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Nonlinear Input-Output Maps
For Bilinear Systems
and Stability

by

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Abstract

The aim of this paper is to introduce two nonlinear input-output representations of bilinear systems. Sufficient conditions for $\mathcal{L}^{\infty}$ -stability are derived. These representations are believed to compare favorably with the standard Volterra series representation.

1 Introduction:

Functional expansions known as Volterra series are one of the most useful tools in nonlinear system theory [5], (see also [7]). Since their introduction by Weiner in the 1940's, papers dealing with the subject appeared periodically. Formulas and/or computational schemes were derived for terms in these expansions.For bilinear systems, explicit formulas for the calculation of the kernels were obtained [1]. Despite its use in realization theory [6], [2], [3] and optimal control [9], to cite a few, the method presented a major drawback, its rate of convergence. One has to compute a large number of terms in order to get a 'reasonable approximation '.

In this paper, we introduce two input-output representations for bilinear systems, and derive some stability results. To our knowledge the method is new, and we believe that, due to the nonlinear nature of the kernels involved, this may lead to a 'reasonable approximation' in a smaller number of terms, a point which will be investigated in future papers.

The paper is organised as follows: In section 2, we shall present a new input-output map which reduces to the standard Volterra series expansion under a suitable assumption. In section 3, a more general input-output map is introduced. Finally, in section 4, stability results are derived from these representations.

In the following, we use the notation below:

If x is an n-vector and A is an nxn matrix, we take as compatible norms:

$$\| x \| = max_{i=1,n} \mid x_i \mid \quad and \quad \| A \| = max_{i=1,n} \sum_{k=1}^{n} \mid a_{ik} \mid$$

2 A nonlinear input-output map for bilinear systems I:

Consider the bilinear system

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{m} u_j N_j x, \qquad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{2.1}$$

where A, N_j, C are constant matrices of suitable dimensions. We shall present in this section a new input-output map to represent this system.

Let $u_0(t) = 1, \;\; t \geq 0$ and $N_0 = A$, then (2.1) becomes

$$\begin{aligned} \dot{x} &= \sum_{j=0}^{m} u_j N_j x, \qquad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{2.2}$$

Consider the change of variable

$$z = e^{-N_k \int_0^t u_k(\tau) d\tau} x \tag{2.3}$$

we have $z(0) = z_0 = x_0$. Differentiating and taking into account (2.2) and (2.3), we get

$$\dot{z} = e^{-N_k \int_0^t u_k(\tau)d\tau} P_k(t) e^{N_k \int_0^t u_k(\tau)d\tau} z \tag{2.4}$$

where $P_k(t) = \sum_{j \neq k} u_j N_j$.

Let $U_k(\tau) = \int_0^\tau u_k(\sigma)d\sigma$

Then,

$$z(t) = z_0 + \int_0^t e^{-N_k U_k(\tau)} P_k(\tau) e^{N_k U_k(\tau)} z(\tau) d\tau \tag{2.5}$$

Using standard Picard iteration, define

$$\begin{aligned} z_0(t) &= z_0 \\ z_l(t) &= \int_0^t e^{-N_k U_k(\tau)} P_k(\tau) e^{N_k U_k(\tau)} z_{l-1}(\tau) d\tau \quad l \geq 1 \end{aligned} \tag{2.6}$$

It is easy to prove that the solution of the integral equation is given by

$$z(t) = \sum_{l \geq 0} z_l(t) \tag{2.7}$$

Thus,

$$\begin{aligned} y(t) &= Ce^{N_k U_k(t)} x_0 \\ &+ \sum_{l \geq 1} \int_0^t \int_0^{\tau_l} \cdots \int_0^{\tau_2} Ce^{N_k [U_k(t) - U_k(\tau_l)]} P_k(\tau_l) e^{N_k [U_k(\tau_l) - U(\tau_{l-1})]} \\ & P_k(\tau_{l-1}) \ldots P_k(\tau_2) e^{N_k [U_k(\tau_2) - U_k(\tau_1)]} \\ & P_k(\tau_1) e^{N_k U_k(\tau_1)} x_0 d\tau_1 \ldots d\tau_l \end{aligned} \tag{2.8}$$

From which we obtain the nonlinear input-output map given by

$$y(t) = Ce^{N_k U_k(t)} x_0$$

$$+ \sum_{l\geq 1}\sum_{j_1\neq k}\cdots\sum_{j_l\neq k}\int_0^t\int_0^{\tau_l}\cdots\int_0^{\tau_2}$$

$$Ce^{N_k[U_k(t)-U_k(\tau_l)]}N_{j_l}e^{N_k[U_k(\tau_l)-U_k(\tau_{l-1})]}N_{j_{l-1}}$$

$$\ldots N_{j_2}e^{N_k[U_k(\tau_2)-U_k(\tau_1)]}N_{j_1}e^{N_kU_k(\tau_1)}x_0$$

$$u_{j_l}(\tau_l)\ldots u_{j_1}(\tau_1)d\tau_1\ldots d\tau_l \tag{2.9}$$

Remarks:

1. The first remark we can make at this point is that '$N_kU_k(t)$' appears as exponent whereas in the standard Volterra representation [8] 'At' appears as exponent. This fact is very important from the stability point of view. We shall see in the next section how we derive a new stability criterion based on this representation.

2. This new representation reduces to the standard Volterra series for $k = 0$.

3 A nonlinear input-output map for bilinear systems II:

Returning to (2.1), We shall present in this section another input-output map to represent this system.

Let Γ be an nxn matrix solution of the equation

$$\dot{\Gamma} = -\Gamma\sum_{j=1}^{m}u_jN_j \quad , \quad \Gamma(0) = I \tag{3.1}$$

where I is the identity matrix, and consider the change of variable

$$z = \Gamma x \tag{3.2}$$

we have $z(0) = z_0 = x_0$. Differentiating and taking into account, we get

$$\begin{aligned}
\dot{z} &= \dot{\Gamma} x + \Gamma \dot{x} \\
&= [-\Gamma \sum_{j=1,m} u_j N_j] x + \Gamma [A x + \sum_{j=1,m} u_j N_j x] \\
&= -\sum_{j=1,m} u_j \Gamma N_j \Gamma^{-1} z + \Gamma A \Gamma^{-1} z + \sum_{j=1,m} u_j \Gamma N_j \Gamma^{-1} z
\end{aligned} \tag{3.3}$$

hence,

$$\dot{z} = \Gamma A \Gamma^{-1} z \ , \qquad z(0) = z_0 \tag{3.4}$$

Again, using standard Picard iteration, define

$$\begin{aligned}
z_0(t) &= z_0 \\
z_k(t) &= \int_0^t \Gamma(\tau) A \Gamma^{-1}(\tau) z_{k-1}(\tau) d\tau \ , \quad k \geq 1
\end{aligned} \tag{3.5}$$

As before, the solution of the integral equation is given by

$$z(t) = \sum_{k \geq 0} z_k(t) \tag{3.6}$$

Defining,

$$\Phi(t, \tau) = \Gamma^{-1}(t) \Gamma(\tau) \tag{3.7}$$

we obtain,

$$\begin{aligned}
y(t) &= C \Phi(t, 0) x_0 \\
&+ \sum_{k \geq 1} \int_0^t \int_0^{\tau_k} \ldots \int_0^{\tau_2} C \Phi(t, \tau_k) A \Phi(\tau_k, \tau_{k-1}) A \\
&\quad \ldots A \Phi(\tau_2, \tau_1) A \Phi(\tau_1, 0) x_0 d\tau_1 \ldots \tau_k
\end{aligned} \tag{3.8}$$

4 Input-Output stability of bilinear systems:

In this section we shall present sufficient conditions for the L^{∞} – *stability* of bilinear systems. We claim the following:

Theorem 1:

A sufficient condition for the system (2.2) to be $L^{\infty} - stable$ is that the following hold:

(i) There exist at least one N_j $(j = 0, \ldots, m)$ (say N_k) having all its eigenvalues with negative real parts,

(ii) $\lim_{t \to \infty} \int_0^t [-\rho_k u_k(\tau) + \alpha_k \parallel P_k(\tau) \parallel] d\tau < \infty$

where $\alpha_k > 0$ and $\rho_k > 0$ are such that $\parallel e^{N_k t} \parallel \leq \alpha_k e^{-\rho_k t}$, $P_k(t) = \sum_{j \neq k} u_j N_j$ and $u \geq 0$.

Furthermore, if the limit is $-\infty$ then y(t) $\rightarrow$ 0 as t $\rightarrow \infty$.

Proof:

It follows from (i) as a standard result in the theory of differential equations that there exist positive constants α_k and ρ_k such that $\parallel e^{N_k t} \parallel \leq \alpha_k e^{-\rho_k t}$.

Combining (2.3) and (2.5) we obtain

$$x(t) = e^{N_k U_k(t)} x_0 + \int_0^t e^{N_k [U_k(t) - U_k(\tau)]} P_k(\tau) x(\tau) d\tau \qquad (4.1)$$

which yields,

$$\| x(t) \| \leq \alpha_k \| x_0 \| e^{-\rho_k U_k(t)} + \alpha_k \int_0^t e^{-\rho_k [U_k(t)-U_k(\tau)]} \| P_k(\tau) \| \| x(\tau) \| d\tau \tag{4.2}$$

Therefore,

$$e^{\rho_k U_k(t)} \| x(t) \| \leq \alpha_k \| x_0 \| + \int_0^t \alpha_k \| P_k(\tau) \| e^{\rho_k U_k(\tau)} \| x(\tau) \| d\tau \tag{4.3}$$

Using Gronwall's lemma, we obtain

$$e^{\rho_k U_k(t)} \| x(t) \| \leq \alpha_k \| x_0 \| exp[\int_0^t \alpha_k \| P_k(\tau) \| d\tau] \tag{4.4}$$

Hence,

$$\| y(t) \| \leq \alpha_k \| C \| \| x_0 \| exp \int_0^t [-\rho_k u_k(\tau) + \alpha_k \| P_k(\tau) \|] d\tau \tag{4.5}$$

Thus the theorem is proved.

Corollary 1:

A sufficient condition for the system (2.2) to be $L^\infty - stable$ is that the following hold:

(i) There exist at least one N_j ($j = 0, \ldots, m$) (say N_k) having all its eigenvalues with negative real parts,

(ii) $\lim_{t\to\infty} \int_0^t [-\rho_k u_k(\tau) + \alpha_k \sum_{j\neq k} | u_j(\tau) | \| N_j \|] d\tau < \infty$

where $\alpha_k > 0$ and $\rho_k > 0$ are such that $\| e^{N_k t} \| \leq \alpha_k e^{-\rho_k t}$, $P_k(t) = \sum_{j\neq k} u_j N_j$ and $u \geq 0$.

Furthermore, if the limit is $-\infty$ then y(t) $\to 0$ as t $\to \infty$.

Theorem 2:

A sufficient condition for the system (2.2) to be $L^\infty - stable$ is that the following hold:

(i) There exist a positive constant α and a non-decreasing function ρ satisfying $\rho(0) = 0$ such that $\| \Phi(t,\tau) \| \leq \alpha e^{-[\rho(t)-\rho(\tau)]}$ for $t \geq \tau \geq 0$

(ii) $\lim_{t\to+\infty}[-\rho(t) + \alpha \| A \| t] \neq \infty$

Furthermore, if the limit is $-\infty$ then $y(t) \to 0$ as $t \to \infty$.

Proof:

Combining (3.2) and (3.4) will yield

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)Ax(\tau)d\tau \tag{4.6}$$

from which we obtain

$$\| x(t) \| \leq \| \Phi(t,0) \| \| x_0 \| + \int_0^t \| \Phi(t,\tau) \| \| A \| \| x(\tau) \| d\tau \tag{4.7}$$

thus,

$$\| x(t) \| \leq \| x_0 \| \alpha e^{-\rho(t)} + \int_0^t \| A \| \alpha e^{-[\rho(t)-\rho(\tau)]} \| x(\tau) \| d\tau \tag{4.8}$$

Therefore,

$$e^{\rho(t)} \| x(t) \| \leq \| x_0 \| \alpha + \int_0^t \| A \| \alpha e^{\rho(\tau)} \| x(\tau) \| d\tau \tag{4.9}$$

Using Gronwall's lemma, we obtain

$$e^{\rho(t)} \parallel x(t) \parallel \leq \alpha \parallel x_0 \parallel e^{\alpha \|A\| t} \tag{4.10}$$

Hence,

$$\parallel y(t) \parallel \leq \alpha \parallel C \parallel \parallel x_0 \parallel e^{-\rho(t)+\alpha \|A\| t} \tag{4.11}$$

Therefore, we have proved the theorem.

Corollary 2:

A sufficient condition for the system to be $L^{\infty} - stable$ is that u_j , $j = 1, \ldots, m$ satisfy the following inequality:

$min\{\rho_1 - \alpha_1 \parallel E_1 + \sum_{j=1}^{m} u_j N_j \parallel , -\rho_2 + \alpha_2 \parallel E_2 - \sum_{j=1}^{m} u_j N_j \parallel\} \geq \alpha_1 \alpha_2 \parallel A \parallel$

where E_1 and E_2 are stable matrices satisfying $\parallel e^{E_i t} \parallel \leq \alpha_i e^{-\rho_i t}$, $t \geq 0$ for some $\alpha_i, \rho_i > 0$.

Proof:

Let Γ be a solution of (3.1) then Γ^{-1} is a solution of the adjoint system

$$\frac{d}{dt}\Gamma^{-1} = (\sum_{j=1}^{m} u_j N_j)\Gamma^{-1} , \quad \Gamma^{-1}(0) = I \tag{4.12}$$

We have

$$\dot{\Gamma} = \Gamma E_1 - \Gamma[E_1 + \sum_{j=1}^{m} u_j N_j] \tag{4.13}$$

thus

$$\Gamma(t) = e^{E_1 t} - \int_0^t \Gamma(t_1)[E_1 + \sum_{j=1}^{m} u_j(t_1) N_j] e^{E_1(t-t_1)} dt_1 \tag{4.14}$$

Similarly, for Γ^{-1}, we obtain

$$\Gamma^{-1}(t) = e^{E_2 t} - \int_0^t e^{E_2(t-t_2)}[E_2 - \sum_{j=1}^{m} u_j(t_2)N_j]\Gamma^{-1}(t_2)dt_2 \tag{4.15}$$

hence,

$$\| \Gamma(t) \| \leq \alpha_1 e^{-\rho_1 t} + \int_0^t \| \Gamma(t_1) \| \| E_1 + \sum_{j=1}^{m} u_j(t_1)N_j \| \alpha_1 e^{-\rho_1(t-t_1)}dt_1 \tag{4.16}$$

and

$$\| \Gamma^{-1}(t) \| \leq \alpha_2 e^{-\rho_2 t} + \int_0^t \alpha_2 e^{-\rho_2(t-t_2)} \| E_2 - \sum_{j=1}^{m} u_j(t_2)N_j \| \| \Gamma^{-1}(t_2) \| dt_2 \tag{4.17}$$

Therefore,

$$e^{\rho_1 t} \| \Gamma(t) \| \leq \alpha_1 + \alpha_1 \int_0^t \| E_1 + \sum_{j=1}^{m} u_j(t_1)N_j \| e^{\rho_1 t_1} \| \Gamma(t_1) \| dt_1 \tag{4.18}$$

$$e^{\rho_2 t} \| \Gamma^{-1}(t) \| \leq \alpha_2 + \alpha_2 \int_0^t e^{\rho_2 t_2} \| \Gamma^{-1}(t_2) \| \| E_2 - \sum_{j=1}^{m} u_j(t_2)N_j \| dt_2 \tag{4.19}$$

Using Gronwall's lemma, we obtain

$$e^{\rho_1 t} \| \Gamma(t) \| \leq \alpha_1 e^{\alpha_1 \int_0^t \|E_1 + \sum_{j=1}^{m} u_j(t_1)N_j\| dt_1} \tag{4.20}$$

$$e^{\rho_2 t} \| \Gamma^{-1}(t) \| \leq \alpha_2 e^{\alpha_2 \int_0^t \|E_2 - \sum_{j=1}^{m} u_j(t_2)N_j\| dt_2} \tag{4.21}$$

which yield,

$$\begin{aligned} \| \Phi(t,\tau) \| &\leq \alpha_1\alpha_2 exp[-\rho_1 t + \alpha_1 \int_0^t \| E_1 + \sum_{j=1}^{m} u_j(t_1)N_j \| dt_1 \\ &- \rho_2\tau + \alpha_2 \int_0^\tau \| E_2 - \sum_{j=1}^{m} u_j(t_2)N_j \| dt_2] \end{aligned} \tag{4.22}$$

Thus,

$$\| \Phi(t,\tau) \| \leq \alpha e^{-[\rho(t)-\rho(\tau)]} \tag{4.23}$$

where $\alpha = \alpha_1\alpha_2$ and,

$$\rho(t) = \int_0^t max\ \{\rho_1 - \alpha_1 \parallel E_1 + \sum_{j=1}^{m} u_j(\tau)N_j \parallel,\ -\rho_2 + \alpha_2 \parallel E_2 - \sum_{j=1}^{m} u_j(\tau)N_j \parallel \}\ d\tau \tag{4.24}$$

Therefore, there exists a non-decreasing function ρ such that $\rho(0) = 0$ and satisfying the hypothesis of theorem 3.

This ends the proof.

Remark:

The sufficient condition in corollary 2 has a nice geometric interpretation in terms of the location of $\sum_{j=1}^{m} u_j N_j$ with respect to the balls centered at $-E_1$ and E_2 with radius $\frac{\rho_1}{\alpha_1} - \alpha_2 \parallel A \parallel$ and $\frac{\rho_2}{\alpha_2} + \alpha_1 \parallel A \parallel$ respectively.

5 Conclusion:

In this paper we have presented two input-output maps for multivariable bilinear systems. Sufficient conditions for the $\mathcal{L}^\infty$ -stability were derived. It is believed that, due to the nonlinear nature of the kernels involved, these representations will compare favorably with the standard Volterra series expansion, a point which will be investigated in future papers.

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