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Existence of Periodic Solutions in n-Dimensional  
Retarded Functional Differential Equations

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### Abstract

The existence of periodic orbits of n-dimensional delay systems of the form  $\dot{x}(t) = -f(x(t-p))$  is proved and applied to systems of the form

$$\dot{x}(t) = -x(t-1)N(x(t)) ,$$

and a certain type of Hamiltonian system .

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## 1. Introduction

Scalar delay equations of the form

$$\dot{x}(t) = -f(x(t-1)) \quad (1.1)$$

have been studied by many authors ; see , for example , Jones (1962) , Nussbaum (1974) , Kaplan and Yorke (1974) . These equations have important applications in population dynamics and nonlinear equations of the form

$$\dot{x}(t) = -x(t-1)N(x(t)) \quad (1.2)$$

can be transformed into equations of type (1.1) , under certain mild conditions. Moreover , it is well-known that these equations are related to nonlinear Hamiltonian systems of the form

$$\dot{x}(t) = -f(y(t))$$

$$\dot{y}(t) = f(x(t))$$

(Kaplan and Yorke , (1974) , Nussbaum and Peitgen (1984)) .

In this paper we shall generalize some of the above results to the multi-dimensional case of systems

$$\dot{x}(t) = -f(x(t-p)) \quad (1.3)$$

where  $x(t) \in \mathbb{R}^n$  for each  $t$  and  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuously differentiable map (with further properties to be introduced later) . We shall generalize the transformation of a system of the form (1.2) to that of (1.3) by solving the nonlinear overdetermined system of partial differential equations

$$\nabla f = N(f) ,$$

where now  $N: \mathbb{R}^n \rightarrow \mathbb{R}^n$  , and  $\nabla f = (\partial f_i / \partial x_j)$  .

## 2. Fixed Point Theory

In this section we shall give a brief introduction to the fixed point index; for more details see Browder (1960) , Nussbaum (1974) , Dold (1965) and Thompson (1969) . If  $X$  is a compact , metric ANR (absolute neighbourhood retract - a space which is such that for any normal space  $Y$  and any map  $f:A \rightarrow X$  , where  $A$  is a closed subset of  $Y$  ,  $f$  can be extended to a neighbourhood of  $A$  in  $Y$ ) and  $f:X \rightarrow X$  is a continuous map , then we define

$$i_X(f, X) = \sum_{i \geq 0} (-1)^i \text{tr}(f_{*,i})$$

where  $f_{*,i}: H_i(X) \rightarrow H_i(X)$  is the induced homology morphism (using Čech homology).

$i_X(f, X)$  is also denoted by  $\Lambda(f)$  and is called the *Lefschetz number* of  $X$  . It is

a classical result of algebraic topology that  $f$  has a fixed point if and only if

$i_X(f, X) \neq 0$  (See Spanier , 1966) . This invariant can be generalized to the case

of a continuous map  $f:G \rightarrow X$  where  $G$  is an open subset of  $X$  for which the set

$S = \{x \in G : f(x) = x\}$  is compact or empty , and leads to an integer-valued

function  $i_X(f, G)$  called the *fixed point index* of  $f$  in  $G$  which satisfies the

properties :

(1) If  $i_X(f, G) \neq 0$  then  $f$  has a fixed point .

(2) If  $S \subseteq G_1 \cup G_2$  where  $G_1$  and  $G_2$  are disjoint open subsets of  $G$  , then

$$i_X(f, G) = i_X(f, G_1) + i_X(f, G_2) .$$

(3) If  $F:G \times [0, 1] \rightarrow X$  is a homotopy such that  $S' = \{(x, t) \in G \times [0, 1] : F(x, t) = x\}$

is compact , then

$$i_X(F_1, G) = i_X(F_0, G) ,$$

where  $F_t(x) = F(x, t)$  .

(4) If  $f: X \rightarrow X$  is continuous, then

$$i_X(f, X) = \Lambda(f).$$

Now let  $X$  be a general topological space,  $x_0 \in X$ , and  $W$  an open neighbourhood of  $x_0$ . Then if  $f: W - \{x_0\} \rightarrow X$  is a continuous map, we say that  $x_0$  is an *ejective point* of  $f$  if there exists an open neighbourhood  $U$  of  $x_0$  such that there exists an integer  $m > 0$  such that  $f^m(x)$  is defined and  $f^m(x) \notin U$  for all  $x \in U - \{x_0\}$ .

### 3. An Abstract Form for the Delay Equation

In order to apply fixed point theory to obtain periodic solutions of the delay equation

$$\dot{x}(t) = -f(x(t-p)) \quad , \quad x(t-p) = \phi(t) \quad , \quad t \in [0, p] \quad (3.1)$$

where  $\phi \in C([0, p]; \mathbb{R}^n)$ , we must express the equation in an abstract form on some space. Nussbaum (1974) uses a space of continuous functions and expresses the solution of (3.1) in terms of the initial function  $\phi$ . We shall use a different approach here which allows a direct consideration of  $n$ -dimensional equations. First note the following simple lemma.

**Lemma 3.1** A necessary condition for  $x(t)$  to be a periodic solution of the equation (3.1) (with period 1) is that

$$\int_{t-1}^t f(x(t-p)) dt = 0 \quad (3.2)$$

**Proof** This follows directly by integrating (3.1), namely

$$x(t) - x(t-1) = - \int_{t-1}^t f(x(t-p)) dt = 0$$

if  $x$  is periodic.  $\square$

Since any solution of (3.1) is differentiable for  $t > 0$ , any periodic solution  $x(t)$  must satisfy

$$x(t) = x(t-1) \quad , \quad \dot{x}(t) = \dot{x}(t-1) \quad , \quad \text{for } t > 1$$

and so any such solution determines an element of the space  $C^1[S^1; \mathbb{R}^n]$ , where  $S^1$  is the unit circle, and conversely. The 'time' axis  $[0, 1]$  is mapped onto  $S^1$  by the map  $t \rightarrow e^{2\pi i t}$ . We shall parameterize  $S^1$  by  $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$ . Let  $X$  denote the Banach space  $C^1[S^1; \mathbb{R}^n] \times C[S^1; \mathbb{R}^n]$ . We now define the 'rotation' operator  $R_p: S^1 \rightarrow S^1$  by

$$R_p(\theta) = (\theta - 2\pi p) \bmod 2\pi \quad , \quad 0 \leq \theta < 2\pi \quad . \quad (3.3)$$

Denote by  $K$  the subspace  $\{(x, 2\pi \dot{x}) : x \in C^1[S^1; \mathbb{R}^n], x(0) = 0\}$  of  $X$  and define the operator  $F$  on  $X$  by

$$F(x, y) = (w, z) \quad (3.4)$$

where

$$z(\theta) = -f(x(R_p(\theta))) \quad , \quad (3.5)$$

and

$$w(\theta) = \frac{1}{2\pi} \int_0^\theta z(\theta_1) d\theta_1 \quad (3.6)$$

(Note that in the definition of  $K$ ,  $\dot{x}$  means  $dx/d\theta$ .)

More generally, consider the equation

$$\dot{x}(t) = -f(x(t-p), x(t-q)) \quad (3.7)$$

with two delays  $p, q$ .  $F$  is defined again by (3.4) where  $z$  in (3.5) is replaced by

$$z(\theta) = -f(x(R_p(\theta)), x(R_q(\theta))) \quad . \quad (3.8)$$

It is even possible to consider neutral equations of the form

$$\dot{x}(t) = -f(x(t-p), x(t+q))$$

in a similar way with (3.5) replaced by

$$z(\theta) = -f(x(R_p(\theta)), x(R_{-q}(\theta))) .$$

#### 4. Existence of Periodic Solutions

In order to prove the existence of a periodic solution of the equation (3.1) (under suitable conditions on  $f$ , to be introduced shortly) we shall use the following theorem, due to Nussbaum (1974).

**Theorem 4.1** Let  $G$  be a closed, bounded, convex infinite-dimensional subset of a Banach space  $X$ ,  $x_0 \in G$  and  $F: G \rightarrow G$  a continuous compact map. Then if  $x_0$  is an ejective point of  $F$  and  $U$  is an open neighbourhood of  $x_0$  such that  $F(x) \neq x$  for  $x \in \bar{U} \setminus \{x_0\}$ , then  $i_G(f, G - \bar{U}) = 1$  and  $f$  has a fixed point in  $G - \bar{U}$ .  $\square$

The first task is to find a closed, bounded, convex (infinite-dimensional) set  $G$  such that  $F: G \rightarrow G$ . For this we shall make the following assumption on  $f$ :

AF1 : There exists a number  $A > 0$  such that

$$\|x\| \leq A \Rightarrow \|f(x)\| \leq A .$$

Then we have

**Lemma 4.2** If  $f$  satisfies AF1 then  $F$  maps the subset

$$G = \{(x, 2\pi\dot{x}) : \|(x, 2\pi\dot{x})\| \leq 2A\}$$

into itself.

**Proof** Since

$$x = \int_0^\theta \dot{x} \, d\theta$$

for  $(x, 2\pi\dot{x}) \in G$  we have

$$\|x\| \leq 2\pi \|\dot{x}\| .$$

Hence

$$2\|x\| \leq 2\pi \|\dot{x}\| + \|x\| = \|(x, 2\pi\dot{x})\| \leq 2A ,$$

and so

$$\|f(x)\| \leq A ,$$

by AF1 . Now ,

$$F(\langle x, 2\pi\dot{x} \rangle)(\theta) = \left( -\frac{1}{2\pi} \int_0^\theta f(x(R_P(\theta))), -f(x(R_P(\theta))) \right)$$

and since  $R_P$  is just a rotation it is clear that

$$\|F(\langle x, 2\pi\dot{x} \rangle)\| \leq 2\|f(x(R_P(\cdot)))\|$$

$$\leq 2A . \quad \square$$

Next we show that  $F:G \rightarrow G$  is a continuous compact map under the assumption

AF2 :  $f$  is continuously differentiable and

$$\|x\| \leq A \Rightarrow \|\nabla f(x)\| \leq B$$

for some  $B > 0$  .

**Lemma 4.3** If  $f$  satisfies AF1 and AF2 , then  $F:G \rightarrow G$  is a continuous compact map.

**Proof** If  $C \subseteq G$  is a bounded set then , for  $(x, 2\pi\dot{x}) \in C$  we have

$$\frac{d}{d\theta} F(x, 2\pi\dot{x}) = \left( -\frac{1}{2\pi} f(x(R_P(\theta))) , -f(x(R_P(\theta)))\dot{x} \frac{dR_P}{d\theta} \right)$$

and so by AF2 the set  $(d/d\theta)C = \{d\zeta/d\theta : \zeta \in C\}$  is bounded . Hence the lemma follows from the Arzela-Ascoli theorem .  $\square$

It remains , therefore , to show that 0 is an ejective point of  $F$  . To do this we restrict attention to the subspace  $G'$  of  $G$  consisting of elements  $(x, 2\pi\dot{x})$  such that  $x$  satisfies the condition

$$C: \quad x(\pi) = 0 , \quad x|_{(0, \pi)} > 0 , \quad x|_{(\pi, 2\pi)} < 0$$

$$x|_{[0, \pi]} \text{ is symmetric about } t=\pi/2$$

$$x|_{[\pi, 2\pi]} \text{ is symmetric about } t=3\pi/2$$

$$x \text{ is monotonically increasing on } [0, \pi/2] \cup [3\pi/2, 2\pi] \\ \text{and monotonically decreasing on } [\pi/2, 3\pi/2] .$$

Furthermore , suppose that  $f$  satisfies the condition

$$\text{AF3 : } \begin{cases} f(x_1, \dots, x_n) > 0 & \text{if } x_1 > 0, \dots, x_n > 0 \\ f(x_1, \dots, x_n) < 0 & \text{if } x_1 < 0, \dots, x_n < 0 \end{cases} ,$$

Then it is easy to check that the restriction  $F'$  of  $F$  to  $G'$  maps  $G'$  into  $G'$ , provided

$$p = 1/4 . \quad (4.1)$$

In order to show that 0 is an ejective point of  $G'$  we require a final condition on  $f$ , namely,

AF4 : For each  $i$ , we have

$$\frac{\partial f}{\partial x_1}(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) > 2\pi$$

for any  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$

such that  $x_j > 0$  for all  $j$  or  $x_j < 0$  for all  $j$ .

Then we have

**Theorem 4.4** Under the conditions AF1-AF4, equation (3.1) has a periodic solution (of period 1) if  $p$  satisfies (4.1).

**Proof**  $G'$  is closed, bounded, convex and infinite-dimensional and so, by theorem 4.1 it is sufficient to show that 0 is an ejective point of  $F$ . If  $0 \neq (x, 2\pi\dot{x}) \in G'$  then, since  $x$  is monotonically increasing on  $[0, \pi/2] \cup [3\pi/2, 2\pi]$  there exists  $i \in \{1, \dots, n\}$  and  $\alpha > 0$  such that  $|x_i(\theta)| \geq |\dot{v}(\theta)|$ ,  $t \in [0, 2\pi]$ , where  $v = \alpha \sin \theta$ . Consider the operation of  $F$  on  $(v, 2\pi\dot{v})$ . Since  $p=1/4$ ,  $F$  first rotates  $v$  counterclockwise by  $\pi/2$ , then applies  $-f$  and finally integrates the resulting function (and multiplies by  $1/2\pi$ ). If  $\alpha$  is sufficiently small, then  $|f(x_1, \dots, x_{i-1}, \alpha v, x_{i+1}, \dots, x_n)| > 2\pi\alpha|\dot{v}|$  by AF4 and AF3 now shows that  $F(v, 2\pi\dot{v}) = (\beta v, 2\pi\beta\dot{v})$  where  $\beta > 1$ . Hence 0 is an ejective point in  $G'$  and the result follows.  $\square$

**Remark 4.1** In Nussbaum (1974) and Kaplan and Yorke (1974), where scalar equations are considered with a delay of  $p=1$ , the condition AF4 is replaced by

$$\frac{\partial f(x)}{\partial x} > \pi/2 .$$

The difference in the growth constants is entirely due to the difference in the delays . If theorem 4.4 is reworked for a delay of 1 then in condition AF4 ,  $2\pi$  is replaced by  $\pi/2$  .

**Remark 4.2** The above proof does not require a delicate consideration of the linearized eigenvalue equation of the delay system used in Nussbaum (1974) .

### 5. Example

Several authors (see Nussbaum , 1974) have considered the scalar equation

$$\dot{x}(t) = -\alpha x(t-1)N(x(t))$$

for some nonlinear function  $N$  . Here we shall generalize this to an  $n$ -dimensional equation

$$\dot{x}(t) = -\alpha N(x(t))x(t-p) \tag{5.1}$$

where  $N$  is a matrix-valued nonlinear function with properties to be specified later . In order to transform such an equation into the form (3.1) we must consider the following overdetermined system of nonlinear partial differential equations :

$$\nabla f(y) = N(f(y)) , f(0)=0 \tag{5.2}$$

for the function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  , where

$$\nabla f = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}$$

is the gradient matrix of  $f$  . We have the following necessary compatibility conditions for the solubility of (5.2) .

**Lemma 5.1** If the equations (5.2) have a solution  $f \in C^2$  , then

$$\sum_p \frac{\partial N_{ij}}{\partial z_p}(z) N_{pk}(z) = \sum_p \frac{\partial N_{jk}}{\partial z_p}(z) N_{pi}(z) \tag{5.3}$$

for any point  $z \in \text{range}(f)$  and for all  $i, j, k$  .

**Proof** Clearly we have

$$\frac{\partial^2 f_1}{\partial y_k \partial y_j} = \frac{\partial^2 f_1}{\partial y_j \partial y_k}$$

and so, from (5.2),

$$\sum_p \frac{\partial N_{ij}(f(y))}{\partial z_p} \frac{\partial f_p(y)}{\partial y_k} = \sum_p \frac{\partial N_{ik}(f(y))}{\partial z_p} \frac{\partial f_p(y)}{\partial y_j} \quad (5.4)$$

However, again by (5.2) we have

$$\frac{\partial f_p(y)}{\partial y_k} = N_{pk}(f(y)) \quad \square$$

In order to find sufficient conditions for the existence of a (nontrivial) solution of (5.2), suppose that  $f(y)$  is a solution of this equation which satisfies (5.4). Then, if  $\eta: I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  is a path in  $\mathbb{R}^n$ , we have

$$\dot{f}(\eta(t)) = \nabla f \dot{\eta}(t)$$

and so

$$f(y) = \int_0^y N(f(\eta)) d\eta$$

where the integral is taken along the path  $\eta$ . However, by (5.4), the integral is independent of path and so  $f$  must satisfy the integral equation

$$f(y) = \int_0^y N(f(y)) dy \quad (5.5)$$

for any path joining 0 to  $y$ .

The following theorem is proved in a similar way to the proof of theorem 4.4.

**Theorem 5.2** Suppose first that  $N$  is such that the set  $G$  of functions  $f \in C^2$  which satisfy  $f(0) = 0$ , AF3 and (5.4) is nonempty, closed, bounded, convex and

infinite-dimensional . Next assume that  $N(z)$  is strictly positive in some neighbourhood of 0 . Then (5.5) has a solution for small enough  $\|y\|$  .  $\square$

(We consider the operator  $\mathcal{N}:f(y) \rightarrow \int_0^y N(f(y))dy$  on  $G$  , which is well-defined by (5.4) and use the stated properties to show that  $\mathcal{N}$  is compact and that the zero function is an ejective point of  $\mathcal{N}$ .)

**Corollary 5.3** Under the conditions of theorem 5.2 , if , in addition ,  $N$  is an analytic function , then  $f$  is analytic and has an analytic extension to some maximal region in  $\mathbb{R}^n$  .  $\square$

If the conditions of corollary 5.3 hold , let  $f$  be a solution of (5.2) and define

$$x = f(y) .$$

Then

$$\begin{aligned} \dot{x}(t) &= \nabla f(y) \cdot \dot{y}(t) \\ &= -\alpha N(x(t))x(t-p) \\ &= -\alpha N(f(y(t)))f(y(t-p)) \end{aligned}$$

if  $x(t)$  satisfies equation (5.1) . Hence , if  $y(t) \in U$  for all  $t$  we have

$$\dot{y}(t) = -\alpha f(y(t-p)) \tag{5.6}$$

and a periodic solution of (5.6) satisfying  $y(t) \in U$  is also a periodic solution of (5.1) .

Consider , for example , the system

$$\dot{x}(t) = -\alpha \begin{pmatrix} 1 + x_1(t) & x_2(t) \\ x_2(t) & 1 + x_1(t) \end{pmatrix} x(t-p) . \tag{5.7}$$

Then condition (5.3) is satisfied by

$$N(z) = \begin{pmatrix} 1 + z_1 & z_2 \\ z_2 & 1 + z_1 \end{pmatrix}$$

and the conditions in (5.4) become

$$\frac{\partial f_1}{\partial y_2} = \frac{\partial f_2}{\partial y_1} \quad (5.8)$$

$$\frac{\partial f_1}{\partial y_1} = \frac{\partial f_2}{\partial y_2} \quad (5.9)$$

The equations (5.2) reduce to

$$\frac{\partial f_1(y_1, y_2)}{\partial y_1} = 1 + f_1(y_1, y_2) \quad (5.10)$$

$$\frac{\partial f_2(y_1, y_2)}{\partial y_1} = f_2(y_1, y_2)$$

with  $f(0,0)=0$  . Solving these equations gives

$$f_1 = -1 + e^{y_1} \cosh(y_2) \quad (5.11)$$

$$f_2 = e^{y_1} \sinh(y_2) . \quad (5.12)$$

Note that  $f_1$  does not satisfy AF3 everywhere , but it does so on the region

$$|y_1| > |y_2| .$$

An easy extension of theorem 4.4 shows that the result remains valid in this case . Condition AF4 now shows that if  $\alpha > 2\pi$  then equation (5.7) has a periodic solution .

As a second application we recall that Kaplan and Yorke (1974) used the two-dimensional Hamiltonian system

$$\dot{x} = -f(y) \quad (5.13)$$

$$\dot{y} = f(x)$$

to prove the existence of periodic orbits in a scalar delay system of the form (3.1) . To do this they use a Lyapunov argument to prove the existence of a

family of periodic solutions of (5.13) . This argument no longer works for  $2n$ -dimensional systems of the form (5.13) . However , we can use theorem 4.4 to reverse the argument and show the existence of periodic solutions of (5.13) from the existence of periodic solutions of (3.1) . In fact , if  $f$  satisfies the conditions of theorem 4.4 and moreover ,  $f$  is odd , i.e.

$$f(-x) = -f(x) \quad , \quad (5.14)$$

then we put

$$y(t) = x(t-p)$$

in (3.1) and obtain

$$\dot{x}(t) = -f(y(t))$$

$$\dot{y}(t) = -f(x(t-2p)) .$$

With  $p=1/4$  , using (5.14) and the fact that

$$x(t-2p) = -x(t) \quad ,$$

the result follows .

## 6. Conclusions

In this paper we have generalized the existence theory of periodic solutions of scalar FDE's to the vector case . This has been achieved by the use of the fixed point index . Using a transformation which depends on the solution of the system

$$\nabla f = N(f)$$

we have given a simple example which generalizes the well-known one-dimensional case .

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