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Realization

and

Generalized Frequency Response

for

Nonlinear Input-Output Maps

by

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Abstract

The purpose of this paper is to develop a realization theory and present the 'generalized frequency response' for an input-output map introduced in a previous paper. Relations with the realization of finite Volterra series will be outlined. A simple example will be given to show that the new representations have improved convergence properties.

where

$$\begin{aligned} ad_X^0\, Y &= Y \\ ad_X\, Y = ad_X^1\, Y &= [X,Y] = XY - YX \qquad (\textit{Lie bracket}) \\ ad_X^{m+1}\, Y &= [X, ad_X^m\, Y] \quad , \quad m \geq 1 \end{aligned} \tag{1.2}$$

2 Nonlinear input-output maps for bilinear systems :

Consider the bilinear system

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{m} u_j N_j x, \qquad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{2.1}$$

where A, N_j, C are constant matrices of suitable dimensions. We shall present in this section a new input-output map to represent this system.

Let $u_0(t) = 1, \;\; t \geq 0$ and $N_0 = A$, then (2.1) becomes

$$\begin{aligned} \dot{x} &= \sum_{j=0}^{m} u_j N_j x, \qquad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{2.2}$$

Consider the change of variable

$$z = e^{-N_k \int_0^t u_k(\tau)d\tau} x \tag{2.3}$$

we have $z(0) = z_0 = x_0$.

Let $U_k(t) = \int_0^t u_k(\tau)d\tau$. It is easy to prove [4] that the nonlinear input-output map is given by

$$
\begin{aligned}
y(t) &= Ce^{N_k U_k(t)}x_0 + \\
&+ \sum_{l\geq 1}\sum_{j_1\neq k}\cdots\sum_{j_l\neq k}\int_0^t\int_0^{\tau_l}\cdots\int_0^{\tau_2} \\
&\quad Ce^{N_k[U_k(t)-U_k(\tau_l)]}N_{j_l}e^{N_k[U_k(\tau_l)-U_k(\tau_{l-1})]}N_{j_{l-1}} \\
&\quad \ldots N_{j_2}e^{N_k[U_k(\tau_2)-U_k(\tau_1)]}N_{j_1}e^{N_k U_k(\tau_1)}x_0 \\
&\quad u_{j_l}(\tau_l)\ldots u_{j_1}(\tau_1)d\tau_1\ldots d\tau_l
\end{aligned}
\tag{2.4}
$$

Returning to (2.1), the other representation [4] is given by

$$
\begin{aligned}
y(t) &= C\Phi(t,0)x_0 \\
&+ \sum_{k\geq 1}\int_0^t\int_0^{\tau_k}\cdots\int_0^{\tau_2} C\Phi(t,\tau_k)A\Phi(\tau_k,\tau_{k-1})A \\
&\quad \ldots A\Phi(\tau_2,\tau_1)A\Phi(\tau_1,0)x_0\; d\tau_1\ldots d\tau_k
\end{aligned}
\tag{2.5}
$$

where, $\Phi(t,\tau) = \Gamma^{-1}(t)\Gamma(\tau)$ and Γ is an $n\times n$ matrix solution of

$$
\dot{\Gamma} \;=\; -\Gamma\sum_{j=1}^{m} u_j N_j \quad,\quad \Gamma(0) = I \tag{2.6}
$$

Example:

As a trivial example, we consider A and N commuting. Thus, the input-output map corresponding to the system

$$
\begin{aligned}
\dot{x} &= Ax + uNx \\
y &= Cx
\end{aligned}
$$

is given by

$$
y(t) = Ce^{At+NU(t)}x_0 \quad . \tag{2.7}
$$

Therefore, the series representation introduced here (see also [4]) and the Volterra series are given by

$$\begin{aligned} y_1(t) &= Ce^{NU(t)} \sum_{k\geq 0} \frac{(At)^k}{k!} x_0 \\ y_2(t) &= Ce^{At} \sum_{k\geq 0} \frac{(NU(t))^k}{k!} x_0 \end{aligned}$$

respectively. Obviously, if a truncation is to be made, y_1 will give a better estimate than y_2 for most u's.

Moreover, in our representation, we have the freedom to select the stable part of the system, whereas in Volterra series representation, we are bound to take A stable in order for the series to converge on $[0, \infty]$ [9]. Of course, one of the main advantages of Volterra series representation is its 'multilinear' nature.

3 Input-Output stability of bilinear systems :

In this section we shall present sufficient conditions for the L^∞ – *stability* of bilinear systems. Detailed proofs can be found in [4]. We claim the following:

Theorem 1:

A sufficient condition for the system (2.2) to be L^∞ – *stable* is that the following hold:

(i) There exist at least one N_j $(j = 0, \ldots, m)$ (say N_k) having all its eigenvalues with negative real parts,

(ii) $\lim_{t\to\infty} \int_0^t [-\rho_k u_k(\tau) + \alpha_k \parallel P_k(\tau) \parallel] d\tau < \infty$

where $\alpha_k > 0$ and $\rho_k > 0$ are such that $\| e^{N_k t} \| \leq \alpha_k e^{-\rho_k t}$, $P_k(t) = \sum_{j \neq k} u_j N_j$ and $u_k \geq 0$.

Furthermore, if the limit is $-\infty$ then y(t) $\rightarrow$ 0 as t $\rightarrow \infty$.

Corollary 1:

A sufficient condition for the system (2.2) to be $L^\infty - stable$ is that the following hold:

(i) There exist at least one N_j $(j = 0, \ldots, m)$ (say N_k) having all its eigenvalues with negative real parts,

(ii) $\lim_{t \rightarrow \infty} \int_0^t [-\rho_k u_k(\tau) + \alpha_k \sum_{j \neq k} \mid u_j(\tau) \mid \| N_j \|] d\tau < \infty$

where $\alpha_k > 0$ and $\rho_k > 0$ are such that $\| e^{N_k t} \| \leq \alpha_k e^{-\rho_k t}$, $P_k(t) = \sum_{j \neq k} u_j N_j$ and $u_k \geq 0$.

Furthermore, if the limit is $-\infty$ then y(t) $\rightarrow$ 0 as t $\rightarrow \infty$.

Theorem 2:

A sufficient condition for the system (2.2) to be $L^\infty - stable$ is that the following hold:

(i) There exist a positive constant α and a non-decreasing function ρ satisfying $\rho(0) = 0$ such that $\| \Phi(t, \tau) \| \leq \alpha e^{-[\rho(t) - \rho(\tau)]}$ for $t \geq \tau \geq 0$

(ii) $\lim_{t \rightarrow +\infty} [-\rho(t) + \alpha \| A \| t] \neq \infty$

Furthermore, if the limit is $-\infty$ then $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Corollary 2:

A sufficient condition for the system to be $L^\infty - stable$ is that u_j , $j = 1, \ldots, m$ satisfy the following inequality:

$min\{\rho_1 - \alpha_1 \parallel E_1 + \sum_{j=1}^m u_j N_j \parallel , -\rho_2 + \alpha_2 \parallel E_2 - \sum_{j=1}^m u_j N_j \parallel\} \geq \alpha_1\alpha_2 \parallel A \parallel$

where E_1 and E_2 are stable matrices satisfying $\parallel e^{E_i t} \parallel \leq \alpha_i e^{-\rho_i t}$, $t \geq 0$ for some $\alpha_i, \rho_i > 0$.

Hint: Take $\alpha = \alpha_1\alpha_2$ and,

$$\rho(t) = \int_0^t max\ \{\rho_1 - \alpha_1 \parallel E_1 + \sum_{j=1}^m u_j(\tau) N_j \parallel , -\rho_2 + \alpha_2 \parallel E_2 - \sum_{j=1}^m u_j(\tau) N_j \parallel \}\ d\tau \tag{3.1}$$

and use theorem 2.

Remark:

The sufficient condition in corollary 2 has a nice geometric interpretation in terms of the location of $\sum_{j=1}^m u_j N_j$ with respect to the balls centered at $-E_1$ and E_2 with radius $\frac{\rho_1}{\alpha_1} - \alpha_2 \parallel A \parallel$ and $\frac{\rho_2}{\alpha_2} + \alpha_1 \parallel A \parallel$ respectively.

4 Realization of a nonlinear i/o map:

Consider the nonlinear input-output map given by

$$\begin{aligned} y(t) \ = \ & \int_0^t \int_0^{\tau_l} \ldots \int_0^{\tau_2} Ce^{N[U(t)-U(\tau_l)]} A e^{N[U(\tau_l)-U(\tau_{l-1})]} A \\ & \ldots A e^{N[U(\tau_2)-U(\tau_1)]} A e^{NU(\tau_1)} x_0 d\tau_1 \ldots d\tau_l \end{aligned} \tag{4.1}$$

where again $U(t) = \int_0^t u(\tau)d\tau$.

Define $z_1, \ldots, z_{l+1}$ by

$$\begin{aligned} z_{l+1}(t) &= e^{NU(t)}x_0 \\ z_l(t) &= \int_0^t e^{N[U(t)-(\tau)]} A z_{l+1}(\tau)d\tau \\ &\vdots \\ z_1(t) &= \int_0^t e^{N[U(t)-U(\tau)]} A z_2(\tau)d\tau \end{aligned} \tag{4.2}$$

Therefore, differentiating with respect to t, we obtain,

$$\begin{aligned} \dot{z} &= \overline{A}z + u\overline{N}z \\ y &= \overline{C}z \end{aligned} \tag{4.3}$$

where

$$\overline{A} = \begin{pmatrix} 0 & A & 0 & \ldots & 0 \\ 0 & 0 & A & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \overline{N} = \begin{pmatrix} N & 0 & 0 & 0 & 0 \\ 0 & N & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \ldots & N \end{pmatrix}$$

$$\overline{C} = (C, 0, \ldots, 0)\ ,\ z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{l+1} \end{pmatrix}\ ,\ z(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_0 \end{pmatrix} = z_0 \tag{4.4}$$

with dimensions $[(l+1)n] \times [(l+1)n]$, $[(l+1)n] \times [(l+1)n]$, $n \times [(l+1)n]$, $(l+1)n$ and $(l+1)n$ respectively.

Note that $\overline{A}$ is nilpotent of degree l , i.e., $\overline{A}^l \neq 0$ and $\overline{A}^{l+1} = 0$.

If we consider now the map obtained from the $(l+1)^{th}$ term of the Volterra series, we obtain,

$$\begin{aligned} \dot{z} &= \overline{\overline{A}}z + u\overline{\overline{N}}z \\ y &= \overline{\overline{C}}z \end{aligned} \tag{4.5}$$

where,

$$\overline{\overline{A}} = \begin{pmatrix} A & 0 & 0 & \ldots & 0 \\ 0 & A & 0 & \ldots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \ldots & A \end{pmatrix}\ ,\ \overline{\overline{N}} = \begin{pmatrix} 0 & N & 0 & \ldots & 0 \\ 0 & 0 & N & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{4.6}$$

and $\overline{\overline{C}} = \overline{C}$.

Therefore, we can see that

$$\begin{aligned} \overline{A} &= \mathcal{S}^+\overline{\overline{A}} \\ \overline{\overline{N}} &= \mathcal{S}^+\overline{N} \\ \overline{C} &= \overline{\overline{C}} \end{aligned} \tag{4.7}$$

where $\mathcal{S}^+$ is the operator

$$\mathcal{S}^+ = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{4.8}$$

An immediate generalization is to consider the input-output map defined by

$$\begin{aligned} y(t) &= e^{NU(t)}x_0 \\ &+ \sum_{k=1}^{l} \int_0^t \int_0^{\tau_k} \dots \int_0^{\tau_2} Ce^{N[U(t)-U(\tau_k)]}A \dots \\ &\dots Ae^{N[U(\tau_2)-U(\tau_1)]}Ae^{NU(\tau_1)}x_0 d\tau_1 \dots d\tau_k \end{aligned} \tag{4.9}$$

If we define, as in section (4), $z_1, \dots z_{l+1}$, we obtain

$$\begin{aligned} \dot{z} &= \overline{A}z + u\overline{N}z \\ y &= C^* z \end{aligned} \tag{4.10}$$

where, this time, $\overline{C}^* = (C, \dots, C)$.

Similar result for the finite Volterra series can be obtained and $\overline{\overline{C}}^* = \overline{C}^*$.

Remark:

There is clearly a one to one map between this representation and the Volterra series representation, and since we know how to solve a minimal realization from the Volterra series, we can deduce a minimal realization for this representation. From the above discussion and the remark we have proved:

Theorem 3:

Given an i/o map of the form (4.9) there exists a minimal bilinear realization.

5 Generalized frequency response:

In [2] (see also [3])we have introduced the notion of the 'generalized frequency response' as follows:

Consider a system S given in terms of an input-output map

$$S : R^n \times L^2_w[0,\infty] \longrightarrow L^2_w[0,\infty] \tag{5.1}$$

defined by

$$y(t) = S(x_0, u(.))(t) \tag{5.2}$$

where u and y are respectively the input and the output of the system and x_0 is the initial state in some given state-space realization. For each fixed initial state x_0, we have a map

$$S_{x_0} = S(x_0, .) : L^2_w[0,\infty] \longrightarrow L^2_w[0,\infty] \tag{5.3}$$

For simplicity, we have assumed scalar input and scalar output.

Let $\mathcal{I}$ denotes the natural isomorphism between $L^2_w[0,\infty]$ and ℓ^2 , then the 'generalized frequency response' for S is the induced map from ℓ^2 to ℓ^2 such that the diagram

$$\begin{array}{ccc} L^2_w[0,\infty] & \xrightarrow{S_{x_0}} & L^2_w[0,\infty] \\ \mathcal{I}\downarrow & & \downarrow\mathcal{I} \\ \ell^2 & \xrightarrow{s_{x_0}} & \ell^2 \end{array}$$

commutes.

In this section, we shall derive the 'generalized frequency response' associated with the input-output map (4.9). First we shall Consider the bilinear system

$$\begin{aligned} \dot{x} &= Ax + uNx, \quad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{5.4}$$

We have,

$$x(t) = e^{NU(t)}x_0 + \int_0^t e^{N[U(t)-U(\tau)]}Ax(\tau)d\tau \tag{5.5}$$

from which we get

$$y(t) = e^{NU(t)}x_0 +$$

$$+ \sum_{k\geq 1}\int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} Ce^{N[U(t)-U(\tau_k)]}A \ldots$$
$$\ldots Ae^{N[U(\tau_2)-U(\tau_1)]}Ae^{NU(\tau_1)}x_0 d\tau_1 \ldots d\tau_k \qquad (5.6)$$

We claim the following:

Theorem 4:

A sufficient condition for the nonlinear operator h defined by (5.6) to map $L^2_w[0,\infty] \cap L^\infty[0,\infty]$ into itself is that the following hold:

(i) N has all its eigenvalues with negative real parts,

(ii) there exists a function η such that $\int_0^t [-\rho u(\tau) + \alpha \parallel A \parallel] d\tau \ \leq \eta(t) < \infty$ with $\lim_{t\to\infty} \eta(t) = -\infty$.

where $\alpha > 0$, $\rho > 0$ are such that $\parallel e^{Nt} \parallel \leq \alpha e^{-\rho t} \quad , t \geq 0$.

Proof:

(5.5) yields

$$\parallel x(t) \parallel \leq \alpha e^{-\rho U(t)} \parallel x_0 \parallel + \alpha \int_0^t e^{-\rho[U(t)-U(\tau)]} \parallel A \parallel \parallel x(\tau) \parallel d\tau \qquad (5.7)$$

Using Gronwall's lemma, we obtain

$$e^{\rho U(t)} \parallel x(t) \parallel \leq \alpha \parallel x_0 \parallel e^{\parallel A \parallel t} \qquad (5.8)$$

Therefore,

$$\begin{aligned} | y(t) | &\leq \alpha \parallel C \parallel \parallel x_0 \parallel e^{\int_0^t [-\rho u(\tau) + \alpha \parallel A \parallel] d\tau} \\ &\leq \alpha \parallel C \parallel \parallel x_0 \parallel e^{\eta(t)} < \infty \end{aligned}$$

hence,

$$\begin{aligned} \| y(t) \|_w^2 &= \int_0^\infty | y(t) |^2 w(t)dt \\ &\leq \alpha^2 \| C \|^2 \| x_0 \|^2 \int_0^\infty w(t) e^{2\eta(t)} dt \end{aligned} \tag{5.9}$$

Using the Cauchy-Bouniakovsky inequality we obtain,

$$\| y \|_w^2 \leq \alpha^2 \| C \|^2 \| x_0 \|^2 \{ \int_0^\infty w^2(t)dt \; . \int_0^\infty e^{4\eta(t)} dt \}^{\frac{1}{2}} < \infty \tag{5.10}$$

Theorem 5:

The 'generalized frequency response' associated with the input-output map (4.9) is the map $s : \ell^2 \longrightarrow \ell^2$ defined by $\overline{y} = s(\overline{u})$ where

$$\begin{aligned} y_k &= C v_k^0 x_0 \\ &+ \sum_{m=1}^{l} \sum_{j_1 \geq 0} \cdots \sum_{j_{m+1} \geq 0} \beta_{j_1 \ldots j_{m+1}}^{m,k} C v_{j_{m+1}}^0 v_{j_m} \ldots v_{j_1} x_0 \end{aligned} \tag{5.11}$$

and $v_k^0 = v_k^0(\overline{u})$ and $v_k = v_k(\overline{u})$ are given by (5.20) and (5.21).

Proof :

Consider v_0 and v defined by

$$\begin{aligned} v_0(t) &= e^{NU(t)} \\ v(t) &= e^{-NU(t)} A e^{NU(t)} \end{aligned} \tag{5.12}$$

We have

$$v_0(t) = \sum_{m \geq 0} N^m \frac{1}{m!} U^m(t) \tag{5.13}$$

and the Baker-Campbell-Hausdorff formula (1.1) yields

$$v(t) = \sum_{m\geq 0}(-1)^m ad_N^m\, A\frac{1}{m!}U^m(t) \tag{5.14}$$

but

$$\frac{1}{m!}U^m(t) = \int_0^t \int_0^{\tau_m} \cdots \int_0^{\tau_2} u(\tau_1)\ldots u(\tau_m)d\tau_1 \ldots d\tau_m \tag{5.15}$$

Thus

$$v_0(t) = \sum_{m\geq 0} N^m \int_0^t \int_0^{\tau_m} \cdots \int_0^{\tau_2} u(\tau_1)\ldots u(\tau_m)d\tau_1 \ldots d\tau_m \tag{5.16}$$

and

$$v(t) = \sum_{m\geq 0}(-1)^m ad_N^m\, A \int_0^t \int_0^{\tau_m} \cdots \int_0^{\tau_2} u(\tau_1)\ldots u(\tau_m)d\tau_1 \ldots d\tau_m \tag{5.17}$$

Assume u, v, and v_0 are in $L^2_w[0,\infty]$, so we can write $u = \sum_{k\geq 0} u_k e_k$,

$v = \sum_{k\geq 0} v_k e_k$, and $v^0 = \sum_{k\geq 0} v^0_k e_k$

Now assume that the basis $\{e_j\}$ satisfies the assumption

(A) the functions

$$\alpha^m_{j_1\ldots j_m} \;:\; t \longrightarrow \textstyle\int_0^t \int_0^{\tau_m} \cdots \int_0^{\tau_2} e_{j_1}(\tau_1)\ldots e_{j_m}(\tau_m)d\tau_1 \ldots d\tau_m$$

and

$$\beta^m_{j_1\ldots j_{m+1}} \;:\; t \longrightarrow e_{j_{m+1}}(t) \textstyle\int_0^t \int_0^{\tau_m} \cdots \int_0^{\tau_2} e_{j_1}(\tau_1)\ldots e_{j_m}(\tau_m)d\tau_1 \ldots d\tau_m$$

belong to $L^2_w[0,\infty]$ for all indices.

Then, under assumption (A), we can write

$$\alpha^{m}_{j_1\cdots j_m}(t) = \sum_{k\geq 0} \alpha^{m,k}_{j_1\cdots j_m} e_k(t) \tag{5.18}$$

and

$$\beta^{m}_{j_1\cdots j_{m+1}}(t) = \sum_{k\geq 0} \beta^{m,k}_{j_1\cdots j_{m+1}} e_k(t) \tag{5.19}$$

Therefore,

$$v_k^0 = \sum_{m\geq 0}\sum_{j_1\geq 0}\cdots\sum_{j_m\geq 0} N^m \alpha^{m,k}_{j_1\cdots j_m} u_{j_1}\cdots u_{j_m} = v_k^0(\overline{u}) \tag{5.20}$$

$$v_k = \sum_{m\geq 0}\sum_{j_1\geq 0}\cdots\sum_{j_m\geq 0} (-1)^m ad_N^m\, A\alpha^{m,k}_{j_1\cdots j_m} u_{j_1}\cdots u_{j_m} = v_k(\overline{u}) \tag{5.21}$$

Thus,

$$y_k = Cv_k^0 x_0 + \sum_{m=1}^{l} \sum_{j_{m+1}\geq 0} \cdots \sum_{j_1\geq 0} \beta^{m,k}_{j_1\cdots j_{m+1}} C v^0_{j_{m+1}} v_{j_m}\cdots v_{j_1} x_0 \tag{5.22}$$

where $y_k =< y\ ,\ e_k >_w$. Hence, we have proved the theorem.

6 Conclusion:

In this paper we have presented a realization theory for an input-output map introduced in [4], outlining the link with the realization of finite Volterra series. We have defined the 'generalized frequency response' [2](see also [3]) for this input-output map. A simple example has been included to show how this representation compares with the Volterra series representation.

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