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Slow Sampling And Stability Of Nonlinear Sampled-Data Systems

Y. Zheng[#], D. H. Owens[*], S. A. Billings[#]

Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield, S1 3JD, England

* Department of Mathematics,
University of Strathclyde,
26, Richmond Street,
Glasgow G1 1XH, Scotland

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Y. Zheng

Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, England

D. H. Owens

Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, Scotland

S. A. Billings

Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, England

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Abstract: This paper studies how a class of sampled-data system may be stabilised by slow sampling. A relationship is obtained between the minimum sampling period and an estimated domain of attraction of the system.

1. Introduction

It has been shown ([1],[2]) that if a sampled-data system has a stable underlying continuous system on a certain domain, it can be stabilised on the same domain by fast enough sampling rates. Inspired by this idea and the consequent results, this paper attempts to study the stability problem which may be thought of as the counter part to that discussed [1] and [2]. Suppose a sampled-data system is stable on a certain domain as its sampling rate tends to zero (i.e., all sampling intervals tend to infinity), one would expect that at sufficiently slow sampling rates, the system could still be stabilised on the same domain. This is to be examined in this paper. First, section 2 will investigate how the sampled-data system may behave as the sampling intervals tend to infinity, or equivalently, how the equilibrium points of the system may behave. Then section 3 will show that if the sampled-data system satisfies certain conditions, then there exists, on a certain domain, at least one sequence $\{x_k\}$ which converges to the origin as the sampling intervals tend to infinity. If this is true, then intuitively, it is reasonable to expect that the motion of the system may converge to such a sequence provided its sampling intervals are sufficiently long, and some additional restrictions may have to be satisfied by the system. This will be studied in section 4. Examples will be taken to demonstrate the method. A certain degree of conservativeness is expected because all the calculations involve norms.

2. The behaviour of a sampled-data system at zero sampling rate

The class of sampled-data systems to be studied can be represented by a block diagram shown in Fig.1.

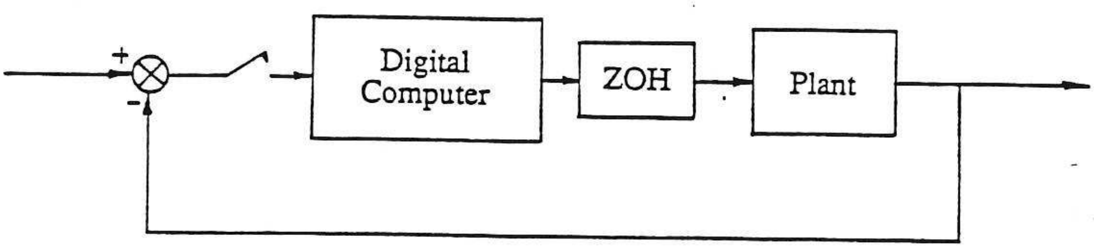

Fig. 1

It is assumed that the plant can be described by the model

$$\dot{x}(t) = Ax(t) + F(x(t)) + Bu(t)$$
$$y(t) = Cx(t)\,, \qquad\qquad x(0)=x_0$$

where $y\in\mathbf{R}^m$, $x\in\mathbf{R}^n$, $u\in\mathbf{R}^l$; A, B and C are the characteristic, input and output matrices of the plant of dimension $(n\times n)$, $(n\times l)$ and $(m\times n)$ respectively; $t_0=0$ without loss of generality; the nonlinearity of the plant $F(x)$ is assumed a polynomial in x of degree at most N, written as

$$F(x) = \sum_{j=2}^{N} F_j(x) , \qquad F_j(\alpha x) = \alpha^j F_j(x), \quad \forall \alpha \in \mathbf{R}, \quad \forall x \in \mathbf{R}^n$$

Assume that the linearised model of the plant is both controllable and observable. The plant is to be controlled by a digital computer, and due to the ZOH device, the plant inputs $u(t)$ are piece-wise constant signals which can be written as

$$u(t) = u_k, \qquad t \in [t_k, t_{k+1}[, \qquad k \geq 0$$

Note that constant sampling rate is *not* assumed. In this paper, the digital controller is assumed to be proportional. Hence the feedback system has a mathematical model

$$\dot{x}(t) = Ax(t) + F(x(t)) + Gx_k , \qquad x(t_k)=x_k, \quad t \in [t_k, t_{k+1}[, \quad k \geq 0 \tag{2.1}$$

where $G=-BKC$; the $(k+1)^{th}$ sampling period is $h_{k+1} = t_{k+1} - t_k$ and $t_k \to \infty$ as $k \to \infty$. Given the above model, the idea behind stabilising the system by slow sampling is very similar to that by fast sampling ([1],[2]). In the later case, the system is first stabilised on some domain D_ξ at the extreme sampling rate of infinity, which corresponds to a continuous system; then the stability behaviour of the system is investigated at some finite sampling interval over the same domain. In this paper, the system will first be stabilised on some domain at the other extreme a sampling rate of zero, which corresponds to the case in which the motion of the system tends to some equilibrium point during any sampling interval; then the stability behaviour of the system will be studied at finite sampling intervals.

It is important to be aware of the fact that for a nonlinear sampled-data system modelled by (2.1), there is no guarantee of the existence of some equilibrium points for a given arbitrary initial condition. For this reason, this section investigates the possible behaviours that system (2.1) may possess at $h_k \to \infty$. Without loss of generality, consider the first sampling interval with $x(0)=x_0$ and $x(h_1)=x_1$. As $h_1 \to \infty$, the system may behave in one of the following ways depending on its initial condition x_0.

(a) There exists some $x_1 \in \mathbf{R}^n$ at which $\dot{x}_1=0$. Therefore, x_1 is an equilibrium point given by

$$0 = Ax_1 + F(x_1) + Gx_0 \tag{2.2}$$

Note that due to the nonlinearity, x_1 is not necessarily unique, i.e., for some $x_0 \in \mathbf{R}^n$, there may exist more than one equilibrium point. Depending on x_0, the motion $x(t,x_0)$ of the system may converge to an asymptotically stable equilibrium point, or it may tend to infinity as $t \to \infty$.

(b) There does not exist any $x_1 \in \mathbf{R}^n$ at which $\dot{x}_1=0$, or equivalently, the equation (2.2) has no real solution for some $x_0 \in \mathbf{R}^n$. This is the case when no real equilibrium point exists in the whole state space with respect to the initial condition x_0.

As far as the stability of the system at zero sampling rate is concerned, the above case (b) is not desirable. A necessary but not sufficient condition on the system (2.1) to avoid case (b) is that the characteristic matrix A must be stable in the continuous sense for the reason that, from (2.1)

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}[\, F(x(\tau)) + Gx_0 \,]d\tau \qquad (2.3) \\ &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}F(x(\tau))d\tau + (e^{At}-I)A^{-1}Gx_0 \end{aligned}$$

if A is not stable, then e^{At} will become unbounded as $t \to \infty$, which results in an unbounded $x(t)$ as $t \to \infty$ at almost all $x_0 \in \mathbf{R}^n$ except in the special case in which $F(x)=0$ and $A^{-1}G = -I$, which results in $x(t)=x_0$ for all $t \geq 0$. Therefore, the following analysis will always assume that A is stable and hence invertible.

The stability study below takes two steps. First, it searches for some conditions that the controller design must satisfy to ensure the existence of a domain containing the origin and at least one sampling sequence $\{x_k\}_\infty$ obtained at $h_k \to \infty$ for all $k \geq 0$ such that provided the initial condition x_0 of the above sequence lies in the above domain, the sampling sequence converges to the origin. Then, as h_k becomes finite, the behaviour of the system is investigated and a search for a minimum sampling period h^* is carried out such that provided $h_k > h^*$, any sampling sequence $\{x_k\}$ will be 'close enough' to the sequence $\{x_k\}_\infty$ and consequently will converge to the origin.

3. The existence of a convergent sequence on a domain at $h_k \to \infty$

This section investigates the relationship between the equilibrium points of the sampled-data system and the sampled datum that results in these equilibrium points, and hence show that if the system satisfies certain conditions, a finite domain exists on which there exists at least one sampling sequence converging to the origin as $h_k \to \infty$.

Suppose D is such a domain, then for any sampled datum $x_s \in D$ at the sampling instant t_s, there must exist at least one equilibrium point which lies in D and whose 'distance' from the origin is shorter than that of x_s.

To find such a domain, first consider the following adjacent problem. Given a matrix equation

$$0 = Az + F(z) + Gx \tag{3.1}$$

where $F(0)=0$ and A is a stable matrix in the continuous sense, investigate the variation of the roots z of the equation with respect to x.

First note that when $x=0$, there exists at least one real root $z=0$. As A is stable and hence invertible, (3.1) can be written as

$$z = -A^{-1}Gx - A^{-1}F(z) \tag{3.2}$$

Now, let $\|.\|$ denote any real norm on the vector space $\mathbf{R}^n$. Taking the norm on both sides of (3.2) yields

$$\|z\| \le \|A^{-1}G\|\ \|x\| + \|A^{-1}\| \sum_{j=2}^{N} M_j \|z\|^j$$

$$= \phi(\|x\|, \|z\|) \tag{3.3}$$

where

$$\phi(\|x\|, \|z\|) = \|A^{-1}G\|\ \|x\| + \|A^{-1}\| \sum_{j=2}^{N} M_j \|z\|^j \tag{3.4}$$

and M_j is defined as

$$M_j = \sup_{||x||=1} ||F_j(x)|| \tag{3.5}$$

From Appendix I, provided $||x||$ is small enough, there exists a finite ζ dependent on $||x||$ such that the inequality (3.3) is satisfied for all $||z|| \le \zeta$ where ζ is given by

$$\zeta = \phi(||x||, \zeta) \tag{3.6}$$

and $\zeta=0$ at $x=0$. In other words, for small enough $||x||$, there exists a region $||z||<\zeta$ within which some of the roots z lie. If it is now required that $||z|| \le \mu||x||$, $0<\mu<1$, this requirement will be met provided

$$\phi(||x||, \mu||x||) \le \mu\, ||x|| \tag{3.7}$$

By substituting the definition of ϕ into the above, dividing both sides by $||x||$ and noting that at $x=0$, the equal sign holds, (3.7) becomes

$$||A^{-1}G|| + ||A^{-1}|| \sum_{j=2}^{N} M_j\, \mu^j\, ||x||^{j-1} \le \mu \tag{3.8}$$

Clearly, this inequality can be satisfied for some finite $||x||$ provided $\mu-||A^{-1}G|| >0$, or

$$\mu > ||A^{-1}G|| \tag{3.9}$$

Under this condition, there exists a well defined domain D_μ of the form

$$D_\mu = \{x \,/\, ||x|| \le r_\mu\}$$

where r_μ is obtained from

$$||A^{-1}G|| + ||A^{-1}|| \sum_{j=2}^{N} M_j\, \mu^j\, r_\mu^{j-1} = \mu \tag{3.10}$$

such that for all $x \in D_\mu$,

$$||z|| \le \mu\, ||x|| , \qquad \forall x \in D_\mu, \quad ||A^{-1}G|| < \mu < 1$$

When this result is applied to the system (2.1) with $h_k=\infty$ for all $k \ge 0$, the implication is obvious: for all $x_0 \in D_\mu$, there exists some equilibrium point x_1 such that

$$||x_1|| \le \mu\, ||x_0|| , \qquad ||A^{-1}G|| < \mu < 1$$

Hence, x_1 lies also within D_μ , and by induction,

$$\|x_k\| \leq \mu \, \|x_{k-1}\|$$
$$\leq \mu^k \, \|x_0\|$$

Obviously, as $k \to \infty$, $\|x_k\| \to 0$, or $x_k \to 0$. The above analysis leads to the following lemma:

Lemma 3.1. Consider the sampled-data system (2.1) where A is stable and invertible. If all sampling intervals of the system tend to infinity, and if the matrix G is chosen such that

$$\|A^{-1}G\| < 1$$

for some real norm defined on the vector space $\mathbf{R}^n$, then there exists at least one sequence $\{x_k\}_\infty$ which is a solution to the equation

$$Ax_{k+1} + F(x_{k+1}) + Gx_k = 0, \qquad k \geq 0 \tag{3.11}$$

and which has the property that for an arbitrary μ lying in the interval

$$\|A^{-1}G\| < \mu < 1 \tag{3.12}$$

if x_0 lies in the domain D_μ which is a connected region

$$\{ \, x \, / \, \|A^{-1}\| \sum_{j=2}^{N} M_j \, \mu^j \, \|x\|^{j-1} \leq \mu - \|A^{-1}G\| \, \} \tag{3.13}$$

of the form

$$D_\mu = \{ \, x \, / \, \|x\| \leq r_\mu \, \} \tag{3.14}$$

where M_j is defined as in (3.5) and r_μ is given by

$$\|A^{-1}\| \sum_{j=2}^{N} M_j \, \mu^j \, r_\mu^{j-1} = \mu - \|A^{-1}G\| \tag{3.15}$$

then,

$$\|x_{k+1}\| \leq \mu \, \|x_k\| \, , \qquad k \geq 0 \tag{3.16}$$

Thus for all $x_0 \in D_\mu$,

$$\lim_{k \to \infty} x_k = 0$$

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Remarks. From the above lemma, the estimation of D_μ is affected by two variables: the controller gain G and the parameter μ. Assuming $\mu \in]\|A^{-1}G\|, 1[$, then it is quite obvious (eq.(3.15)) that a high controller gain reflected by a higher $\|A^{-1}G\|$ reduces the value of r_μ and hence decreases the domain D_μ. The effect of μ on D_μ is not so obvious, as μ appears on both sides of eq.(3.15). However, it is not difficult to see that $r_\mu=0$ at $\mu=\|A^{-1}G\|$, and when $\mu>\|A^{-1}G\|$, there always exists an $r_\mu>0$. It can be shown that in general r_μ increases with μ to a maximum, and then decreases to zero as $\mu \rightarrow \infty$. Hence for each r_μ, two values of μ are possible one of which may be greater than unity. They represent two upper bounds on $\|z\|$. It is sufficient to take the lowest upper bound as the upper bound of $\|z\|$, and this upper bound increases monotonically with r_μ.

Before leaving this section, it is emphasised that lemma 3.1 only serves to specify the conditions for the existence of some sequence $\{x_k\}_\infty$ converging to the origin. It did not exclude the possibility of having more than one such sequence, and it did not specify the stability property of each element in the sequence. For example, x_k may be an equilibrium point resulting from x_{k-1}, but x_k may be unstable. Now if a sampled-data system is to be stabilised by slow sampling rates, it means that the motion trajectory of the sampled-data system during any sampling interval converges to an equilibrium point which is 'closer' to the origin. This has two implications: (i) this equilibrium point must be stable; (ii) the sampled datum at the beginning of the sampling interval must lie inside the DOA of this equilibrium point. When these conditions are satisfied, then it is quite obvious that sufficiently long sampling intervals guarantee the stability of the system.

4. Relationship between sampling periods and the estimated DOA

This section aims to find a domain having the following properties: for every sampled datum x_s in it, there exists a unique equilibrium point z which is closer to the origin and that x_s lies in the DOA of z. Once such a domain is found, the search for a minimum sampling interval that ensures the system stability is an easy task.

The analysis below first introduces an error signal measuring the vector difference between the motion $x(t,x_0)$ of the system and one of its equilibrium points z. The idea behind it is to show that provided z is sufficiently close to the origin, there exists a Lyapunov function in the neighbourhood of z. This implies that the corresponding z is an asymptotically stable equilibrium point. It then shows that on certain domain, x_0 actually lies in the DOA of z. Furthermore, if x_0 also lies on D_μ (given by lemma 3.1), then $||z|| \le \mu ||x_0||$. Thus, $x(t,x_0)$ will converge to this z and the sampled-data system will be asymptotically stable at $h_k=\infty$.

Hence, the analysis starts by considering the error term $e(t)$ defined as

$$e(t) = x(t, x_0) - z \tag{4.1}$$

where $x(t,x_0)$ is the motion of the system with the initial condition $x(0)=x_0$ and z satisfies

$$0 = Az + F(z) + Gx_0 \tag{4.2}$$

for some x_0. A is a stable matrix as required before. Let the $(n \times n)$ matrix P satisfy

$$A^T P + P A = -I_n \tag{4.3}$$

Hence P is a positive definite matrix. Let V be a quadratic mapping V: $\mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$V(e) = e^T P\, e, \qquad P = P^T > 0 \tag{4.4}$$

Also as before, let the P-matrix norm be defined as

$$||x||_P = (x^T P\, x)^{1/2} = V(x)^{1/2} \tag{4.5}$$

Now, consider how the error signal $e(t)$ behaves by considering

$$\begin{aligned} \dot{e} = \dot{x} &= Ax + F(x) + Gx_0 \\ &= A(x-z) + F(x) - F(z) \\ &= Ae + F(e+z) - F(z) \end{aligned} \tag{4.6}$$

Along the solution $e(t)$, $\dfrac{dV}{dt}$ is found to be

$$\dot{V} = \dot{e}^T P\ e + e^T P\ \dot{e}$$
$$= -e^T e + 2e^T P\ [F(e+z) - F(z)] \tag{4.7}$$

If it is possible to restrict z in some way so that there exists a domain on which $\dot{V}<0$, then from the La Salle's theorem, $e(t)$ is asymptotically stable on this domain. As before, one approach is to find the upper bound of $\dot{V}$ in terms of the norms of e and z. Now, the nonlinear terms have the following expansion:

$$e^T P\ [F(e+z) - F(z)] = e^T P \sum_{j=2}^{N} [F_j(e+z) - F_j(z)] \tag{4.8}$$

First, consider the second order nonlinearity at $j=2$, it can be written in the form of a bilinear function as

$$F_2(x) = g(x,x)$$

where $g(x,x)$ is bilinear and therefore,

$$g(e+z,e+z) = g(e,e) + g(e,z) + g(z,e) + g(z,z)$$

Hence, the second order nonlinear term can be expanded as

$$e^T P\ [F_2(e+z) - F_2(z)] = e^T P\ [g(e,e) + g(e,z) + g(z,e)]$$
$$\leq \eta_t^2\ \|z\|_P\ M_{21} + \eta_t^3\ M_{22} \tag{4.9}$$

where

$$\eta_t = \|e(t)\|_P = V(e(t))^{1/2} \tag{4.10}$$

$$M_{21} = \sup_{\substack{\|z\|_P=1 \\ \eta_t=1}} \|g(e,z) + g(z,e)\|_P$$

$$M_{22} = \sup_{\eta_t=1} \|g(e,e)\|_P$$

More generally and following the same principle, the j^{th} order nonlinearity can be written as

$$F_j(x) = g(x,x,\ldots,x)$$

with j arguments in g, and

$$e^T P\ [F_j(e+z) - F_j(z)] \leq \eta_t \sum_{l=1}^{j} M_{jl} \|z\|_P^{j-l}\ \eta_t^l$$

Hence eq.(4.7) becomes

$$\dot{V} \le -\frac{V}{\bar{\sigma}(P)} + 2\sum_{j=2}^{N}\sum_{l=1}^{j} M_{jl}\, \|z\|_P^{j-l}\, \eta_t^{l+1}$$
$$= -\left(\frac{1}{\bar{\sigma}(P)} - 2\sum_{j=2}^{N}\sum_{l=1}^{j} M_{jl}\, \|z\|_P^{j-l}\, \eta_t^{l-1}\right) V \tag{4.11}$$

Note that the above inequality is true for all z satisfying (4.2). However, V will be a Lyapunov function only if the quantity inside the bracket of the inequality (4.11) be kept positive at all times, and this puts restrictions on $\|z\|_P$ as well as on η_t. To find these restrictions, define

$$S(\|z\|_P,\eta_t) = 2\sum_{j=2}^{N}\sum_{l=1}^{j} M_{jl}\, \|z\|_P^{j-l}\, \eta_t^{l-1} \tag{4.12}$$

and note that S has the following properties:

(i) $S(0,0)=0$;

(ii) S is continuous and increases monotonically with respect to its arguments;

Thus, S can be made as small as required by making both of its arguments small enough. The requirement that

$$\dot{V} \le -\xi V, \qquad 0<\xi<\frac{1}{\bar{\sigma}(P)} \tag{4.13}$$

will be satisfied provided S is sufficiently small that

$$\frac{1}{\bar{\sigma}(P)} - S(\|z\|_P,\eta_t) \ge \xi$$

or

$$S(\|z\|_P,\eta_t) \le \frac{1}{\bar{\sigma}(P)} - \xi \tag{4.14}$$

This restriction implies that if an equilibrium point z is close enough to the origin, it is possible to find a finite error norm $\eta_t=\|x(t,x_0)-z\|_P$ and hence a finite domain about z on which the quadratic mapping V is a Lyapunov function of the error signal $e(t)$. Furthermore, V is exponentially bounded on that domain about z. This domain is therefore an estimated DOA for the corresponding equilibrium point z. It is important to observe at this stage that if (4.14) is satisfied initially at $t=0$, i.e., if

$$S(\|z\|_P,\eta_0) \leq \frac{1}{\bar{\sigma}(P)} - \xi$$

then, $\dot{V} \leq -\xi V$ initially, which implies that V, and hence η_t, decreases initially. This decrease results in an even smaller S. Therefore (4.14) will always be satisfied if it is satisfied *initially*, and hence (4.13) can be written as

$$\|x(t,\, x_0) - z\|_P \leq e^{-\frac{\xi t}{2}} \|x_0 - z\|_P \tag{4.15}$$

This is stated by the following lemma.

Lemma 4.1. Consider the sampled-data system (2.1). Let the quadratic mapping V and the P-matrix norm be defined as in (4.4) and (4.5) respectively. Let the mapping S be defined as in (4.12). Let z denote an equilibrium point given by

$$0 = Az + F(z) + Gx_0, \qquad x(0) = x_0$$

and let $e(t)$ denote the error signal

$$e(t) = x(t,\, x_0) - z$$

If z is such that

$$S(\|z\|_P,\, \eta_0) \leq \frac{1}{\bar{\sigma}(P)} - \xi\,, \qquad 0 < \xi < \frac{1}{\bar{\sigma}(P)} \tag{4.16}$$

where $\eta_0 = \|x_0 - z\|_P$, then V is a Lyapunov function of $e(t)$ over the domain

$$D_{z\xi} = \{\, e \,/\, \|e\|_P \leq \eta_0 \,\}$$

and this domain is an estimated DOA of the corresponding equilibrium point.

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The next step is to find those conditions which ensure that x_0 actually lies in $D_{z\xi}$. It has been stated in lemma 3.1 that provided the system (2.1) satisfies the condition

$$\|A^{-1}G\|_P < 1$$

there exists a domain D_μ and that for all $x_0 \in D_\mu$, at least one equilibrium point of the system lies in the domain

$$D_z = \{\ z\ /\ \|z\|_P \le \mu\|x_0\|_P,\ \ \|A^{-1}G\|_P<\mu<1,\ \ x_0\in D_\mu\ \} \tag{4.17}$$

The upper bound on η_0 can thus be found from

$$\begin{aligned}\eta_0 = \|x_0-z\|_P &\le \|x_0\|_P + \|z\|_P \\ &\le \|x_0\|_P + \mu\|x_0\|_P \\ &= (1+\mu)\ \|x_0\|_P\end{aligned} \tag{4.18}$$

for all $x_0\in D_\mu$. If now for some $x_0\in D_\mu$,

$$S(\mu\|x_0\|_P,\ (1+\mu)\|x_0\|_P)\ \le\ \frac{1}{\bar{\sigma}(P)} - \xi \tag{4.19}$$

this implies that for every z that is bounded by $\|z\|_P \le \mu\|x_0\|_P$, it has an estimated DOA

$$D_{z\xi} = \{\ e\ /\ \|e\|_P \le (1+\mu)\|x_0\|_P\ \}$$

Therefore, the corresponding x_0 lies in the DOA of z and hence $x(t,x_0)$ will converge to z as $t\to\infty$. Substituting the definition of S into (4.19) yields:

$$2\sum_{j=2}^{N}\sum_{l=1}^{j} M_{jl}\ \mu^{j-l}\ (1+\mu)^{l-1}\ \|x_0\|_P^{j-1}\ \le\ \frac{1}{\bar{\sigma}(P)} - \xi$$

and the domain D_ξ on which this inequality is satisfied is given by

$$D_\xi = \{\ x\ /\ \|x\|_P \le r_\xi\ \}$$

where r_ξ is calculated from

$$2\sum_{j=2}^{N}\sum_{l=1}^{j} M_{jl}\ \mu^{j-l}\ (1+\mu)^{l-1}\ r_\xi^{j-1}\ =\ \frac{1}{\bar{\sigma}(P)} - \xi$$

It is clear that if x_0 lies in the intersection of the above two domains, i.e.,

$$x_0 \in D^* = D_\mu \cap D_\xi$$

then $x(t,x_0)\to z$ as $t\to\infty$ and $x_k\to 0$ as $k\to\infty$. Hence the following theorem has been proved.

Theorem 4.2 (Asymptotic stability at infinite sampling intervals). Consider the sampled-data system (2.1) where A is stable and invertible. Assume that all sampling intervals of the system tend to infinity and the matrix G is such that

$$\|A^{-1}G\|_P\ <\ 1$$

Let the P-matrix norm be defined as in eq.(4.5). Let D_μ be a domain of the form

$$D_\mu = \{\, x \,/\, \|x\|_P \le r_\mu \,\} \tag{4.20}$$

where r_μ is given by

$$\|A^{-1}\|_P \sum_{j=2}^{N} M_j\, \mu^j\, r_\mu^{j-1} = \mu - \|A^{-1}G\|_P\,, \qquad \|A^{-1}G\|_P < \mu < 1 \tag{4.21}$$

and let D_ξ be a domain of the form

$$D_\xi = \{\, x \,/\, \|x\|_P \le r_\xi \,\} \tag{4.22}$$

where r_ξ is given by

$$2\sum_{j=2}^{N} \sum_{l=1}^{j} M_{jl}\, \mu^{j-l}\, (1+\mu)^{l-1}\, r_\xi^{j-1} = \frac{1}{\overline{\sigma}(P)} - \xi\,, \quad 0<\xi<\frac{1}{\overline{\sigma}(P)} \tag{4.23}$$

The sampled-data system is asymptotically stable on the domain

$$D^* = D_\mu \cap D_\xi \tag{4.24}$$

at $h_k = +\infty$, $\forall k \ge 0$.

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Having found the domain upon which the sampled-data system is asymptotically stable as $h_k \to \infty$, it is now easy to show that the system is also asymptotically stable on D^* at finite sampling intervals. Denote $x(t)=x(t,x_0)$. Then,

$$\begin{aligned} \|x(t)\|_P &= \|x(t)-z+z\|_P \\ &\le \|x(t)-z\|_P + \|z\|_P \end{aligned}$$

If $x_0 \in D^*$, then according to the above theorem,

$$\begin{aligned} \|x(t)\|_P &\le e^{-\frac{\xi t}{2}}\, \|x_0 - z\|_P + \mu\, \|x_0\|_P \\ &\le e^{-\frac{\xi t}{2}}\, (\|x_0\|_P + \mu \|x_0\|_P) + \mu\, \|x_0\|_P \\ &= [e^{-\frac{\xi t}{2}}\, (1+\mu) + \mu]\, \|x_0\|_P \end{aligned}$$

If h_1 is such that

$$e^{-\frac{\xi h_1}{2}}\, (1+\mu) + \mu \le \varepsilon\,, \qquad \mu<\varepsilon<1 \tag{4.25}$$

then

$$\|x_1\|_P = \|x(h_1)\|_P \le \varepsilon\ \|x_0\|_P$$

and hence $x_1 \in D^*$. Imposing this conditions on all sampling intervals h_k , it is clear that at the k^{th} sampling instance,

$$\begin{aligned}\|x_k\|_P &\le \varepsilon \|x_{k-1}\|_P \\ &\le \varepsilon^k \|x_0\|_P\ , \qquad \mu<\varepsilon<1\end{aligned}$$

which results in

$$\lim_{k\to\infty} \|x_k\|_P = 0$$

Hence the restriction upon h_k can be found from (4.25) by replacing h_1 by h_k, which yields:

$$h_k \ge -\frac{2}{\xi}\ ln\ (\frac{\varepsilon-\mu}{1+\mu}), \qquad \mu<\varepsilon<1$$

The minimum sampling interval h^* is thus given by

$$h^* = -\frac{2}{\xi}\ ln\ (\frac{\varepsilon-\mu}{1+\mu})$$

The following stability theorem is a result of the stability study carried out in this paper.

Theorem 4.3 (Asymptotic stability by slow sampling). The sampled-data system (2.1) is asymptotically stable in the domain D^* given by theorem 4.2 provided all sampling intervals h_k of the system are such that

$$h_k \ge h^*\ , \qquad k=1,2,...$$

where

$$h^* = -\frac{2}{\xi}\ ln\ (\frac{\varepsilon-\mu}{1+\mu}), \qquad \mu<\varepsilon<1 \tag{4.26}$$

μ and ξ are defined in eq.s (3.12) and (4.13) respectively.

o

Example 4.1 Consider the second order system

$$\dot{x}(t) = \begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix} x(t) + \begin{bmatrix}1/4x_\alpha^2\\ 0\end{bmatrix} + \begin{bmatrix}0 & 1/2\\ 0 & 0\end{bmatrix} x_k\,, \qquad x(t_k)=x_k, \quad t\in[t_k,\, t_{k+1}[, \quad k\geq 0$$

where $x = (x_\alpha, x_\beta)^T$. Note that $(x_1, x_2, \ldots x_i, \ldots)$ are reserved to denote sampled data rather than the components of a data point. Hence,

$$P = \begin{bmatrix}1/2 & 0\\ 0 & 1/6\end{bmatrix}$$

From theorem 4.2, r_μ is found to be

$$r_\mu = \frac{4\times(\mu-\frac{\sqrt{3}}{2})}{3\sqrt{2}\mu^2}\,, \qquad \frac{\sqrt{3}}{2}<\mu<1$$

and by choosing

$$\xi = \frac{1}{8}\times\frac{1}{\overline{\sigma}(P)} = 0.25$$

r_ξ is given by

$$r_\xi = \frac{3.5}{3\sqrt{2}(3\mu + 1)}$$

At μ=0.990 for example,

$$r_\mu = 0.119\,, \qquad r_\xi = 0.208$$

Then on the domain

$$D^* = \{\, x \,/\, \|x\|_P \leq 0.119 \,\}$$

the sampled-data system is asymptotically stable at zero sampling rate. The actual domain on which the system is stable at zero sampling rate can be easily found, and is shown in Fig.2. Compared with the above estimated domain, it is seen that theorem 4.2 can give rather conservative results.

Now according to theorem 4.3, the system will be asymptotically stable on the domain D^* for all $h\geq h^*$ where h^* is given by

$$h^* = -\frac{2}{0.25}\, ln\!\left(\frac{\varepsilon-0.990}{1+0.990}\right), \qquad 0.990<\varepsilon<1$$

At $\varepsilon = 0.999$, $h^* \approx 43$. It can be shown that the actual sampling sequence $\{x_k\}$ will eventually converge to zero as $k\rightarrow\infty$ irrespective to sampling intervals.

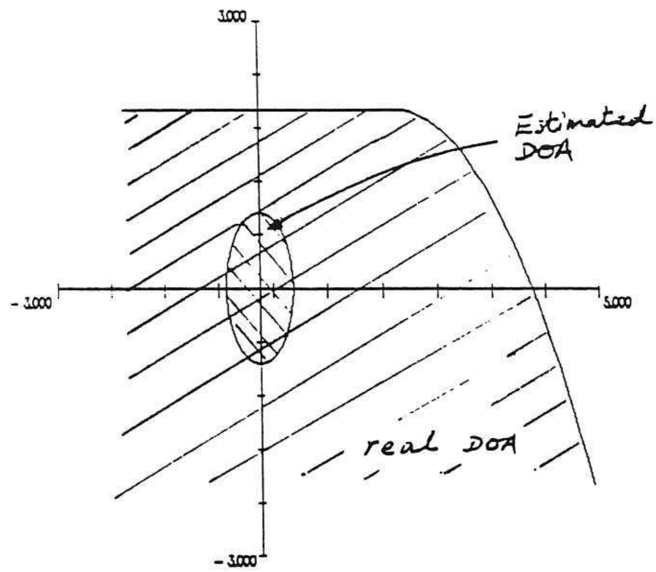

Fig. 2

Although the stability criteria derived in this paper are rather conservative, as is shown by the above example, it is important to bear in mind that in most general cases, it will not be possible to calculate the actual domain of attraction of a sampled-data system or to relate it to the sampling rate of the system.

5. Conclusions

This paper has studied sufficient conditions for stabilising the sampled-data system (2.1) by slow sampling. The principle idea behind the analysis was quite simple. If a sampled-data system could be stabilised over a domain with an infinite sampling period, then with a sufficiently long sampling period, the system could still be stabilised. The restrictions that must be met by the system are that the A matrix of the system should be asymptotically stable, and that the proportional controller gain G satisfies

$$\|A^{-1}G\|_P < 1$$

It is emphasised that the above analysis allows random sampling rates provided all sampling periods are within the 'allowable' range which ensures the system stability. Examples taken above showed how the stability criteria could be easily applied, even though the results may be conservative in some cases.

References

[1] Owens, D.H., Zheng, Y. and Billings, S.A., 'Fast sampling and stability of nonlinear sampled-data systems -- part I: Existence theorems', (submitted for publication).

[2] Zheng, Y., Owens, D.H. and Billings, S.A., 'Fast sampling and stability of nonlinear sampled-data systems -- part II: Sampling rate estimations', (submitted for publication).

APPENDIX

A Lemma

Lemma. Let $\phi(\alpha, \beta)$ be a mapping ϕ: $\mathbf{R}\times\mathbf{R}\rightarrow\mathbf{R}$ with the following properties:

(1) ϕ is defined and continuous $\forall\alpha\geq 0$, $\forall\beta\geq 0$, with continuous first partial derivative with respect to β;

(2) ϕ increases monotonically with respect to α and β, with $\phi(0, 0) = 0$;

(3) $\dfrac{\partial\phi}{\partial\beta}$ also increases monotonically with respect to α and β, with

$$\frac{\partial\phi(0,0)}{\partial\beta} = 0$$

then, for small enough α, there exists a β^* dependent on α such that

$$\beta < \phi(\alpha,\beta), \qquad \forall\beta\in[0,\beta^*)$$

where

$$\beta^* = \phi(\alpha,\beta^*)$$

Proof. Define

$$\Phi(\alpha, \beta) = \phi(\alpha, \beta) - \beta \tag{A.1}$$

on a compact domain $L\times L$ where

$$L = \{\, x \,/\, 0 \leq x \leq l \,\}$$

and l can be arbitrarily large. Then Φ is continuous on $L\times L$ and hence it is uniformly continuous on $L\times L$. The following proof first shows that for some small enough α, there exists a β' such that at β', $\Phi<0$. It then shows that for the same α, there exists a β'' such that for all $\beta<\beta''$, $\Phi>0$. It then concludes that there must exist a β^* lying in the region $\beta''<\beta^*<\beta'$ such that for all $\beta<\beta^*$, $\Phi>0$ and at $\beta=\beta^*$, $\Phi=0$. The proof is thus complete.

Consider now

$$\frac{\partial\Phi}{\partial\beta} = \frac{\partial\phi}{\partial\beta} - 1$$

From the properties of ϕ, it is clear that $\frac{\partial\Phi}{\partial\beta}$ is continuous on $L{\times}L$, and hence uniformly continuous on $L{\times}L$, and that it increases *monotonically* with respect to α and β, with

$$\frac{\partial\Phi(0,0)}{\partial\beta} = \frac{\partial\phi(0,0)}{\partial\beta} - 1 = -1$$

Thus, for each $\delta>0$, there exists an $\varepsilon>0$ such that

$$\frac{\partial\Phi(0,\beta)}{\partial\beta} - \frac{\partial\Phi(0,0)}{\partial\beta} < \delta$$

provided $\beta < \varepsilon$ and $\beta\in L$. Set $\delta = \frac{\Delta}{2}$ where $0<\Delta<1$, and let the corresponding $\varepsilon=\varepsilon_\Delta$. Then, provided $\beta<\varepsilon_\Delta$,

$$\frac{\partial\Phi(0,\beta)}{\partial\beta} < \frac{\Delta}{2} - 1 \ (<0)$$

Observe that because $\frac{\partial\Phi}{\partial\beta}$ increases *monotonically* with β, this inequality holds for all $\beta<\varepsilon_\Delta$. Thus, by integrating both sides,

$$\Phi(0,\beta) < -(1 - \frac{\Delta}{2})\beta\,, \quad \forall\beta\in[0,\varepsilon_\Delta[$$

In other words, there exists an ε_Δ such that for all $\beta<\varepsilon_\Delta$, $\Phi(0,\beta)$ is negative. Now, from its definition (A.1), the mapping Φ increases monotonically with α, and Φ is uniformly continuous on $\alpha\in L$. Hence at some fixed $\beta=\beta'<\varepsilon_\Delta$, for each $\delta>0$, there exists an $\varepsilon>0$ such that

$$\Phi(\alpha,\beta') - \Phi(0,\beta') < \delta$$

provided $\alpha<\varepsilon$ and $\alpha\in L$. Now, set $\delta=(1-\Delta)\beta'$ and let the corresponding $\varepsilon=\varepsilon'$. Then at $\beta=\beta'<\varepsilon_\Delta$, provided $\alpha<\varepsilon'$,

$$\begin{aligned}\Phi(\alpha,\beta') &< \delta + \Phi(0,\beta')\\ &< (1-\Delta)\beta' - (1-\frac{\Delta}{2})\beta'\\ &= -\frac{\Delta}{2}\,\beta' \end{aligned} \qquad (A.2)$$

In other words, provided α is small enough, there exists some $\beta'<\varepsilon_\Delta$ at which $\Phi(\alpha,\beta')$ can still be kept negative.

Now, return to the definition (A.1) again and note that at $\beta=0$,

$$\Phi(\alpha,0) = \phi(\alpha,0) > 0, \qquad 0<\alpha\leq l$$

Then, from uniform continuity of Φ on $\beta\in L$, it is clear that for each $\delta(\alpha)>0$, there exists an $\xi(\alpha)>0$ such that

$$|\Phi(\alpha,\beta) - \Phi(\alpha,0)| < \delta(\alpha)$$

provided $\beta<\varepsilon(\alpha)$, $\beta\in L$. Set $\delta(\alpha)=r(\alpha)<\Phi(\alpha,0)$ and let the corresponding $\varepsilon(\alpha)=\varepsilon_r$. Then for $\beta\leq\beta''<\varepsilon_r$,

$$0 < \Phi(\alpha,0)-r < \Phi(\alpha,\beta) < \Phi(\alpha,0)+r$$

Thus, for each $0<\alpha\leq l$, there exists a $\beta''<\varepsilon_r$ such that for all $\beta\in[0,\beta'']$, $\Phi>0$. This implies that for each $\alpha\in(0,\varepsilon')$, there exist a $\beta''<\varepsilon_r$ and a range $\beta\in[0,\beta'']$ such that $\Phi(\alpha,\beta)>0$. But, eqn.(A.2) says that for each $\alpha\in(0,\varepsilon')$, there exists a β' such that $\Phi(\alpha,\beta')<0$. Hence, there must exist a $\beta^*\in(\beta'',\beta')$ such that $\Phi(\alpha,\beta)>0$ for $\alpha\in(0,\varepsilon')$ and $\beta\in[0,\beta^*)$, and at $\beta=\beta^*$,

$$\Phi(\alpha,\beta^*) = 0$$

This indicates, from (A.1), that for all $\beta\in[0,\beta^*)$,

$$\beta < \phi(\alpha,\beta)$$

and at $\beta=\beta^*$,

$$\beta^* = \phi(\alpha,\beta^*)$$

Q.E.D.