Monograph:
Banks, S.P. and Chanane, B. (1987) On Frequency Response for Nonlinear Systems. Research Report. Acse Report 323 . Dept of Automatic Control and System Engineering. University of Sheffield

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ON FREQUENCY RESPONSE FOR NONLINEAR SYSTEMS

by

S. P. Banks and B. Chanane

Department of Control Engineering
University Of Sheffield
Mappin Street
SHEFFIELD, S1 3JD

Research Report No. 323 December 1987

Abstract

In this paper we define a 'generalized frequency response' for a nonlinear input-output map S_{x_0} as the mapping $\mathcal{I} o \mathcal{F} o S_{x_0} o \mathcal{F}^{-1} o \mathcal{I}^{-1}$ from ℓ^2 to ℓ^2 where $\mathcal{F}$ is the Fourier transform and $\mathcal{I}$ the usual isomorphism from $L^2[-\infty, \infty]$ to ℓ^2. Realization results relative to linear and bilinear systems are presented. Also sufficient conditions for $\mathcal{L}^2$-stability of bilinear systems are derived.

equivalent.

The paper is organised as follows: In section 2 we shall consider sufficient conditions for a bilinear system to be $\mathcal{L}^2$-stable. In section 3 we define a 'GFR' for a nonlinear input-output map. We shall illustrate the 'GFR' for linear and bilinear systems. Finally a realization theory will be presented in section 4. We shall make use in the sequel of the following notation: The Fourier transform F of a square integrable function f is defined by

$$F(i\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t)\, e^{-i\omega t}\, dt$$

If x is an n-vector and A is an nxn-matrix, we take as compatible norms:

$$\| x \| = \max_{i=1,n} | x_i | \text{ and } \| A \| = \max_{i=1,n} \sum_{k=1}^{n} | a_{i,k} |$$

$\|\cdot\|_1$ and $\|\cdot\|_{2w}$ are the norms associated with the standard L^1 and L^2_w spaces, the space of absolutely integrable functions and the space of square integrable functions (with weight w) respectively. Whereas, ℓ^2 designates the space of square summable sequences.

2. $\mathcal{L}^2$-stability of bilinear systems:

In this section we shall present sufficient conditions for the $\mathcal{L}^2$-stability of bilinear systems. By $\mathcal{L}^2$-stability we understand that: $\forall u \in \Omega \subset L^2_w[0,\infty] \longrightarrow y \in L^2_w[0,\infty]$ where u and y are respectively the input and the output of the system and Ω is defined by the sufficient conditions. The weight function w is such that $w(t)e^{-2\alpha t} \in L^1[0,\infty]$ for some $\alpha \geq 0$, $w(t) \geq 0$ for $t \geq 0$.

Consider for simplicity the single input-single output system described by

$$\begin{cases} \dot{x} = Ax + uNx + Bu \quad , \quad x(0) = 0 \\ y = Cx \end{cases} \tag{2.1}$$

where A, N, B, C are constant matrices of suitable dimensions.

As is well known [3], the input-output map is given by:

$$y(t) = Ce^{At}x_0$$
$$+ \sum_{j\geq 1} \int_0^t \int_0^{\sigma_j} \cdots \int_0^{\sigma_2} v_j(t,\sigma_1,\cdots,\sigma_j)u(\sigma_1)\cdots u(\sigma_j)d\sigma_1\cdots d\sigma_j$$
$$+ \sum_{j\geq 1} \int_0^t \int_0^{\sigma_j} \cdots \int_0^{\sigma_2} w_j(t,\sigma_1,\cdots,\sigma_j)u(\sigma_1)\cdots u(\sigma_j)d\sigma_1\cdots d\sigma_j \tag{2.2}$$

where

$$v_1(t,\sigma_1) = Ce^{A(t-\sigma_1)}B$$
$$v_j(t,\sigma_1,\cdots,\sigma_j) = Ce^{A(t-\sigma_j)}Ne^{A(\sigma_j-\sigma_{j-1})}N\cdots Ne^{A(\sigma_2-\sigma_1)}B \quad , \quad j>1$$

and

$$w_1(t,\sigma_1) = Ce^{A(t-\sigma_1)}x_0$$
$$w_j(t,\sigma_1,\cdots,\sigma_j) = Ce^{A(t-\sigma_j)}Ne^{A(\sigma_j-\sigma_{j-1})}N\cdots Ne^{A(\sigma_2-\sigma_1)}Ne^{A\sigma_1}x_0$$
$$j>1.$$

Theorem 1 :

A sufficient condition for the system (2.1) to be $\mathcal{L}^2$-stable is that the following holds:

(i) the eigenvalues of A have real parts less than $-\alpha$,

(ii) u , $e^{\alpha t}u \in L^1[0, \infty]$.

Proof :

It follows from (i) as a standard result in the theory of differential equations that there exists a positive constant K such that

$$\|e^{At}\| \leq Ke^{-\alpha t} \quad , \quad t \geq 0 .$$

We have,

$$\int_0^\infty w(t)y^2(t)dt \le \int_0^\infty w(t)\cdot\Big[\|C\|\cdot\|x_0\|\cdot K\cdot e^{-\alpha t}$$

$$+\|C\|\cdot\|B\|\cdot K\cdot\int_0^t e^{-(t-\sigma_1)\alpha}\ |u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\ge 2}\int_0^t\int_0^{\sigma_j}\cdots\int_0^{\sigma_2}\|C\|\cdot\|B\|\cdot\|N\|^{j-1}e^{-(t-\sigma_1)\alpha}$$

$$\cdot K^j|u(\sigma_1)|\cdots|u(\sigma_j)|d\sigma_1\cdots d\sigma_j$$

$$+\|C\|\cdot\|x_0\|\cdot K\cdot\int_0^t e^{-(t-\sigma_1)\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\ge 2}\int_0^t\int_0^{\sigma_j}\cdots\int_0^{\sigma_2}\|C\|\cdot\|x_0\|\cdot\|N\|^j\cdot K^j\cdot e^{-\alpha t}$$

$$\cdot|u(\sigma_1)|\cdots|u(\sigma_j)|d\sigma_1\cdots d\sigma_j\Big]^2\ dt \qquad (2.3)$$

therefore,

$$\int_0^\infty w(t)y^2(t)dt \le \int_0^\infty w(t)e^{-2\alpha t}dt\cdot\Big[\|C\|\cdot\|x_0\|\cdot K$$

$$+\|C\|\cdot\|B\|\cdot K\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\ge 2}\|C\|\cdot\|B\|\cdot\|N\|^{j-1}\cdot K^j\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1\cdot\frac{\|u\|_1^{j-1}}{(j-1)!}$$

$$+\|C\|\cdot\|x_0\|\cdot K\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\ge 2}\|C\|\cdot\|x_0\|\cdot\|N\|^j\cdot K^j\cdot\frac{\|u\|_1^{j-2}}{j!}\Big]^2 \qquad (2.4)$$

hence,

$$\|y\|_{2w}^2 \le \int_0^\infty w(t)e^{-2\alpha t}dt\cdot\|C\|^2\cdot\Big[K\cdot\|x_0\| + K\cdot\|B\|\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+K\cdot\|B\|\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1\cdot(e^{K\cdot\|N\|\cdot\|u\|_1}-1)$$

$$+K\cdot\|x_0\|\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\|x_0\|\cdot(e^{K\cdot\|N\|\cdot\|u\|_1}-1-K\cdot\|N\|\cdot\|u\|_1\Big]^2 < \infty \qquad (2.5)$$

3.Generalized frequency response of Nonlinear input-output maps:

Consider a system S given in terms of an input-output map

$$S : R^n x L_w^2[0, \infty] \longrightarrow L_w^2[0, \infty] \qquad (3.1)$$

defined by $$y(t) = S(x_0 , u(\cdot))(t) \qquad (3.2)$$

where u and y are respectively the input and the output of the system and x_0 is the initial state in some given state-space realization. For each fixed initial state x_0, we have a map

$$S_{x_0} \triangleq S(x_0 , \cdot) : L_w^2[0, \infty] \longrightarrow L_w^2[0, \infty] \qquad (3.3)$$

For simplicity, we have assumed scalar input and scalar output.

In a recent paper [1], we introduced the notion of a 'Generalized Frequency Response' by using the natural isomorphism $\mathcal{J}$ between $L_w^2[0,\infty]$ and ℓ^2 in the time-domain, and then defining the 'G.F.R' as the induced map from ℓ^2 to ℓ^2 such that the diagram

$$\begin{array}{ccc} L_w^2[0, \infty] & \xrightarrow{S_{x_0}} & L_w^2[0, \infty] \\ \downarrow \mathcal{J} & & \downarrow \mathcal{J} \\ \ell^2 & \xrightarrow{s_{x_0}} & \ell^2 \end{array} \qquad (3.4)$$

commutes.

Alternatively, we can use the natural isomorphism between $L^2[-\infty,\infty]$ and ℓ^2 in the frequency domain (i.e., we operate on the Fourier transforms of the input and output) and then defining the 'G.F.R' as the induced map from ℓ^2 to ℓ^2 such that the diagram

$$\begin{array}{ccc} L^2[0,\infty] & \xrightarrow{S_{x_0}} & L^2[0,\infty] \\ \mathcal{F}\downarrow & & \mathcal{F}\downarrow \\ L^2[-\infty,\infty] & \xrightarrow{\hat{S}_{x_0}} & L^2[-\infty,\infty] \\ \mathcal{I}\downarrow & & \mathcal{I}\downarrow \\ \ell^2 & \xrightarrow{s_{x_0}} & \ell^2 \end{array} \tag{3.5}$$

commutes.

Let $\{e_j\}_{j\geq 0}$ be a basis of $L^2[0,\infty]$ and $E_j=\mathcal{F}\{e_j\}$, $j\geq 0$. Recalling the fact that the scalar product is invariant under the Fourier transform, we deduce that $\{E_j\}_{j\geq 0}$ is a basis of $L^2[-\infty,\infty]$. Let $\mathcal{I}$ denotes the usual isomorphism

$$\mathcal{I}: L^2[-\infty,\infty] \longrightarrow \ell^2$$

given by
$$\mathcal{I}(F) = \{F_j\}_{j\geq 0} \tag{3.6}$$

where
$$F \in L^2[-\infty,\infty] \;,\; F = \sum_{j\geq 0} F_j E_j$$

explicitly s_{x_0} is given by

$$s_{x_0} = \mathcal{I} o \mathcal{F} o S_{x_0} o \mathcal{F}^{-1} o \mathcal{I}^{-1} \tag{3.7}$$

that is
$$s_{x_0}(\{U_k\}_{k\geq 0}) = \{Y_j\}_{j\geq 0} \tag{3.8}$$

where $\quad U_k = \langle\, \mathcal{F}(u), E_k\rangle \;,\; Y_j = \langle\, \mathcal{F}(S_{x_0}(u)), E_j\rangle$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2[-\infty,\infty]$. We have

$$U_k = \langle u, e_k\rangle \text{ and } Y_j = \langle S_{x_0}(u), e_j\rangle$$

Let assume the systems at hand are $\mathcal{L}^2$-stable. We shall illustrate the expression (3.7) for the linear and the bilinear input-output maps.

<u>Example 1: Linear systems</u>

Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu \quad , \quad x(0) = x_0 \\ y = Cx \end{cases} \tag{3.9}$$

then the input-output map is given by

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \tag{3.10}$$

In this case

$$S(x_0,u(\cdot))(t) = g_0(t) + (g*u)(t) \tag{3.11}$$

where $g_0(t) = Ce^{At}x_0$, $g(t) = Ce^{At}B$

and $*$ denotes the convolution operator.

Taking the Fourier transform of both sides of (3.10) we obtain

$$Y(i\omega) = G_0(i\omega) + G(i\omega)\cdot U(i\omega) \tag{3.12}$$

Let $x_0 = 0$ and introduce a basis $\{E_j\}_{j\geq 0}$ of $L^2[-\infty,\infty]$.

We obtain, $$Y_l = \sum_{j\geq 0} G_{lj}\cdot U_j \quad , \ l\geq 0 \tag{3.13}$$

where $Y(i\omega) = \sum_{l\geq 0} Y_l\cdot E_l(i\omega)$, $U(i\omega) = \sum_{j\geq 0} U_j\cdot E_j(i\omega)$

$$G_{lj} = \langle\, G(i\omega)\cdot E_j(i\omega),\ E_l(i\omega)\,\rangle$$

We therefore see that the matrix representation of the linear operator $s : \ell^2 \longrightarrow \ell^2$ for the linear system above, with respect to the basis $(1,0,0,\cdots),(0,1,0,\cdots),\cdots$ of ℓ^2 is $(G_{lj})_{l,j\geq 0}$.

Example 2: Bilinear systems

Consider the bilinear system (2.1). Let $N = N_1\cdot N_2$ where the dimensions of N, N_1, N_2 are respectively nxn, nxm, mxn, and

$$\begin{aligned} g_0(t) &= Ce^{At}B \quad , & g_1(t) &= Ce^{At}N_1 \\ g_2(t) &= N_2e^{At}N_1 \quad , & g_3(t) &= N_2e^{At}B \\ g(t) &= Ce^{At}x_0 \quad , & g_4(t) &= N_2e^{At}x_0 \end{aligned} \tag{3.14}$$

Equation (2.2) can therefore be rewritten as

$$\begin{aligned} y(t) = g(t) &+ (g_0*u)(t) \\ &+ \sum_{j\geq 2} (g_1*[u(g_2*[u(\cdots u(g_2*[u(g_2*[u(g_3*u)])])\cdots)])])(t) \\ &+ (g*u)(t) \\ + \sum_{j\geq 2} & (g_1*[u(g_2*[u(\cdots u(g_2*[u(g_2*[u(g_2*(g_4\cdot u))])])\cdots)])])(t) \end{aligned} \tag{3.15}$$

where g_2 appears (j-2) times in the first summation and (j-1) times in the second.

Taking the Fourier transform of both sides of (3.15) and using its properties we obtain

$$Y(i\omega) = G(i\omega) + G_0(i\omega)\cdot U(i\omega) + \sum_{j\geq 2} \frac{1}{\sqrt{2\pi}^{\,j-1}} (G_1\cdot[U*(G_2\cdot[U*(\cdots$$
$$\cdots U*(G_2\cdot[U*(G_2\cdot[U*(G_3\ U)])])\cdots)])])(i\omega) + G(i\omega)\cdot U(i\omega)$$
$$+ \sum_{j\geq 2} \frac{1}{\sqrt{2\pi}^{\,j}} (G_1\cdot[U*(G_2\cdot[U*(\cdots$$
$$\cdots (G_2\cdot[U*(G_2\cdot[U*(G_2\cdot(G_4*U))])])\cdots)])])(i\omega) \qquad (3.16)$$

Let $x_0 = 0$ and introduce a basis $\{E_j\}_{j\geq 0}$ of $L^2[-\infty,\infty]$.

If $U(i\omega) = \sum_{j\geq 0} U_j E_j(i\omega)$ then (3.16) becomes

$$Y(i\omega) = \sum_{k_1\geq 0} U_{k_1} G_0(i\omega) E_{k_1}(i\omega)$$
$$+ \sum_{j\geq 2} \sum_{k_1\geq 0} \cdots \sum_{k_j\geq 0} U_{k_1}\cdots U_{k_j}\cdot \frac{1}{\sqrt{2\pi}^{\,j-1}} (G_1\cdot[E_{k_j}*(G_2\cdot[E_{k_{j-1}}$$
$$*(\cdots E_{k_4}*(G_2\cdot[E_{k_3}*(G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})])])\cdots)])])(i\omega) \qquad (3.17)$$

Therefore,

$$Y_1 = \sum_{j\geq 1} \sum_{k_1\geq 0} \cdots \sum_{k_j\geq 0} V_{jk_1\cdots k_j 1}\cdot U_{k_1}\cdots U_{k_j} \qquad (3.18)$$

where $$V_{1k_1 1} = \langle G_0(i\omega)E_{k_1}(i\omega)\ ,\ E_1(i\omega)\rangle \qquad (3.19)$$

and

$$V_{jk_1\cdots k_j 1} = \frac{1}{\sqrt{2\pi}^{\,j-1}} \cdot\langle (G_1\cdot[E_{k_j}*(G_2\cdot[E_{k_{j-1}}*(\cdots$$
$$\cdots E_{k_4}*(G_2\cdot[E_{k_3}*(G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})])])\cdots$$
$$\cdots)])])(i\omega), E_1(i\omega)\rangle\ , \quad \text{for} \quad j>1 \qquad (3.20)$$

Hence, the diagram (3.5) induces the map $s_{x_0} : \ell^2 \longrightarrow \ell^2$ given by

$$s_{x_0}((U_0,U_1,\cdots))_1 = \sum_{j\geq 1} \sum_{k_1\geq 0} \cdots \sum_{k_j\geq 0} V_{jk_1\cdots k_j 1}\cdot U_{k_1}\cdots U_{k_j} \qquad (3.21)$$

4. Realization theory:

In this section we shall consider the problem of the realizability and the state space realization of an analytic map $s : \ell^2 \longrightarrow \ell^2$ which defines a ' Generalized Frequency Response'. We shall present conditions under which such s is realized by a linear or a bilinear system.

4.A: Linear system :

Theorem 2 :

A necessary and sufficient condition for a sequence of numbers $\{G_{lj}\}_{l,j\geq 0}$ to be the 'Generalized Frequency Response ' of a linear system with zero initial condition (with respect to a given basis $\{E_k\}_{k\geq 0}$ of $L^2[-\infty,\infty]$) is that there exists a strictly proper rational function $G(i\omega)$ such that

$$\sum_{l\geq 0} G_{lj}\cdot E_l(i\omega) = G(i\omega)\cdot E_j(i\omega) \qquad (4.1)$$

for all $j\geq 0$. $G(i\omega)$ is then the Fourier transform of the impulse response of the linear system.

Proof :

immediate and shall be ommited.

4.B Bilinear systems :

Let e_k , $k\geq 0$ be the Laguerre functions defined by

$$e_k(t) = e^{-t/2} \sum_{m=0}^{k} \frac{(-1)^m}{m!} \binom{k}{m} t^m$$

They constitute a complete orthonormal basis for $L^2[0,\infty]$.

Consider E_k the Fourier transform of e_k , we have

$$E_k(i\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\frac{1}{2} + i\omega} \left[\frac{-\frac{1}{2} + i\omega}{\frac{1}{2} + i\omega}\right]^k$$

$\{E_k\}_{k\geq 0}$ is therefore a complete basis for $L^2[-\infty,\infty]$.

Remark 1 : The coefficients in the expansion of $F(i\omega)$ with respect to the basis $\{E_k\}_{k\geq 0}$ are the coefficients in the Taylor expansion of $\overline{F}$

$$\overline{F}(z) = \sqrt{2\pi}\, \frac{1}{1-z}\, F\left(\frac{1}{2}\, \frac{1+z}{1-z}\right)$$

at z=0. So $\quad F_k = \frac{\sqrt{2\pi}}{k!}\left[\frac{1}{1-z}\,F\left(\frac{1}{2}\,\frac{1+z}{1-z}\right)\right]^{(k)}_{z=0}$

where $F(i\omega) = \sum_{k\geq 0} F_k\cdot E_k(i\omega)$.

Theorem 3 :

A necessary and sufficient condition for a sequence of numbers $\{V_{jk_1\cdots k_j l}\}_{j\geq 1,k_1,\cdots,k_j,l\geq 0}$ to be the ' Generalized Frequency Response ' of a bilinear system with zero initial condition (with respect to the given basis $\{E_k\}_{k\geq 0}$ of $L^2[-\infty,\infty]$) is that there exist four matrices $G_0(i\omega)$, $G_1(i\omega)$, $G_2(i\omega)$, $G_3(i\omega)$ with dimensions respectively 1x1, 1xm, mxm, mx1 of strictly proper rational functions such that

$$\text{(i)}\quad \sum_{l\geq 0} V_{1k_1 l}\, E_l(i\omega) = G_0(i\omega)\cdot E_{k_1}(i\omega) \tag{4.2}$$

$$\text{(ii)}\quad \sum_{l\geq 0} V_{jk_1\cdots k_j l}E_l(i\omega) = \frac{1}{k_j!\cdots k_2!}\; D_j^{k_j}\cdots D_2^{k_2}\cdot$$
$$\cdot[G_1(i\omega)\cdot G_2(i\omega-z_j)\cdot G_2(i\omega-z_j-z_{j-1})\cdots$$
$$\cdots G_2(i\omega-z_j-\cdots-z_3)\cdot G_3(i\omega-z_j-\cdots-z_2)\cdot$$
$$\cdot E_{k_1}(i\omega-z_j-\cdots-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}\cdots(-\tfrac{1}{2}+z_j)^{k_j}]\Big|_{z_2=\cdots=z_j=-\frac{1}{2}}$$
$$\text{for all } k_1,\cdots,k_j\geq 0\ ,\ j>1 \tag{4.3}$$

where $D_j^{k_j} = \partial^{k_j}/\partial z_j^{k_j}$.

Proof :

(i) (4.2) can readily be obtained from (3.9) by premultiplying by $E_l(i\omega)$ and summing over $l\geq 0$. Whereas (3.14) yields $G_0(i\omega) = C(i\omega-A)^{-1}B$.

(ii) From (3.20) we obtain

$$\sum_{l\geq 0} V_{jk_1\cdots k_j l}E_l(i\omega) = \frac{1}{\sqrt{2\pi}^{\,j-1}}\,(G_1\cdot[E_{k_j}*(G_2\cdot[E_{k_{j-1}}*(\cdots$$
$$\cdots E_{k_4}*(G_2\cdot[E_{k_3}*(G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})])])\cdots)])])(i\omega)$$

Starting from the inner square bracket and proceeding outward, we obtain, after a change of variable $i\Omega = z$ and the use of residue theory,

$$-\{E_{k_2}*(G_3\cdot E_{k_1})\}(i\omega) = \frac{\sqrt{2\pi}}{k_2!}D_2^{k_2}[G_3(i\omega-z_2)\cdot E_{k_1}(i\omega-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}]\big|_{z_2=-\frac{1}{2}}$$

$$-\{E_{k_3}*\{G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})]\}\}(i\omega) = \int_{-\infty}^{\infty}E_{k_3}(i\Omega)G_2(i\omega-i\Omega)\frac{\sqrt{2\pi}}{k_2!}\cdot$$

$$\cdot D_2^{k_2}\left[G_3(i\omega-i\Omega-z_2)\cdot E_{k_1}(i\omega-i\Omega-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}\right]_{z_2=-\frac{1}{2}}d\Omega$$

$$= \frac{\sqrt{2\pi}}{k_2!}(-i)\int_{-i\infty}^{i\infty}E_{k_3}(z_3)G_2(i\omega-z_3)\cdot$$

$$D_2^{k_2}\left[G_3(i\omega-z_3-z_2)\cdot E_{k_1}(i\omega-z_3-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}\right]_{z_2=-\frac{1}{2}}dz_3$$

$$= \frac{(\sqrt{2\pi})^2}{k_3!k_2!}D_3^{k_3}\left[G_2(i\omega-z_3)\cdot\right.$$

$$\cdot D_2^{k_2}\left[G_3(i\omega-z_3-z_2)\cdot E_{k_1}(i\omega-z_3-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}\right]_{z_2=-\frac{1}{2}}\cdot$$

$$\left.\cdot(-\tfrac{1}{2}+z_3)^{k_3}\right]_{z_3=-\frac{1}{2}}$$

$$-\{E_{k_j}*(G_2\cdot[E_{k_{j-1}}*(\cdots E_{k_3}*G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})]\cdots)])\}(i\omega) =$$

$$\frac{\sqrt{2\pi}^{\,j-1}}{k_j!\cdots k_2!}D_j^{k_j}\left[G_2(i\omega-z_j)\cdot D_{j-1}^{k_{j-1}}\left[G_2(i\omega-z_j-z_{j-1})\cdots\right.\right.$$

$$\cdots D_3^{k_3}\left[G_2(i\omega-z_j-\cdots-z_3)\cdot D_2^{k_2}\left[G_3(i\omega-z_j-\cdots-z_2)\cdot\right.\right.$$

$$\left.\cdot E_{k_1}(i\omega-z_j-\cdots-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}\right](-\tfrac{1}{2}+z_3)^{k_3}\Big]$$

$$\left.\left.\cdots(-\tfrac{1}{2}+z_{j-1})^{k_{j-1}}\right]\cdot(-\tfrac{1}{2}+z_j)^{k_j}\right]_{z_1=\cdots=z_j=-\frac{1}{2}}$$

Hence, (4.3).

From (3.14) we get

$$G_0(i\omega) = C(i\omega I-A)^{-1}B \quad , \quad G_1(i\omega) = C(i\omega I-A)^{-1}N_1$$
$$G_2(i\omega) = N_2(i\omega I-A)^{-1}N_1 \quad , \quad G_3(i\omega) = N_2(i\omega I-A)^{-1}B \qquad (4.6)$$