t)$ and $\{v_k\}$ at $t=t_s$ respectively and write the eq.s (3.19a) and (3.19b) as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = (A_c+G_c) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + F(x,w) + G_c \begin{bmatrix} x_s-x(t) \\ v_s-w(t) \end{bmatrix} \qquad (3.26)$$

The argument for the stability of this system on D_ξ follows exactly from section 2 that if there exists an h^* such that for all $t\in[0,h^*[$ and $z=(x,w)^T\in D_\xi$,

$$\|z(t)\|_P \leq e^{\frac{-rt}{2}} \|z_s\|_P \qquad (3.27)$$

for some $r\geq 0$, then $z(t)$ would be a stable solution in D_ξ. Here, only the case $r=0$ will be discussed as it sufficiently illustrates the modifications involved in searching for the h^*. The case $r>0$ merely adds more restrictions on the value of h^* but the mathematics is much more involved.

The procedure of finding an h^* follows from the previous section. The crucial difference is that the error term cannot be written as $z_s-z(t)$, but as

$$e(t)=(x_s-x(t),\ v_s-w(t))^T \qquad (3.28)$$

and correspondingly, the definition for η_t becomes

$$\eta_t = \sup_{0\leq t'\leq t} \|(x_s-x(t'),v_s-w(t'))\|_P \qquad (3.29)$$

The upper bound for $\|z(t)\|_P$ is found as before, but because $e(t)$ is no longer equal to $z_s-z(t)$, the upper bound for the term $\|z(t')\|_P$ during the time period $t'\in[0,t]$ has to be written in terms of η_t and $\|z_s\|_P$ according to the following inequalities. First,

$$\begin{aligned} \|z\|_P = \|(x,w)\|_P &= \|(x_s,w_s) - (x_s-x,w_s-w)\|_P \\ &= \|(x_s,w_s) - (x_s-x,v_s-w) + (0,v_s-w_s)\|_P \\ &\leq \|(x_s,w_s)\|_P + \|(x_s-x,v_s-w)\|_P + \|(0,v_s-w_s)\|_P \end{aligned} \qquad (3.30)$$

From (3.19c),

$$\begin{aligned} (0,v_s-w_s)^T &= h(0,\ (I-\Delta)K_i x_s)^T \\ &= h \begin{bmatrix} 0 & 0 \\ (I-\Delta)K_i & 0 \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix} \\ &= h\, K_\Delta\, (x_s,\ w_s)^T \end{aligned} \qquad (3.31)$$

where

$$K_\Delta = \begin{bmatrix} 0 & 0 \\ (I-\Delta)K_i & 0 \end{bmatrix} \tag{3.32}$$

Thus the last term in (3.30) is bounded by

$$\|(0, v_s - w_s)\|_P \le h\, \|K_\Delta\|_P\, \|(x_s, w_s)\|_P \tag{3.33}$$

and therefore for $0 \le t' \le t$, $t \le h$, $\|z(t')\|_P$ is bounded by

$$\|z\|_P \le (1 + h\|K_\Delta\|_P)\, \|z_s\|_P + \eta_t \tag{3.34}$$

The upper bound for $\|z(t)\|_P$ hence becomes

$$\hat{U}(t, h, \|z_s\|_P, \eta_t) = e^{-\frac{t}{2\bar{\sigma}(P)}} \|z_s\|_P + 2\bar{\sigma}(P)(1 - e^{-\frac{t}{2\bar{\sigma}(P)}})[\sum_{j=2}^{N} M_j((1 + h\|K_\Delta\|_P)\|z_s\|_P + \eta_t)^j + \|G\|_P \eta_t] \tag{3.35}$$

Clearly, $\hat{U}$ depends on the sampling period h and increases monotonically with it, whereas in section 2, the upper bound U for $\|x(t)\|_P$ is independent of h_k. This induces major modifications in the process of searching for a maximum sampling period h^*, as will be seen below.

Now, the sampled-data system will have solutions on D_ξ if there exists an $h^*>0$ such that for all $h \le h^*$, $t \in [0, h[$,

$$\hat{U}(t, h, \|z_s\|_P, \eta_t) \le \|z_s\|_P, \qquad \forall z_s \in D_\xi \tag{3.36}$$

Substitute the definition for $\hat{U}$, define

$$\lambda_t = \frac{\eta_t}{\|z_s\|_P} \tag{3.37}$$

and after rearranging, (3.36) becomes

$$2\bar{\sigma}(P)[\sum_{j=2}^{N} M_j (1 + h\|K_\Delta\|_P + \lambda_t)^j \|z_s\|_P^{j-1} + \|G\|_P \lambda_t] \le 1 \tag{3.38}$$

Now, define a mapping $\hat{\phi}$ as

$$\hat{\phi}(h, \lambda_t, \|z_s\|_P) = 2\bar{\sigma}(P)[\sum_{j=2}^{N} M_j (1 + h\|K_\Delta\|_P + \lambda_t)^j \|z_s\|_P^{j-1} + \|G\|_P \lambda_t] \tag{3.39}$$

which is a modification of the mapping ϕ defined by (2.32) and is again dependent on the sampling period h. Hence (3.38) becomes

$$\hat{\phi}(h, \lambda_\tau, \|z_s\|_P) \leq 1 \tag{3.40}$$

According to corollary 2.4, $z(t, z_s)$ will be a stable solution on D_ξ if there exists an h^* and a corresponding λ^* such that (3.40) holds for all $z_s \in D_\xi$, $h \leq h^*$ and $t \in [0, h[$. To prove that such h^* and λ^* exist, note first that $\hat{\phi}$ is continuous and monotonically increasing with respect to all of its arguments. Also, λ_t is nondecreasing with t. Hence on the domain D_ξ, (3.40) will be satisfied for all $t \in [0, h[$, $\lambda_t \in [0, \lambda_h[$ if

$$\hat{\phi}(h, \lambda_h, d_\xi^{1/2}) \leq 1 \tag{3.41}$$

Furthermore, because

$$\hat{\phi}(0, 0, d_\xi^{1/2}) = 2\overline{\sigma}(P) \sum_{j=2}^{N} M_j \, d_\xi^{\frac{j-1}{2}} = 1 - \xi\overline{\sigma}(P) \tag{3.42}$$

there exists a compact domain $\hat{\Omega}$ on $\mathbf{R}\times\mathbf{R}$ such that for all $(h, \lambda_h) \in \hat{\Omega}$, (3.41) is satisfied, and the boundary of $\hat{\Omega}$ is given by

$$\hat{\phi}(h', \lambda'_h, d_\xi^{1/2}) = 1 \tag{3.43}$$

To search for a relationship between λ_t and t in terms of the system parameters, the same procedure as that of section 2 is carried out. But due to the way the error term $e(t)$ is defined in this case, the inequality (2.39) is modified, yielding

$$\frac{\|z_s - z(t)\|_P}{\|z_s\|_P} \leq \hat{\psi}(t, h, \lambda_\tau, \|z_s\|_P) \tag{3.44}$$

where

$$\hat{\psi}(t, h, \lambda_\tau, \|z_s\|_P) = 2\overline{\sigma}(P)(1 - e^{-\frac{t}{2\overline{\sigma}(P)}}) \, [\|A_c + G_c\|_P + \sum_{j=2}^{N} M_j \, (1 + h\|K_\Delta\|_P + \lambda_\tau)^j \, \|z_s\|_P^{j-1} + \|G\|_P \, \lambda_\tau] \tag{3.45}$$

Now, it would be desirable if the left-hand side of (3.44) could be written in terms of λ_τ. This can be done by noting that

$$\|(x_s-x,v_s-w)\|_P = \|(x_s-x,w_s-w) + (0,v_s-w_s)\|_P$$
$$\leq \|(x_s-x,w_s-w)\|_P + \|(0,v_s-w_s)\|_P$$
$$\leq \|z_s-z\|_P + h\,\|K_\Delta\|_P\,\|z_s\|_P \qquad (from\ (3.19c)) \tag{3.46}$$

which, when divided by $\|z_s\|_P$ on both sides, taking the supremum during the time interval $[0,t]$, yields

$$\lambda_t \;\leq\; \hat{\psi}(t,\, h,\, \lambda_t,\, \|z_s\|_P) + h\|K_\Delta\|_P \tag{3.47}$$

for all $z_s \in \mathbf{R}^{n+1}|0$, $t \in [0,\, h[$, $h>0$. Note $\hat{\psi}$ is continuous and increases monotonically with its arguments, and that λ_t is nondecreasing with t. Hence (3.47) also implies that

$$\lambda_h \;\leq\; \hat{\psi}(h,\, h,\, \lambda_h,\, \|z_s\|_P) + h\|K_\Delta\|_P, \qquad z_s \in \mathbf{R}^{n+1}|0 \tag{3.48}$$

By defining

$$\hat{\psi}_s(h,\, \lambda_h) = \hat{\psi}(h,\, h,\, \lambda_h,\, \|z_s\|_P) + h\|K_\Delta\|_P$$

it can be shown from Appendix II that provided h is sufficiently small, there exists a $\hat{\lambda}_h>0$ such that for all $\lambda_h \leq \hat{\lambda}_h$, (3.48) holds. Thus, if λ'_h and h', as given by (3.43), are such that on the domain D_ξ,

$$\lambda'_h \;\geq\; \hat{\psi}(h',\, h',\, \lambda'_h,\, d_\xi^{1/2}) + h'\,\|K_\Delta\|_P$$
$$= (1-e^{-\frac{h'}{2\overline{\sigma}(P)}})\,(2\overline{\sigma}(P)\|A_c+G_c\|_P + 1) + h'\,\|K_\Delta\|_P \tag{3.49}$$

then (3.41) will be satisfied for all λ_h bounded in the range $0 \leq \lambda_h \leq \hat{\lambda}_h$, $\lambda_h \leq \lambda'_h$. The maximum of such an h', denoted by h^*, is obtained when

$$\lambda^* \;=\; (1-e^{-\frac{h^*}{2\overline{\sigma}(P)}})\,(2\overline{\sigma}(P)\|A_c+G_c\|_P + 1) + h^*\,\|K_\Delta\|_P \tag{3.50}$$

where λ^* is given by

$$\hat{\phi}(\lambda^*,\, h^*,\, d_\xi^{1/2}) \;=\; 1 \tag{3.51}$$

In other words, h^* is obtained by solving (3.50) and (3.51) simultaneously, eliminating λ^*. It can therefore be concluded that $z(t)$ is stable on D_ξ provided the sampling period $h \leq h^*$. However, this is not quite the end of the stability study because $w(t)$ is just a dummy variable. The next step is to find a domain on which the state vector

(x_k, v_k) is stable. By noting from (3.18) that

$$\begin{bmatrix} x_k \\ v_k \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ h(I_n-\Delta)K_i & I_l \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \tag{3.52}$$

there exists a domain $\hat{D}$ of the form

$$\hat{D} = \{ \theta \;/\; \theta=Rz, \;\; \forall z\in D_\xi \} \tag{3.53}$$

where

$$R = \begin{bmatrix} I_n & 0 \\ h(I-\Delta)K_i & I_l \end{bmatrix} \tag{3.54}$$

on which (x_k, v_k) is stable provided $h \leq h^*$. Note that $\hat{D}$ is dependent on both the choice of Δ and the system's sampling period h.

It is quite obvious that the choice of Δ should be such that it gives the 'best' control action. From the point of view of system stability, the choice of Δ should result in the largest sampling interval possible upon the same estimated DOA, or the largest DOA for the same maximum sampling interval. As it can be seen from both (3.50) and (3.51), by keeping λ^* unchanged, a decrease in $\|K_\Delta\|_P$ increases h^*, which means that by choosing Δ so that $\|I-\Delta\|_P$ is closer to zero, h^* can be increased. It is not difficult to see that if $\Delta=I$, then $\|K_\Delta\|_P=0$ which gives the maximum possible h^* on the same domain D_ξ. Hence as far as the dummy state variable (x, w) is concerned, the choice of $\Delta=I$ gives the optimum stability behaviour.

For the real state vector (x_k, v_k) however, $\hat{D}$ is dependent on Δ. The effect of Δ is to transform the domain D_ξ by 'distorting' it along some direction, and the degree of such distortion is increased by choosing Δ such that $I-\Delta \approx I$, increasing the effect of the off-diagonal terms. The resultant domain $\hat{D}$ is shaped differently from D_ξ that the two domains cannot in general be compared in size, except in the case $\Delta=I$ in which $\hat{D} = D_\xi$. Note that when $\Delta=I$, the system (3.19) is converted back to the system (3.12) and hence both theorems 2.6 and 2.8 can be applied to examine the stability of the system.

4. Conclusions

The stability study carried out in this part of the paper established a relationship between sampling rate and some estimated DOA of the systems for a class of nonlinear sampled-data systems. The analysis allows variable sampling rate in the system. Two types of control strategies were considered: proportional-control and proportional-plus-integral-control.

Through examples, it was seen that the results could be rather conservative. The conservativeness was caused mainly by the use of norms throughout the study. It may be possible to reduce the conservativeness by using a different normed space. For instance, the matrix P was defined by

$$(A+G)^T P + P(A+G) = -I_n$$

It is evident that a more general equation which can be used to determine P is

$$(A+G)^T P + P(A+G) = -Q$$

where Q is some positive definite matrix. P thus can be varied by varying the matrix Q in the hope that a larger DOA could be resulted without significantly increasing the sampling rates of the system, or a wider range of sampling rates could be obtained over the same domain. Nevertheless, the simplicity of finding a sampling period in terms of the system parameters and its estimated DOA with little computation necessary makes the above result valuable as a first approximation to the real or a less conservative evaluation.

References

[1] Zheng, Y., Owens, D.H. and Billings, S.A., 'Fast sampling and stability of nonlinear sampled-data systems – part I: estimation theorems', (submitted for publication).

[2] Zheng, Y., Owens, D.H. and Billings, S.A., 'Slow sampling and stability of nonlinear sampled-data systems', (submitted for publication).

APPENDIX I

Five Inequalities

Let the mapping $V: \mathbf{R}^n \to \mathbf{R}_+$ be defined as in (2.8), the P-matrix be defined as in (2.7), and $F(x)$ be defined as in (1.2) respectively.

Inequality 1.

$$-x^T x \le -\frac{1}{\bar{\sigma}(P)} V(x) \tag{AI.1}$$

where

$$\bar{\sigma}(P) = \textit{the largest eigenvalue of the matrix } P. \tag{AI.2}$$

o

Proof. Since P is a real positive definite matrix, its eigenvalues $\lambda_i > 0,\ \forall i$. Hence, $\bar{\sigma}(P) = max\{\lambda_i(P)\},\ i=1,2,\ldots,n$. Furthermore, its eigenvector matrix U is an unitary matrix, i.e., $UU^T = I_n$. Write

$$P = U \Lambda U^T$$

where

$$\Lambda = diag\ \{\ \lambda_1, \lambda_2, \ldots, \lambda_n\ \}$$

Then,

$$\begin{aligned} x^T P\, x &= x^T U \Lambda U^T x \\ &= y^T \Lambda\, y \end{aligned}$$

where $y = U^T x$. But,

$$\begin{aligned} y^T \Lambda\, y = \sum_{i=1}^{n} \lambda_i\, y_i^2 \ &\le\ \bar{\sigma}(P) \sum_{i=1}^{n} y_i^2 \\ &= \bar{\sigma}(P)\, y^T y = \bar{\sigma}(P)\, x^T U\, U^T x \\ &= \bar{\sigma}(P)\, x^T x \end{aligned}$$

Hence, $x^T P\, x \le \overline{\sigma}(P)\, x^T x$, or,

$$- x^T x \le - \frac{1}{\overline{\sigma}(P)}\, x^T P\, x$$
$$= - \frac{1}{\overline{\sigma}(P)}\, V(x)$$

Q.E.D.

Inequality 2.

$$\|F_j(x)\|_P \le \|x\|_P^j\, M_j \tag{AI.3}$$

where

$$M_j = \sup_{\|x\|_P=1} \|F_j(x)\|_P \tag{AI.4}$$

o

Proof. From the definition of $F_j(x)$ in (1.2), it is clear that

$$\|F_j(x)\|_P = \|\, \|x\|_P^j\, F_j(\frac{x}{\|x\|_P})\, \|_P$$
$$\le \|x\|_P^j \sup_{\|y\|_P=1} \|F_j(y)\|_P$$
$$= \|x\|_P^j\, \hat{M}_j$$

where

$$\hat{M}_j = \sup_{\|y\|_P=1} \|F_j(y)\|_P$$

But this $\hat{M}_j$ is exactly the same as the M_j defined by the eqn.(AI.4).

Q.E.D.

Inequality 3.

$$x^T P\, F_j \le \|x\|_P \|F_j\|_P \tag{AI.5}$$

o

Proof. The operation $<.|.>_P : x,y \to x^T P\, y$ is known as an inner product. Hence, from *Schwartz inequality,*

$$\begin{aligned} x^T P\, F_j = <x|F_j>_P &\le <x|x>_P^{1/2} <F_j|F_j>_P^{1/2} \\ &= (x^T P\, x)^{1/2} (F_j^T P\, F_j)^{1/2} \\ &= ||x||_P ||F_j||_P \end{aligned}$$

Q.E.D.

Inequality 4. Let P be a Lyapunov function matrix of the stable autonomous linear system

$$\dot{x}(t) = \Phi x, \qquad x(0) = x_0 \tag{AI.6}$$

such that

$$\Phi^T P + P\,\Phi = -I \tag{AI.7}$$

then,

$$||\, e^{\Phi t}\, ||_P \le e^{-\frac{t}{2\bar{\sigma}(P)}} \tag{AI.8}$$

where $\bar{\sigma}(P)$ is defined in (AI.2)

o

Proof. With P given by (AI.7), $V = x^T P\, x$ is a Lyapunov function of the linear system (AI.6). Then along the motion of the system,

$$\begin{aligned} \dot{V}(x) &= x^T(\Phi^T P + P\Phi)x \\ &= -\,x^T x \;\le\; -\frac{1}{\bar{\sigma}(P)}\, V(x) \end{aligned} \tag{AI.9}$$

from inequality 1. By denoting $V(x_0) = V_0$, (AI.9) implies

$$V(x) \le e^{-\frac{t}{\bar{\sigma}(P)}}\, V_0 \tag{AI.10}$$

As the solution to the system (AI.6) is

$$x(t) = e^{\Phi t} x_0$$

and that $V(x)=x^TPx$, (AI.10) becomes

$$x_0^T(e^{\Phi t})^T P (e^{\Phi t}) x_0 \le e^{-\frac{t}{\bar{\sigma}(P)}} x_0^T P\, x_0 \qquad \text{(AI.11)}$$

Hence,

$$\begin{aligned} \| e^{\Phi t} \|_P &= \sup_{\|x\|_P=1} \| e^{\Phi t} x \|_P \\ &= \sup_{\|x\|_P=1} (x^T (e^{\Phi t})^T P\, e^{\Phi t} x)^{1/2} \\ &\le \sup_{\|x\|_P=1} (e^{-\frac{t}{\bar{\sigma}(P)}} x^T P\, x)^{1/2} \qquad \textit{(from (AI.11))} \\ &= e^{-\frac{t}{2\bar{\sigma}(P)}} \end{aligned}$$

Q.E.D.

Inequality 5. Consider again the stable system (AI.6) and let P be still defined as in (AI.7). Then,

$$\| I - e^{\Phi t} \|_P \le 2\, \bar{\sigma}(P)\, \|\Phi\|_P \,(1 - e^{-\frac{t}{2\bar{\sigma}(P)}}) \qquad \text{(AI.12)}$$

o

Proof. Transform the autonomous system (AI.6) by introducing a new state variable

$$\theta = x_0 - x$$

Then,

$$\theta = x_0 - e^{\Phi t} x_0 = (I - e^{\Phi t}) x_0 \qquad \text{(AI.13)}$$

and

$$\begin{aligned} \dot{\theta} &= -\dot{x} = -\Phi\, x \\ &= \Phi(\theta - x_0) \end{aligned}$$

By noting that $\theta_0 = 0$, the above differential equation has the solution

$$\theta = -\int_0^t e^{\Phi(t-\tau)}\,\Phi\, x_0\, d\tau \tag{AI.14}$$

Thus,

$$\| I-e^{\Phi t} \|_P = \sup_{\|x_0\|_P=1} \| (I-e^{\Phi t})x_0 \|_P$$

By comparing (AI.13) and (AI.14), the above equation can be written as

$$\|I-e^{\Phi t}\|_P = \sup_{\|x_0\|_P=1} \| \int_0^t e^{\Phi(t-\tau)}\,\Phi\, x_0\, d\tau \|_P$$

Now, use the familiar property of integrals that

$$\| \int_{t_0}^t f(\tau)\, d\tau \| \le \int_{t_0}^t \|f(\tau)\|\, d\tau$$

the P-matrix norm of $(I-e^{\Phi t})$ is bounded by

$$\|I-e^{\Phi t}\|_P \le \sup_{\|x_0\|_P=1} \int_0^t \|e^{\Phi(t-\tau)}\|_P\, \|\Phi\|_P\, \|x_0\|_P\, d\tau$$

Thus from inequality 4,

$$\| I-e^{\Phi t} \|_P \le \int_0^t e^{-\frac{t-\tau}{2\bar{\sigma}(P)}}\, \|\Phi\|_P\, d\tau$$

$$= 2\,\bar{\sigma}(P)\, \|\Phi\|\, (\, 1-e^{-\frac{t}{2\bar{\sigma}(P)}}\,)$$

Q.E.D.

APPENDIX II

A Lemma

Lemma. Let $\phi(\alpha, \beta)$ be a mapping ϕ: $\mathbf{R}\times\mathbf{R}\rightarrow\mathbf{R}$ with the following properties:

(1) ϕ is defined and continuous $\forall\alpha\geq 0$, $\forall\beta\geq 0$, with continuous first partial derivative with respect to β;

(2) ϕ increases monotonically with respect to α and β, with $\phi(0, 0) = 0$;

(3) $\frac{\partial\phi}{\partial\beta}$ also increases monotonically with respect to α and β, with

$$\frac{\partial\phi(0,0)}{\partial\beta} = 0$$

then, for small enough α, there exists a β^* dependent on α such that

$$\beta < \phi(\alpha,\beta), \qquad \forall\beta\in[0,\beta^*)$$

where

$$\beta^* = \phi(\alpha,\beta^*)$$

Proof. Define

$$\Phi(\alpha, \beta) = \phi(\alpha, \beta) - \beta \tag{AII.1}$$

on a compact domain $L\times L$ where

$$L = \{ x \,/\, 0 \leq x \leq l \}$$

and l can be arbitrarily large. Then Φ is continuous on $L\times L$ and hence it is uniformly continuous on $L\times L$. The following proof first shows that for some small enough α, there exists a β' such that at β', $\Phi<0$. It then shows that for the same α, there exists a β'' such that for all $\beta<\beta''$, $\Phi>0$. It then concludes that there must exist a β^* lying in the region $\beta''<\beta^*<\beta'$ such that for all $\beta<\beta^*$, $\Phi>0$ and at $\beta=\beta^*$, $\Phi=0$. The proof is thus complete.

Consider now

$$\frac{\partial\Phi}{\partial\beta} = \frac{\partial\phi}{\partial\beta} - 1$$

From the properties of ϕ, it is clear that $\frac{\partial\Phi}{\partial\beta}$ is continuous on $L\times L$, and hence uniformly continuous on $L\times L$, and that it increases *monotonically* with respect to α and β, with

$$\frac{\partial\Phi(0,0)}{\partial\beta} = \frac{\partial\phi(0,0)}{\partial\beta} - 1 = -1$$

Thus, for each $\delta>0$, there exists an $\varepsilon>0$ such that

$$\frac{\partial\Phi(0,\beta)}{\partial\beta} - \frac{\partial\Phi(0,0)}{\partial\beta} < \delta$$

provided $\beta < \varepsilon$ and $\beta\in L$. Set $\delta = \frac{\Delta}{2}$ where $0<\Delta<1$, and let the corresponding $\varepsilon=\varepsilon_\Delta$. Then, provided $\beta<\varepsilon_\Delta$,

$$\frac{\partial\Phi(0,\beta)}{\partial\beta} < \frac{\Delta}{2} - 1 \ (<0)$$

Observe that because $\frac{\partial\Phi}{\partial\beta}$ increases *monotonically* with β, this inequality holds for all $\beta<\varepsilon_\Delta$. Thus, by integrating both sides,

$$\Phi(0,\beta) < -(1 - \frac{\Delta}{2})\beta\ , \quad \forall\beta\in [0,\varepsilon_\Delta[$$

In other words, there exists an ε_Δ such that for all $\beta<\varepsilon_\Delta$, $\Phi(0,\beta)$ is negative. Now, from its definition (AII.1), the mapping Φ increases monotonically with α, and Φ is uniformly continuous on $\alpha\in L$. Hence at some fixed $\beta=\beta'<\varepsilon_\Delta$, for each $\delta>0$, there exists an $\varepsilon>0$ such that

$$\Phi(\alpha,\beta') - \Phi(0,\beta') < \delta$$

provided $\alpha<\varepsilon$ and $\alpha\in L$. Now, set $\delta=(1-\Delta)\beta'$ and let the corresponding $\varepsilon=\varepsilon'$. Then at $\beta=\beta'<\varepsilon_\Delta$, provided $\alpha<\varepsilon'$,

$$\begin{aligned}\Phi(\alpha,\beta') &< \delta + \Phi(0,\beta') \\ &< (1-\Delta)\beta' - (1-\frac{\Delta}{2})\beta' \\ &= -\frac{\Delta}{2}\,\beta' \end{aligned} \qquad \text{(AII.2)}$$