---

**Monograph:**
Billings, S.A. and Tsang, K.M. (1988) Spectral Analysis for Nonlinear Spectral Analysis Part 1: Parametric Nonlinear Spectral Analysis. Research Report. Acse Report 337 . Dept of Automatic Control and System Engineering. University of Sheffield

---

# SPECTRAL ANALYSIS FOR NONLINEAR SYSTEMS

## Part I - Parametric Nonlinear Spectral Analysis

S A Billings

K M Tsang

Department of Control Engineering
University of Sheffield
Mappin Street, Sheffield S1 3JD

July 1988

Research Report No 337

Abstract

A new theory of spectral analysis for nonlinear systems is introduced. The method consists of estimating the parameters in a NARMAX model description of the system and then computing the generalised frequency response functions directly from the estimated model. The paper is divided into three parts. Part I introduces a methodology for estimating the models and computing the frequency response functions. Part II concentrates on the interpretation of the nonlinear frequency response functions and Part III describes a series of case study examples and illustrates in detail how to apply the algorithms to real systems.

## 1. Introduction

Although linear spectral analysis is a well established tool in signal processing, that finds wide application in almost every branch of science and engineering, its usefulness is limited when the system under investigation is nonlinear. Indeed it is easy to construct simple nonlinear examples where linear spectral analysis gives completely misleading results [Billings and Tsang 1988]. This is a severe limitation given that most real systems are nonlinear to some extent. It is these restrictions, coupled with the current interest in measuring, analysing and exploiting nonlinearity, which have provided the impetus to develop spectral estimators for nonlinear systems which can be applied almost as easily and routinely as the linear algorithms.

Existing nonlinear methods are based on transforming the Volterra kernels to the frequency domain to yield generalised transfer functions [Schetzen 1980, Marmarelis and Marmarelis 1978]. Whilst this approach provides a suitable means of characterizing nonlinear systems in the frequency domain [Bedrosian and Rice 1971; Bussgang, Ehrman and Graham 1974] the measurement techniques which are available are all based on extending the classical linear FFT based algorithms to higher dimensions [Brillinger and Rosenblatt 1967; Vinh, Chouychai and Djouder 1987; Billings 1980]. This inevitably leads to considerable complexity and an explosion in computational cost to the extent that the only systems identified with this approach have contained just the first two Volterra kernels. These restrictions, unless they can be overcome, suggest that this non-parametric approach may not be upwardly extendable to general classes of multi-input multi-output nonlinear systems.

The present paper introduces a methodology for analysing unknown nonlinear systems in the frequency domain which has been developed as an alternative to the non-parametric algorithms[Billings and Tsang 1987; Billings, Tsang and Tomlinson 1988]. The method consists of estimating the parameters in a NARMAX (Nonlinear AutoRegressive Moving Average model with eXogenous inputs) description of the system [Leontaritis and Billings 1985; Chen and Billings 1988] and then computing the generalised frequency response functions directly from the estimated model. The first part of this approach consists of an estimation procedure that can be applied to a wide class of nonlinear systems. This can be divided into four stages; testing for non-

linearity in the data, identifying the system structure or which terms to include in the model, estimating the system parameters and validating the model. The generalised frequency response functions can then be obtained directly from the estimated NARMAX model by discarding the noise model, which was estimated to ensure unbiased system parameters, and using the probing method [Bedrosian and Rice 1971] to obtain analytical expressions for the frequency response functions. Because the NARMAX methodology provides a model of the system which is essentially a discrete-time version of the describing differential equation any order of generalised frequency response function can be obtained. The advantages of this approach are that it works for small data lengths (typically a few hundred data points), the system can be described by a small parameter set and there are no restrictions on the form of the input except that it should be persistently exciting.

The cornerstone to this whole procedure is the estimation of an accurate parametric description of the system. Whereas previous algorithms were based on an optimum prediction error estimator [Leontaritis and Billings 1988; Billings and Chen 1988] much simpler alternatives have now been derived. These are based on an orthogonal estimation algorithm [Korenberg, Billings and Liu 1988; Billings, Korenberg and Chen 1988] which, even for complex nonlinear models, consist of just a few lines of simple code. The algorithm detects both the system structure and provides estimates of the unknown parameters and forms the estimation engine for the current approach.

The purpose of the present paper is to introduce a unified procedure for estimating nonlinear frequency response functions. The main objective is to demonstrate how easy it is to code and use these algorithms and the emphasis is placed on the practical application rather than the theory underlying the methods which is available elsewhere. The paper is divided into three parts. In Part I the estimation algorithms are introduced and illustrated using simulated examples. Part II concentrates on the interpretation of nonlinear frequency response functions and illustrates by example properties of the nonlinear frequency response functions including harmonics, gain compression/expansion, desensitization, intermodulation and other related characteristics. Part III describes a series of case study examples and illustrates in detail how to apply the algorithms to real systems and how to interpret the results.

## 2. Nonlinear Frequency Response Functions

The importance of studying spectral analysis for nonlinear systems can best be demonstrated by considering what can happen if linear methods are blindly applied to data generated from nonlinear systems. For example estimation of the frequency response function for a system represented by the model

$$dy/dt + ay(t) = bu(t) + cu^2(t) \tag{1}$$

where u(t) is a signal whose third order moments are zero (eg zeromean Gaussian or sine wave input) yields

$$\hat{H}_1(j\omega) = \frac{S_{uy}(j\omega)}{S_{uu}(\omega)} = \frac{b}{j\omega+a} \tag{2}$$

where $S_{uy}(j\omega)$ represents the cross-spectral density.

The estimate is completely independent of the nonlinear term $cu^2(t)$ and shows the limitations of linear methods when applied to nonlinear systems.

Efforts to extend spectral analysis to the nonlinear case have to date concentrated on the functional series methods and higher order spectra. Although several authors [Bedrosian and Rice 1971; Bussgang, Ehrman and Graham 1974, Chua and Ng 1979; Brillinger and Rosenblatt 1969] have considered the theoretical analysis of such ideas very few practical results have been reported. In the few cases where a simulated or practical system has been analysed only the first and second order functions were estimated on the assumption that all higher order terms were zero. This was because of the inherent difficulties of estimating nonlinear frequency response functions by extending the traditional FFT windowing algorithms to work in many dimensions, the necessity of a huge data set, special inputs and an excessive computational requirement. All these difficulties can however be overcome by merging the theoretical ideas of generalised frequency response functions with the NARMAX model description and estimation procedures.

### 2.1. Generalised Frequency Response Functions

The traditional description of nonlinear systems has been based on the Volterra series [Schetzen 1980; Marmarelis and Marmarelis 1978]

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int h_n(\tau_1,\tau_2,\cdots\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i \qquad (3)$$

where $h_n(\tau_1,\tau_2\cdots\tau_n)$ is the n'th order Volterra kernel which can be visualised as a nonlinear impulse response of order n. Equation (3) can be expressed as

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \qquad (4)$$

$$\text{where} \quad y_n(t) = \int_{-\infty}^{\infty} \cdots \int h_n(\tau_1,\cdots\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i \qquad (5)$$

Taking the multiple Fourier transform of the n'th order Volterra kernel yields the n'th order generalised frequency response function [Bedrosian and Rice 1971; Bussgang, Ehrman and Graham 1974]

$$H_n(f_1,f_2,\cdots f_n) = \int_{-\infty}^{\infty} \cdots \int h_n(\tau_1,\cdots\tau_n) e^{-j2\pi(f_1\tau_1+\cdots+f_n\tau_n)} d\tau_1\cdots d\tau_n \qquad (6)$$

Conversely, the nonlinear impulse response of order n follows from the nonlinear transfer function of order n by the inverse Fourier transform

$$h_n(\tau_1,\cdots\tau_n) = \int_{-\infty}^{\infty} \cdots \int H_n(f_1,\cdots f_n) e^{j2\pi(f_1\tau_1+\cdots f_n\tau_n)} df_1\cdots df_n \qquad (7)$$

Substituting eqn (7) into eqn (5) and carrying out the multiple integrals on $\tau_1,\cdots\tau_n$ gives

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int H_n(f_1,\cdots f_n) \prod_{i=1}^{n} U(f_i) e^{j2\pi f_i t} df_i \qquad (8)$$

where U(f) represents the input spectrum.

The total system response eqn (3) can be thought of in terms of each component which makes up eqn (4). This consists of the linear portion characterized by the linear frequency response function $H_1(f)$, the quadratic contribution to the response characterized by the second order frequency response function $H_2(f_1,f_2)$ and so on.

Notice that $h_n(\tau_1,\cdots\tau_n)$ and its transform $H_n(f_1,\cdots f_n)$ may not be symmetric functions of their arguments. Even so all kernels or transforms that differ only by the permutations of their arguments are equivalent in representing the system because in each case the output $y_n(t)$ in eqn's (5)

or (8) would be identical. It is however convenient to work with symmetric transforms where the order of the arguments in $H_n(f_1,\ldots f_n)$ can be arbitrarily interchanged and this will be assumed in the analysis that follows. If a kernel or transform is not symmetric it can be replaced by a symmetrized kernel and associated transform defined as [Schetzen 1980]

$$h_n(\tau_1,\ldots\tau_n)_{sym} = \frac{1}{n!} \sum_{\#} h_n(\tau_1,\tau_2\ldots\tau_n) \tag{9}$$

$$H_n(f_1,\ldots f_n)_{sym} = \frac{1}{n!} \sum_{\#} H_n(f_1,\ldots f_n) \tag{10}$$

where # denotes that the summation is over all permutations of the arguments of $h_n(\cdot)$ or $H_n(\cdot)$ respectively.

It is apparent from eqn (6) that the properties of spectral conjugation hold

$$H_n^*(f_1,\ldots f_n) = H_n(-f_1,\ldots -f_n) \tag{11}$$

where * denotes complex conjugation. It therefore follows that $H_1(-f) = H_1(f)$, $H_2(-f_1,f_2) = H_2^*(f_1,-f_2)$ and $H_2(-f_1,-f_2) = H_2^*(f_1,f_2)$. Therefore plots of $H_1(f)$ $0 \leq f \leq \infty$, and $H_2(f_1,f_2)$ $0 \leq f_1 \leq \infty$, $-\infty \leq f_2 \leq \infty$ will fully characterize $H_1(f_1)$ and $H_2(f_1,f_2)$ respectively. It is of course impossible to plot the third order function $H_3(f_1,f_2,f_3)$ directly. By fixing one of the frequency variables however $f_3 = f_c$ say slices of the function can be displayed [Billings, Tsang and Tomlinson 1988]. The plot of $H_3(f_1,f_2,f_c)$ can then be plotted in a similar way to the second order response function $H_2(f_1,f_2)$. An alternative way would be to set one of the frequency variables equal to another $f_3 = f_1$ say. Similar procedures can be used to obtain frequency plots of higher order response functions.

## 2.2. Computation of Nonlinear Frequency Response Functions

The probing or harmonic input method [Bedrosian and Rice 1971] can be justified by considering the steady-state output of a nonlinear system with several exponential inputs. Let the input u(t) be a sum of K exponentials

$$u(k) = \sum_{k=1}^{K} A_k e^{j2\pi f_k t} \tag{12}$$

where $A_k$ represent the amplitudes and $f_k$ may be any positive or negative real number. From eqn (5) the n'th order output can be expressed as [Chua and Ng 1979]

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int h_n(\tau_1,\ldots\tau_n) \prod_{i=1}^{n} \sum_{k=1}^{K} A_k e^{j2\pi f_k(t-\tau_i)} d\tau_i \tag{13}$$

$$= \sum_{k_1=1}^{K} \cdots \sum_{k_n=1}^{K} \prod_{i=1}^{n} A_{k_i} e^{j2\pi f_{k_1} t} \int_{-\infty}^{\infty} \cdots \int h_n(\tau_1,\ldots\tau_n) \prod_{i=1}^{n} e^{-j2\pi f_{k_i}\tau_i} d\tau_i \tag{14}$$

Substituting from eqn (6) yields

$$y_n(t) = \sum_{k_1=1}^{K} \cdots \sum_{k_n=1}^{K} [A_{k_1}\ldots A_{k_n} H_n(f_{k_1},\ldots f_{k_k})] e^{j2\pi(f_{k_1}+\ldots+f_{k_n})t} \tag{15}$$

where different terms in this equation may give rise to the same output frequency, a common phenomena of nonlinear systems, and each permutation of $f_{k_1}\ldots f_{k_n}$ in the argument of $H_n(\cdot)$ gives rise to a term in the n'th order output.

The results of eqn (15) and hence the probing method of determining the generalised frequency response functions, can be made much more transparent by noting that the Fourier transform of the input eqn (12) will be a sum of delta functions. So that using eqn (8) with the input eqn (12) and setting $K = n$, $A_k = 1$ for all $k = 1,2,\ldots n$ yields

$$y_n(t) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} H_n(f_{k_1},\ldots f_{k_n}).e^{j2\pi(f_{k_1}+\ldots+f_{k_n})t} \tag{16}$$

which also follows directly from (15). There are n! terms in $y_n(t)$ eqn (16) with frequency $f_1+f_2\ldots+f_n$ each corresponding to a permutation of $f_1,\ldots f_n$ in the argument of $H_n(\cdot)$. Consequently if y(t) contains no component with frequency $f_1+\ldots+f_n$ other than those n! terms in $y_n(t)$ then it follows from (10) and (16) (compare also with eqn (18)) that the symmetrized n'th order nonlinear transfer function $H_n(f_1,\ldots f_n)_{sym}$ can be obtained by the harmonic or probing input method by equating coefficients

of $n!\exp\{j2\pi(f_1+\ldots+f_n)t\}$ in the system output when the input is defined as in eqn (12) with $K = n$ and $A_{k_i} = 1$ [Bedrosian and Rice 1971; Bussgang, Ehrman and Graham 1974]. It can be shown [Chua and Ng 1979] that this procedure will be valid providing the set of frequencies $\{f_1,\ldots f_k\}$ form a frequency base. This means that there is no set of rational numbers $\{r_1,\ldots r_K\}$ (not all zero) such that

$$r_1f_1 + r_2f_2 \ldots + r_Kf_K = 0$$

Since any output frequency $f_{k_1}+f_{k_2}\ldots+f_{k_n}$ in eqn (15) can be written as $m_1f_1+m_2f_2\ldots+m_Kf_K$ with a suitable choice of coefficients $m_i$, $i = 1,2\ldots K$ that are nonnegative integers ($m_i \geq 0$) it is convenient to define a module vector $M = (m_1,\ldots m_K)$ so that the corresponding output frequency for each module vector M can be written as

$$f_M = m_1f_1 + \ldots + m_Kf_K$$

Further, if $m_1+m_2+\ldots+m_K = n$ for $n>0$ then $f_M$ is an n'th order output frequency generated by $H_n(f_1,\ldots f_n)$.

If the input frequencies constitute a frequency base then the sum of all terms with frequency $f_M$ in eqn (15) for the n'th output component $y_n(t)$ will be denoted by $y_n(t;f_M)$ and is given by [Bussgang, Ehrman and Graham 1974; Chua and Ng 1979]

$$y_n(t;f_M) = n!\left[\prod_{k=1}^{K}\frac{A_k^{m_k}}{m_k!}\right] H_n(m_1\{f_1\},\ldots,m_k\{f_k\})e^{j2\pi f_M t} \qquad (17)$$

where $m_k\{f_k\}$ denotes $m_k$ consecutive arguments in $H_n(\cdot)$ with the same frequency $f_k$ and $\sum_{k=1}^{K} m_k = n$. Since $y_n(t)$ consists of all possible frequency mixes that satisfy $\sum_{k=1}^{K} m_k = n$ eqn (15) can be expressed as

$$y_n(t) = \sum_{\substack{\text{all}\\ \text{possible}\\ M}} y_n(t;f_M) = \sum_{m_1=0}^{n} \ldots \sum_{m_k=0}^{n} y_n(t;f_M) \qquad (18)$$

where in general the terms do not have overlapping frequency components. Notice that whilst $y_n(t,f_M)$ is complex terms in eqn (18) occur in conjugate pairs so that $y_n(t)$ is real.

Because the probing method for computing the $H_i(\cdot)$'s is analytic and cannot be used as a basis for measurements on real systems consider the more realistic input composed of K different sinusoids [Chua and Ng 1979], often called a K-tone input

$$u(t) = \sum_{i=1}^{K} |A_i| \cos(2\pi f_i t + \angle A_i) = \sum_{i=1}^{K} \left[ \frac{A_i}{2} e^{j2\pi f_i t} + \frac{A_i^*}{2} e^{-j2\pi f_i t} \right] \quad (19)$$

where $|A_i|$ is the amplitude, $\angle A_i$ the phase and $A_i^*$ is the complex conjugate of $A_i$. By definition $A_{-i} = A_i^*$ and $f_{-i} = -f_i$ so that eqn (19) can be expressed as

$$u(t) = \sum_{\substack{i=-K \\ i\neq 0}}^{K} \frac{A_i}{2} e^{j2\pi f_i t} \quad (20)$$

The n'th order module vector of input frequencies becomes $M = (m_{-K},\ldots m_{-1},m_1,\ldots m_K)$ where $m_i$, $i = \pm1,\pm2,\ldots\pm K$ are nonnegative integers with $\sum_{\substack{i=-K \\ i\neq 0}}^{K} m_i = n$. The input frequencies $\{f_{-K},\ldots,f_{-1},f_1,\ldots f_K\}$ do not in this case form a frequency base because $f_i+f_{-i} = 0$ and

$$f_M = \sum_{\substack{i=-K \\ i\neq 0}}^{K} m_i f_i = \sum_{i=1}^{K} (m_i f_i + m_{-i} f_{-i}) = \sum_{i=1}^{K} (m_i - m_{-i}) f_i \quad (21)$$

Substituting eqn's (20) and (21) into eqn (17) yields

$$y_n(t;f_M) = \frac{n!}{2^n} \left[ \prod_{\substack{i=-K \\ i\neq 0}}^{K} \frac{A_i^{m_i}}{m_i!} \right]$$

$$\times H_n(m_{-K}\{f_{-K}\}\ldots m_{-1}\{f_{-1}\},m_1\{f_1\},\ldots m_K\{f_K\})$$

$$\times e^{j2\pi f_M t} \quad (22)$$

Equation (22) illustrates that when a sum of K sinusoids is applied to a nonlinear system additional output frequencies are generated by the n'th order frequency response function of the system consisting of all possible

combinations of the input frequencies $\{-f_K,\ldots-f_1,f_1,\ldots f_K\}$ taken n at a time. Because $\{f_{-K},\ldots f_{-1},f_1,\ldots f_K\}$ is not a frequency base different module vectors of the same order can produce the same output frequency. Equation (22) can be used to predict how harmonic, intermodulation, desensitization and compression terms can arise in the output response of nonlinear systems and these effects will be illustrated by example in Part II of this paper.

## 2.2. Examples of the Probing Method

It was shown in the previous section that the harmonic input or probing method can be used to determine the symmetrized n'th order transfer function $H_i(f_1,\ldots f_n)_{sym}$ by equating coefficients of $n!\exp\{j2\pi(f_1+\ldots+f_n)t\}$ in the system output for an input defined by eqn (12). The procedure is recursive and is best illustrated by an example [Bussgang, Ehrman and Graham 1974].

Consider a system described by the differential equation

$$\gamma_3 u(t) = \frac{dy(t)}{dt} + \gamma_1 y(t) + \gamma_2 y^2(t) \tag{23}$$

where $\gamma_i$, i = 1,2,3 are constants.

This could for example represent a nonlinear circuit [Bussgang, Ehrman and Graham 1974] consisting of a linear resistor, a nonlinear resistor and a capacitor all in parallel and supplied by a current source u(t). The successive nonlinear frequency response functions can be obtained by using a series of probing inputs.

The procedure begins by setting K = 1, $A_k$ = 1, $f_k$ = f in eqn (12) to define the first probing input

$$u(t) = e^{j2\pi ft} \tag{24}$$

From eqn's (14) and (16) with n = 1

$$y(t) = H_1(f)e^{j2\pi ft}$$

and hence

$$\frac{dy}{dt} = j2\pi f H_1(f)e^{j2\pi ft} \tag{25}$$

Substituting into eqn (23) gives

$$\gamma_3 e^{j2\pi ft} = j2\pi f H_1(f) e^{j2\pi ft} + \gamma_1 H_1(f) e^{j2\pi ft} + \gamma_2 [H_1(f) e^{j2\pi ft}]^2$$

and equating coefficients of $e^{j2\pi ft}$ on both sides yields

$$H_1(f) = \frac{\gamma_3}{j2\pi f + \gamma_1} \tag{26}$$

Probing with two inputs by setting K = 2, $A_k = 1 \; \forall \; k$ in eqn (12)

$$u(t) = e^{j2\pi f_1 t} + e^{j2\pi f_2 t} \tag{27}$$

From eqn's (14) and (16) with n = 2

$$y(t) = H_1(f_1) e^{j2\pi f_1 t} + H_1(f_2) e^{j2\pi f_2 t} + 2! H_2(f_1, f_2) e^{j2\pi (f_1+f_2)t}$$
$$+ H_2(f_1, f_1) e^{j2\pi (2f_1)t} + H_2(f_2, f_2) e^{j2\pi (2f_2)t} \tag{28}$$

and the derivative is

$$\frac{dy}{dt} = j2\pi f_1 H_1(f_1) e^{j2\pi f_1 t} + j2\pi f_2 H_2(f_2) e^{j2\pi f_2 t}$$
$$+ j2! 2\pi (f_1+f_2) H_2(f_1, f_2) e^{j2\pi (f_1+f_2)t}$$
$$+ j2\pi (2f_1) H_2(f_1, f_2) e^{j2\pi (2f_1)t} + j2\pi (2f_2) H_2(f_2, f_2) e^{j2\pi (2f_2)t} \tag{29}$$

Substituting eqn's (27), (28) and (29) into (23) and equating coefficients of $2! e^{j2\pi (f_1+f_2)t}$ yields

$$0 = \{j2\pi (f_1+f_2) + \gamma_1\} H_2(f_1, f_2) + \gamma_2 H_1(f_1) H_1(f_2) \tag{30}$$

$$H_2(f_1, f_2) = - \frac{\gamma_2 H_1(f_1) H_1(f_2)}{j2\pi (f_1+f_2) + \gamma_1}$$

and hence from eqn (26)

$$H_2(f_1, f_2) = - \frac{\gamma_2}{\gamma_3} H_1(f_1) H_1(f_2) H_1(f_1+f_2) \tag{31}$$

Continuing the procedure by probing with three exponentials and equating coefficients of $3!e^{j2\pi(f_1+f_2+f_3)t}$ on both sides of eqn (23) yields

$$H_3(f_1,f_2,f_3) = -\frac{2\gamma_2}{3\gamma_3}\{H_1(f_1)H_2(f_2,f_3)+H_1(f_2)H_2(f_1,f_3)$$
$$+H_1(f_3)H_2(f_1,f_2)\}H_1(f_1+f_2+f_3) \qquad (32)$$

The procedure can be continued indefinitely to find at each step higher order nonlinear frequency response functions in terms of the lower order functions.

Even though the defining equation (23) is a very simple nonlinear system, there is only one nonlinear term, the Volterra expansion is infinite because the nonlinearity is in the output. This clearly demonstrates the disadvantages of identifying systems based on a Volterra description and shows the advantages of the NARMAX model which is based on expanding the current output in terms of both past inputs and outputs. Notice also that a classical linear frequency response analysis which computed $S_{uy}(j\omega)/S_{uu}(j\omega)$ would not provide an estimate of $H_1(f)$ eqn (26) because many of the other terms in the Volterra expansion would contribute. The degree that each higher order kernel contributed would be dependent on the statistical properties of the input so that the results obtained would only be valid for the specific input used in the experiment. This would be a severe limitation since it would mean the results could not be used to predict the system response to other inputs, the estimates would be input sensitive. The NARMAX method avoids all these limitations, but before it is introduced consider the evaluation of the generalised frequency response functions given that a NARMAX model has been identified.

Consider the application of the probing method to the NARMAX model

$$y(k) = ay(k-1) + bu(k-1) + cy^2(k-1) \qquad (33)$$

Implementing the procedure almost exactly as in the continuous time case the system is probed initially with a single exponential

$$u(k) = e^{j2\pi fk}$$

The Volterra series eqn (3) will be as before except the integrals now become summations. Equation (16) can therefore be applied directly with $n = 1$ such that

$$y(k) = H_1(f)e^{j2\pi fk} \tag{34}$$

and the delayed inputs and outputs are

$$u(k-1) = e^{j2\pi f(k-1)}$$

$$y(k-1) = H_1(f)e^{j2\pi f(k-1)} \tag{35}$$

Substituting into eqn (33)

$$H_1(f)e^{j2\pi fk} = aH_1(f)e^{j2\pi f(k-1)} + be^{j2\pi f(k-1)} + c[H_1(f)e^{j2\pi f(k-1)}]^2$$

and equating coefficients of $e^{j2\pi fk}$ on both sides yields

$$H_1(f) = \frac{be^{-2j\pi f}}{1-ae^{-j2\pi f}} \tag{36}$$

Probing with two exponentials

$$u(k) = e^{j2\pi f_1 k} + e^{j2\pi f_2 k} \tag{37}$$

The output response is from eqn (16)

$$y(k) = H_1(f_1)e^{j2\pi f_1 k} + H_1(f_2)e^{j2\pi f_2 k} + 2!H_2(f_1,f_2)e^{j2\pi(f_1+f_2)k} + H_2(f_1,f_1)e^{j2\pi(2f_1)k} + H_2(f_2,f_2)e^{j2\pi(2f_2)k} \tag{38}$$

The delayed input u(k-1) and output y(k-1) follow directly as in the case above so that substituting into eqn (33) and equating coefficients of $2!e^{j2\pi(f_1+f_2)k}$ yields

$$H_2(f_1,f_2) = \frac{cH_1(f_1)H_1(f_2)e^{-j2\pi(f_1+f_2)}}{1-ae^{-j2\pi(f_1+f_2)}} \tag{39}$$

As in the continuous time case the probing can be continued to determine higher order nonlinear frequency response functions. Clearly therefore

any order of generalised frequency response function can be found analytically as above provided efficient algorithms can be developed to identify models of the form of eqn (33). This is the subject of the next section.

## 3. The NARMAX Methodology

The NARMAX approach is a parameter estimation methodology for identifying both the structure and the parameters of unknown nonlinear systems. The method consists of several stages including testing for nonlinearities in the data prior to analysis, structure detection(which terms should be included in the model), parameter estimation, model validation and testing. Each of these stages is designed to support, verify and if necessary update the results from a previous stage in a manner which increases the probability of identifying an accurate description of the system under investigation. It is this handshaking and interaction between the stages in the identification that makes the method so powerful.

The theory associated with each of the stages in the NARMAX methodology is well documented in the literature and the objective below will be to provide a short overview of the algorithms with the aim of demonstrating how they can be implemented.

### 3.1. Testing for Nonlinearity

It is pointless applying powerful nonlinear identification algorithms if the system under test is linear. The first stage in any analysis should therefore indicate if the data was generated from a linear or a nonlinear system. Various algorithms are available which test for nonlinearity and these include two correlation tests [Billings and Tsang 1988], a filter detection method [Peyton Jones and Billings 1988] and the Hilbert transform test [Tomlinson 1987]. These will not be discussed in the present publication because of space limitations. They are however all easy to implement and provide valuable information which should be verified by the NARMAX estimation and model validation stages as described below.

## 3.2. The NARMAX Model

The NARMAX model (Nonlinear AutoRegressive Moving Average model with eXogenous inputs) for a single input single output system takes the form

$$y(k) = \alpha + F^{\ell}[y(k-1),\ldots y(k-n_y)u(k-d),\ldots u(k-n_u),$$
$$\varepsilon(k-1),\ldots\varepsilon(k-n_\varepsilon)] + \varepsilon(k) \qquad (40)$$

where $y(k)$, $u(k)$, $\varepsilon(k)$ are the sampled output input and prediction error sequences respectively, $d$ is the system time delay, $\ell$ the degree of nonlinearity and $\alpha$ is a constant term.

Although in the present study $F[\cdot]$ will be taken to be a polynomial the theory states that $F[\cdot]$ is some nonlinear function and it may be that other choices for $F[\cdot]$ are more appropriate in certain situations. These can easily be accommodated as part of the NARMAX methodology [Billings and Chen 1988; Chen and Billings 1988].

The NARMAX model which was first introduced in 1981 [Billings and Leontaritis 1981] and rigorously derived by Leontaritis and Billings [1985] can be visualised as the discrete-time equivalent of the differential equation that would represent the system in continuous time [Chen and Billings 1988]. The Hammerstein, Wiener, bilinear and Volterra models can all be interpreted as subclasses of the NARMAX.

The prediction error terms $\varepsilon(\cdot)$ are included in the NARMAX model to accommodate noise which because the system is nonlinear could be additive or multiplicative. Both types of noise can be represented by the NARMAX which provides estimates of the process and noise models. Although the noise model is not usually used in nonlinear spectral analysis it must be estimated to ensure the process model is unbiased. Failure to fit a noise model will in general lead to biased estimates; the model will predict well over the data used in estimation but may be totally inappropriate for predicting the system response to alternative inputs. It is therefore very misleading to accept an estimated model on the basis of its prediction accuracy over the estimation set only. Unfortunately, this is a practice that appears to be widespread and which will often lead to the acceptance of incorrect models and hence may result in inappropriate designs.

## 3.3 The Orthogonal Estimator

Detecting which terms are significant and should be included in the model is vitally important. Although several structure detection and parameter estimation algorithms have been developed as part of the NARMAX methodology [Leontaritis and Billings 1987] only the orthogonal estimator [Korenberg, Billings and Liu 1988; Billings, Korenberg and Chen 1988] will be described here because it is both the simplest to implement and to use. The strength of the algorithm is that it allows each coefficient in the model to be estimated independently of the other terms and provides an indication of the contribution that the term makes to the system output. An m-dimensional estimation problem can therefore be reduced to m one dimensional problems.

Represent the NARMAX model eqn (40) by the regression equation

$$y(k) = \sum_{i=1}^{n_\Theta} p_i(k)\Theta_i + \varepsilon(k) \tag{41}$$

where $p_i(t)$ represents a term in the NARMAX and no two $p_i(t)$'s are identical. For example, the NARMAX model

$$\begin{aligned} y(k) = {} & \Theta_1 y(k-1)+\Theta_2 u(k-1)+\Theta_3 u(k-1)y(k-1) \\ & +\Theta_4 u(k-1)\varepsilon(k-1)+\Theta_5\varepsilon(k-1)+\varepsilon(k) \end{aligned} \tag{42}$$

could be described by eqn (41) by defining

$$p_1(k) = y(k-1),\ p_2(k) = u(k-1),\ p_3(k) = u(k-1)y(k-1)$$

$$p_4(k) = u(k-1)\varepsilon(k-1),\ p_5(k) = \varepsilon(k-1)$$

Notice that the term $\varepsilon(k)$ will always be present, it represents the prediction errors and as such does not appear as a term in $p_i(k)$. Using the model in eqn (42) as an example the process model is defined by the terms which do not involve $\varepsilon(\cdot)$. The process model associated with the NARMAX of eqn (42) is therefore defined by the first three terms and the noise model is defined by all terms which include an $\varepsilon(\cdot)$, that is the last three terms in eqn (42).

Rather than estimating the parameters $\Theta_i$ directly from eqn (41) the orthogonal algorithm operates on an equivalent auxiliary model

$$y(k) = \sum_{i=1}^{n_\Theta} g_i w_i(k) + \varepsilon(k) \tag{43}$$

The parameters $g_i$ in eqn (43) are estimated by implementing the orthogonal estimator:-

(a) Set $w_1(t) = p_1(t)$ and $\hat{g}_1 = \dfrac{\sum_{k=1}^{N} w_1(k)y(k)}{\sum_{k=1}^{N} w_1^2(k)}$ (44)

(b) Set $j = 2$ and compute

$$\alpha_{ij} = \frac{\sum_{k=1}^{N} w_i(k)p_j(k)}{\sum_{k=1}^{N} w_i^2(k)} \qquad \text{for } i = 1,2,\ldots,j-1 \tag{45}$$

$$w_j(k) = p_j(k) - \sum_{i=1}^{j-1} \alpha_{ij} w_i(k) \tag{46}$$

$$\hat{g}_j = \frac{\sum_{k=1}^{N} w_j(k)y(k)}{\sum_{k=1}^{N} w_j^2(k)} \tag{47}$$

Increment j and compute eqn's (45), (46) and (47). When the case $j = n_\Theta$ has been completed go to (c).

(c) Compute the NARMAX parameters $\Theta_i$ backwards using

$$\hat{\Theta}_{n_\Theta} = \hat{g}_{n_\Theta}$$

$$\hat{\Theta}_i = g_i - \sum_{j=i+1}^{n_\Theta} \alpha_{ij}\hat{\Theta}_j \qquad \text{for } i = n_\Theta-1, n_\Theta-2, \ldots 1 \tag{48}$$

Equations (44) to (48) define the orthogonal estimator where N represents the number of data points. The algorithm is remarkably simple and easy to implement. The auxiliary regressors $w_i(t)$ are orthogonal so that additional terms can be added to the model without the need to recompute all the previous $\hat{g}_j$, $j<i$. Numerical ill-conditioning can be avoided by deleting $w_j(k)$ if $\sum_{k=1}^{N} w_j^2(k)$ is less than some threshold (typically $10^{-5}$). The standard deviation of the estimates are given by [Korenberg, Billings, Liu 1988]

$$\text{Std}(\hat{\theta}_i) = \sigma \sqrt{\sum_{j=i}^{n_\theta} \frac{t_{ij}^2}{\sum_{k=1}^{N} w_j^2(k)}} \qquad i = 1,2,\ldots n_\theta \qquad (49)$$

where

$$t_{ij} = \begin{cases} 1 & , \quad i = j, \; j = 1,2,\ldots n_\theta \\ -\sum_{\ell=i+1}^{j} \alpha_{i\ell} t_{\ell j} & \quad 0<i<j, \; j = 1,2,\ldots n_\theta \end{cases} \qquad (50)$$

and $\sigma^2 = E[\varepsilon^2(k)]$. In practice the prediction errors must be estimated from eqn (43)

$$\hat{\varepsilon}(k) = y(k) - \sum_{i=1}^{n_\theta} g_i w_i(k) \qquad (51)$$

and an estimate of $\sigma$ can then be obtained as

$$\hat{\sigma} = \sqrt{\frac{1}{N-n_\theta} \sum_{k=1}^{N} \varepsilon^2(k)} \qquad (52)$$

By estimating all the process parameter terms first and then computing the prediction errors eqn (51) and estimating the noise model the orthogonal algorithm presented above provides an efficient method of estimating all the unknown coefficients in the NARMAX model eqn (41). Our objective however was not just to estimate the parameters, but to detect which terms should be included within the model. This can be achieved by computing the error reduction ratio [Korenberg, Billings and Liu 1988] for the i'th term as

$$\varepsilon RR_i = \frac{\hat{g}_i^2 \sum_{k=1}^{N} w_i^2(k)}{\sum_{k=1}^{N} y^2(k)} \times 100 \qquad (53)$$

$\varepsilon RR_i$ provides a measure of the reduction in mean squared error which would result by including the i'th term $\theta_i p_i(k)$ in the NARMAX (associated with $g_i w_i(k)$ in the auxiliary model) expressed as a percentage reduction in the total mean squared error. Usually $\varepsilon RR_i$ is tested against a threshold and the i'th term is only included in the model if $\varepsilon RR_i$ exceeds the threshold.

Separate thresholds of $C_d$ and $C_{de}$ for the process and noise terms respectively have been found to be appropriate.

Notice that $\varepsilon RR_i$ only gives an indication and not a definitive measure of which terms to include in the model. If the position or order of each $p_i(k)$ in eqn (41) were changed a different value of $\varepsilon RR_i$ would be obtained. In general, a term which is introduced at an early stage in the regression will have a larger $\varepsilon RR_i$ than would be obtained if it were re-ordered to enter as a candidate term at a later stage. This difficulty can be overcome by using the forward regression version of the above algorithm [Billings, Korenberg and Chen 1987]. To introduce the idea of this algorithm consider initially the case where only process model terms exist. Consider all the possible $p_i(k)$, $i = 1,2,\ldots n_\Theta$ as candidates for $w_1(k)$, and for $i = 1,\ldots n_\Theta$ calculate

$$w_1^{(i)}(k) = p_i(k) \; ; \; \hat{g}_1^{(i)} = \frac{\sum_{k=1}^{N} w_1^{(i)}(k)y(k)}{\sum_{k=1}^{N} (w_1^{(i)}(k))^2} \; ; \; [\varepsilon RR]_1^{(i)} = \frac{(\hat{g}_1^{(i)})^2 \sum_{k=1}^{N} (w_1^{(i)}(k))^2}{\sum_{k=1}^{N} y^2(k)} \tag{54}$$

Find the maximum of $[\varepsilon RR]_1^{(i)}$, say $[\varepsilon RR]_1^{(j)} = \max\{[\varepsilon RR]_1^{(i)}, 1 \leq i \leq n_\Theta\}$. Then the first term to include in the model is selected as $w_1(k) = w_1^{(j)}(k)$ (associated with $p_j(k)$) with $\hat{g}_1 = \hat{g}_1^{(j)}$ and $[\varepsilon RR]_1 = [\varepsilon RR]_1^{(j)}$.

At the second step all the $p_i(k)$, $i = 1,\ldots n_\Theta$, $i \neq j$ are considered as possible candidates for $w_1(k)$. For $i = 1,\ldots n_\Theta$, $i \neq j$ compute

$$w_2^{(i)}(k) = p_i(k) - \alpha_{12}^{(i)} w_1(k)$$

$$\hat{g}_2^{(i)} = \frac{\sum_{k=1}^{N} w_2^{(i)}(k)y(k)}{\sum_{k=1}^{N} (w_2^{(i)}(k))^2} \; ; \; [\varepsilon RR]_2^{(j)} = \frac{(\hat{g}_2^{(i)})^2 \sum_{k=1}^{N} (w_2^{(i)}(k))^2}{\sum_{k=1}^{N} y^2(k)} \tag{55}$$

where

$$\alpha_{12}^{(i)} = \frac{\sum_{k=1}^{N} w_1(k)p_i(k)}{\sum_{k=1}^{N} w_1^2(k)} \tag{56}$$

Find the maximum of $[\varepsilon RR]_2^{(i)}$, say $[\varepsilon RR]_2^{(\ell)} = \max\{[\varepsilon RR]_2^{(i)}, 1 \leq i \leq n_\theta, i \neq j\}$. Then the second term $w_2(k) = w_2^{(\ell)}(k) = p_\ell(k) - \alpha_{12} w_1(k)$, (associated with $p_\ell(k)$) is selected with $\alpha_{12} = \alpha_{12}^{(\ell)}$, $\hat{g}_2 = \hat{g}_2^{(\ell)}$ and $[\varepsilon RR]_2 = [\varepsilon RR]_2^{(\ell)}$. The procedure is terminated when at any step q say

$$[\varepsilon RR]_q < C_d \text{ or } C_{de} \text{ as appropriate} \tag{57}$$

or when the total parameter set has been searched.

The forward regression algorithm with a noise model estimator can now be summarized as:-

(a) Select initial values of $n_n$, $n_y$, $n_\varepsilon$, d and $\ell$ in eqn (40) and use the forward regression algorithm to select and estimate terms in the process model. The selection is terminated using the rule defined in eqn (57).

(b) Calculate the residuals

$$\hat{\varepsilon}(k) = y(k) - \sum_{j=1}^{n_p} g_j w_j(k)$$

where $n_p$ is the number of process model terms.

(c) Using $\hat{\varepsilon}(k)$ apply the forward regression algorithm to the noise model and determine the next noise term which has the maximum $\varepsilon RR$. If this $\varepsilon RR$ is less than $C_{de}$ then go to (f).

(d) Add the selected noise term to the model, recompute the residuals

$$\hat{\varepsilon}(k) = y(k) - \sum_{j=1}^{n_c} g_j w_j(k)$$

with the new term included where $n_c = n_p$ + the number of noise model terms.

(e) Re-apply the orthogonal estimator for each of the current noise model terms.
Go to (d) and repeat until the parameters converge to constant values then go to (c).

(f) Compute the NARMAX coefficient using eqn (48).

In practice repeating steps (d) and (e) 4 or 5 times should be sufficient to achieve convergence. Notice that because of the orthogonality property the estimation of the process parameters in step (a) is totally decoupled from the noise model updates. The algorithm is particularly simple to

code because it just involves repeated computation of eqn's (45) to (47).

The advantage of the forward regression algorithm is that it provides an efficient yet simple method to search through and select the terms which should be included in the model. If the original model set is large however the search can be time consuming. There are simple ways to minimize this and these will be discussed in the examples. An alternative is to implement a simpler form of the algorithm, called forward inclusion, to dramatically increase the speed. This is based on applying the algorithm defined by eqn's (44) to (48) directly but introducing the terms into the estimator in the order:- constant term, u(k-1),y(k-1), u(k-2),y(k-2)..., nonlinear process model terms in order of ascending nonlinearity, $\varepsilon(k-1),\varepsilon(k-2),...$, nonlinear noise terms in order of ascending nonlinearity. As each term is introduced $\varepsilon RR_i$ eqn (53) is calculated and if it is less than $C_d$ or $C_{de}$ respectively the relevance term is immediately deleted from the model. The choice of $C_d$ and $C_{de}$ are data dependent but typical values are 0.001 to 0.5 for $C_d$ and 0.0001 to 0.1 for $C_{de}$. This algorithm is not optimal in any sense but it is very quick and it works well in practice. Furthermore any deficiencies in the model structure which result because of the order dependancy of $\varepsilon RR$ or an inappropriate choice of $C_d$ or $C_{de}$ will be detected by the model validation tests. Notice that the ordering and implementation described above is far preferable than proposed in the original version of the method [Korenberg, Billings, Liu 1988].

## 3.4 Model Validation

Model validation is designed to test if the model structure and parameter estimates are correct. When the system is nonlinear the prediction errors $\varepsilon(k)$ should be unpredictable from all linear and nonlinear combinations of past inputs and outputs and this will hold iff [Billings and Voon 1986]

$$\begin{aligned} \varphi_{\varepsilon\varepsilon}(\tau) &= \delta(\tau) \\ \varphi_{u\varepsilon}(\tau) &= 0 \;\forall \tau \\ \varphi_{u^2,\varepsilon}(\tau) &= 0 \;\forall \tau \\ \varphi_{u^2,\varepsilon^2}(\tau) &= 0 \;\forall \tau \end{aligned} \qquad (58)$$

$$\varphi_{\varepsilon\varepsilon u}(\tau) = 0 \forall \tau \geq 0$$

where the dash (') indicates the mean has been removed. If any significant terms have been omitted from the model or any parameters incorrectly estimated $\hat{\varepsilon}(k)$ will not satisfy the above tests and this will indicate that the orthogonal algorithm should be re-entered to investigate the deficiency. The form of the functions in eqn (58) can often indicate the type of deficiency [Billings and Voon 1986; Leontaritis and Billings 1987].

## 3.5 A Recipe for Estimation

The NARMAX methodology can be implemented by following a simple recipe. This will be briefly summarised below and described in detail using the examples and case studies in Parts II and III.

In order to fit a NARMAX a few hundred data samples (typically 500-1000) are required. The input does not have to have any special form but obviously should be sufficient to excite the properties of interest in the system [Leontaritis and Billings 1987b].

Initially the data may be tested to detect nonlinearity. Whilst this step can be easily omitted it does provide information which should be confirmed by the NARMAX term selection procedure. If any a-priori information is available about the form of model this can be used to focus the term selection procedure. Otherwise use either the forward inclusion or forward regression algorithms to detect significant terms and provide parameter estimates. It may be advantageous in some cases to use the forward inclusion method initially with small $C_d$ and $C_{de}$ to rapidly search and provide a reduced set of terms which can then be optimally searched using forward regression. The final model should then be validated using the correlation tests. If any deficiencies are revealed the estimation stage should be re-entered to investigate if the model can be improved in any way. When an acceptable model has been obtained the noise model can be discarded and the process model can be used to compute the nonlinear frequency response functions or to predict the response to any input [Billings, Tsang and Tomlinson 1988].

The results from each of the stages in the recipe should reinforce results from earlier stages. It is this interaction that adds to the methods robustness and ensures that any mistakes by the user are trapped and signaled by the following stages in the procedure.

## 4. An Illustrative Example

In order to demonstrate the application of the algorithms described above the system described by eqn (23) will be used as a simulated example. Allocating some values to the constant parameters in eqn (23) defined the system as

$$\frac{dy}{dt} = 5u(t) - 5y(t) - 0.8y^2(t) \qquad (59)$$

This differential equation was simulated on a Vidac analogue computer. A zero mean Gaussian noise signal of bandwidth 5Hz was used as input and the input/output signals were sampled with a ten bit A/D converter sampling at 32ms to provide five hundred pairs of sampled input/output sequences which are illustrated in Fig.1.

### 4.1 Linear Identification

Although tests for nonlinearity (not shown) indicated that the system was nonlinear it is almost always worthwhile identifying the best linear model initially. There are two reasons for this. Firstly, if the nonlinear detection tests had not been used, or were in error, and the system were linear this would be the correct procedure to follow. If the system is nonlinear the model validity tests should indicate that the estimated linear model is deficient. Secondly, fitting a linear model as a first step almost always indicates to the investigator a suitable choice for $n_u$, $n_y$ and $n_\varepsilon$. This is valuable information since it defines a smaller search space for the nonlinear terms and it helps considerably with the choice of $n_\varepsilon$ which can be difficult to select using $C_{de}$ only.

The initial analysis involved fitting linear models of various orders ($n_u = n_y = 1,2,3,4$) and delays ($d = 0,1,2,3$) and computing the loss functions (sum of squared errors) of the fitted models using the forward inclusion algorithm with $\ell = 1$. The results are summarised in table 1. Inspection of table 1 shows that the time delay is almost certainly zero and that an appropriate model order would be $n_u = n_y = 3$. This latter decision is made because, whilst the loss function will always monotonically decrease with increasing $n_u$, $n_y$ its rate of decrease should level out when the true system order is reached. Having established a good initial guess of $n_u$, $n_y$ and d the forward regression algorithm was applied for a third order model, $d = 0$, $n_\varepsilon = 3$, $C_d = 0.001$, $C_{de} = 0.0001$ with five iterations on the noise model. The results are illustrated in table 2 and the model

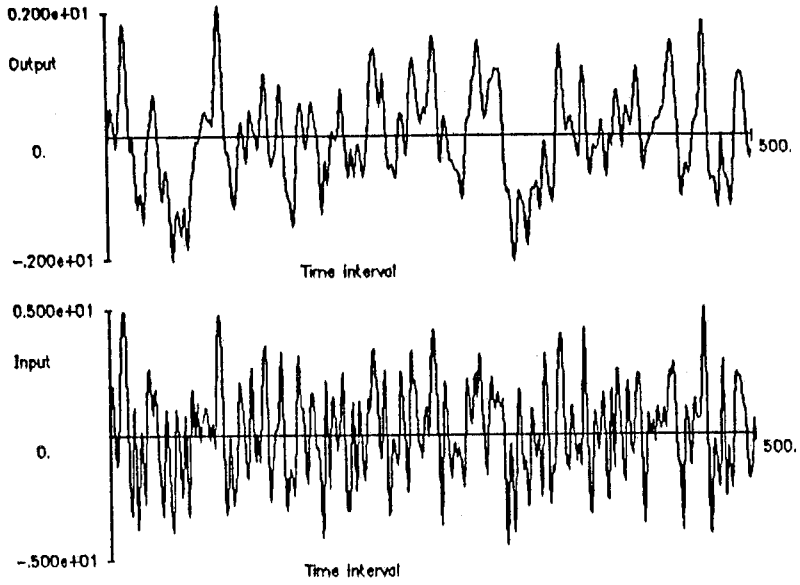


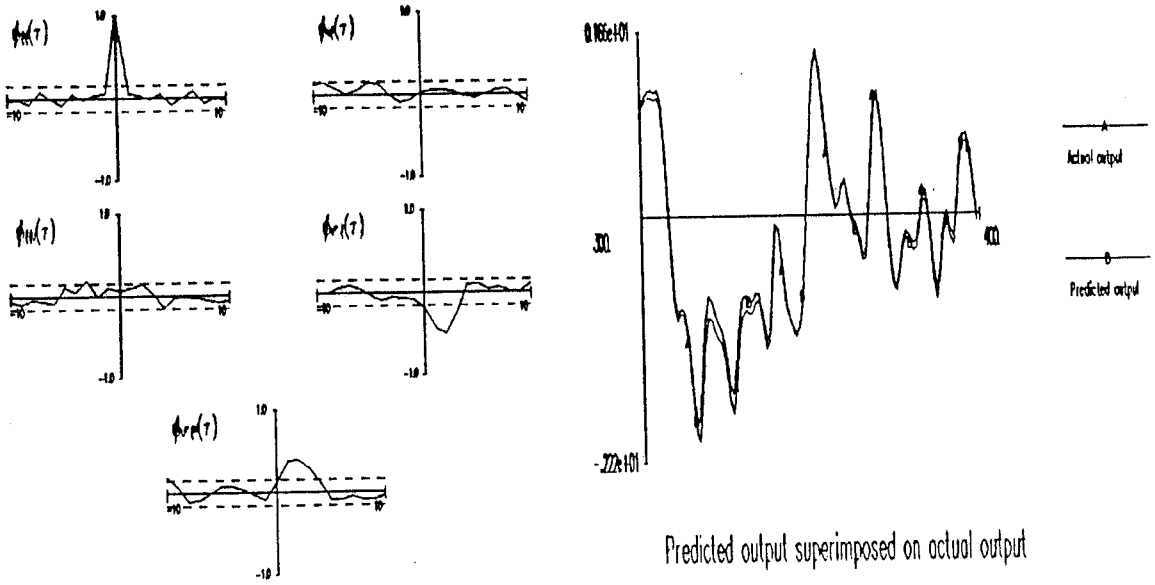

Fig.1 Input and output of the data records

| order | delay | | | |
|---|---|---|---|---|
| | 0 | 1 | 2 | 3 |
| 1 | -2.519 | -2.394 | -1.474 | -1.349 |
| 2 | -3.779 | -2.855 | -2.626 | -2.227 |
| 3 | -4.078 | -3.140 | -2.655 | -2.404 |
| 4 | -4.089 | -3.155 | -2.655 | -2.469 |

Table 1 Linear model loss functions

| Terms | Estimates | ERR | Stdev |
|---|---|---|---|
| y(t- 1)= | 0.1232e+01 | (0.886e+02) | (0.718e-01) |
| u(t- 0)= | 0.6999e-01 | (0.104e+02) | (0.130e-02) |
| u(t- 1)= | 0.7342e-01 | (0.934e+00) | (0.607e-02) |
| Constant term= | -.6913e-02 | (0.354e-01) | (0.142e-02) |
| u(t- 2)= | -.6836e-01 | (0.217e-01) | (0.839e-02) |
| y(t- 2)= | -.2025e+00 | (0.170e-01) | (0.540e-01) |
| y(t- 3)= | -.1032e+00 | (0.490e-02) | (0.191e-01) |
| e(t- 2)= | 0.1127e+00 | (0.132e-03) | (0.671e-01) |
| e(t- 1)= | 0.1025e+00 | (0.135e-03) | (0.850e-01) |

Table 2 Linear model estimation




(a) Model validity tests

(b) Output prediction

Figure 2

validity tests eqn (58) are shown in Fig.2a. Notice that in Fig.2a the traditional linear model validity tests indicate that the model is adequate $\varphi_{\xi\xi}(\tau) = \delta(\tau)$, $\varphi_{u\xi}(\tau) = 0$ ($\xi(t) = \hat{\varepsilon}(t)$) and the predicted output of the model in Fig.2b looks quite good. Inspection of the nonlinear model validity tests $\varphi_{u^2,\xi}(\tau)$ and $\varphi_{u^2,\xi}(\tau)$ however shows that these are well outside the 95% confidence bounds indicating that nonlinear terms are missing from the model, and that the parameter estimates are highly biased.

## 4.2 Nonlinear Identification

The linear identification results suggest that $n_u = n_y = n_\varepsilon = 3$ and $d = 0$ would be a good starting point for the nonlinear search. Inputting these values into the forward regression estimator setting the degree of nonlinearity $\ell = 2$ initially, $C_d = 0.001$, $C_{de} = 0.0001$ with five iterations of the noise model produced the results in Table 3a. There are fifty five possible terms in the model with this specification and the algorithm has indicated that only those in Table 3a are significant. A comparison with the linear model Table 2 shows that the additional terms $y^2(k-1)$ and $\xi^2(k-3)$ have been included. For the purpose of illustration the nonlinear estimates

in Table 3a were then optimised using a prediction error algorithm [Leontaritis and Billings 1988] to produce the estimates in Table 3b. A comparison of Tables 3a and 3b shows that there is little difference between the two demonstrating the accuracy of the orthogonal methods. Fig 3 shows the model validity tests of the model in Table 3b and since all the plots are within the 95% confidence bands this indicates that the model is now acceptable. The predicted output of the nonlinear model is illustrated in Fig 3b. A comparison of Figs 2(a),(b) the linear model case with Figs 3(a),(b) clearly shows the improvement that has been achieved by the addition of just one nonlinear process model term.

Estimates of the nonlinear frequency response functions can now be computed from the estimated nonlinear model by applying the probing method described in section 2.2. The probing method is applied after discarding the noise model to yield from Table 3(b) the process model

$$y(k) = 0.1758y(k-1)+0.0623u(k)+0.1616u(k-1)$$
$$-0.03839y^2(k-1)+0.569y(k-2)+0.03143u(k-2) \qquad (60)$$

Probing eqn (60) yields the frequency response functions illustrated in Fig 4. All the frequency response plots are against normalised frequency. Multiplying the normalised frequency by the sampling frequency (31.25Hz in this example) gives Hertz. Hence $H_1(f)$ in Fig 4(a) is plotted for the range of normalised frequencies $0 \leq f \leq 0.16$ which corresponds to 0 to 5Hz in actual frequency. $H_2(f_1,f_2)$ is plotted for normalised frequencies $0 \leq f_1 \leq 0.16$, $-0.16 \leq f_2 \leq 0.16$ and $H_3(f_1,f_2,f_3)$ is displayed over the same range but with $f_1 = f_3$. Because the model eqn (60) contains a nonlinear term in $y(\cdot)$ $H_4(\cdot)$, $H_5(\cdot)$ etc will exist and although these can easily be computed analytically from the NARMAX it is difficult to see how to display them. The true frequency response functions of eqn (59) are illustrated in Part II of the paper and a comparison of these with the estimates in Fig 4 demonstrates the effectiveness of the current algorithms.

| Terms | Estimates | ERR | Stdev |
|---|---|---|---|
| y(t- 1)= | 0.1572e+00 | (0.886e+02) | (0.354e-01) |
| u(t- 0)= | 0.6230e-01 | (0.104e+02) | (0.659e-03) |
| u(t- 1)= | 0.1633e+00 | (0.934e+00) | (0.347e-02) |
| y(t- 1)*y(t- 1)= | -.3822e-01 | (0.845e-01) | (0.972e-03) |
| y(t- 2)= | 0.5849e+00 | (0.234e-01) | (0.298e-01) |
| u(t- 2)= | 0.3275e-01 | (0.290e-02) | (0.257e-02) |
| e(t- 2)= | -.3902e+00 | (0.158e-02) | (0.494e-01) |
| e(t- 3)*e(t- 3)= | -.1049e+02 | (0.274e-03) | (0.329e+01) |
| e(t- 3)= | -.1005e+00 | (0.114e-03) | (0.480e-01) |

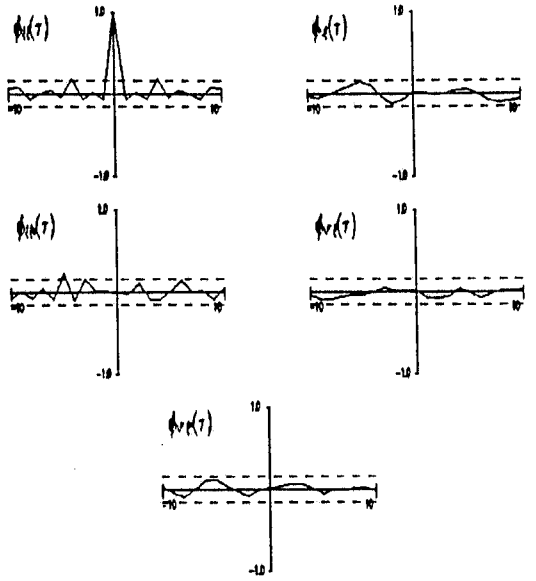
(a) Initial nonlinear estimate

Loss function = -.470686e+01

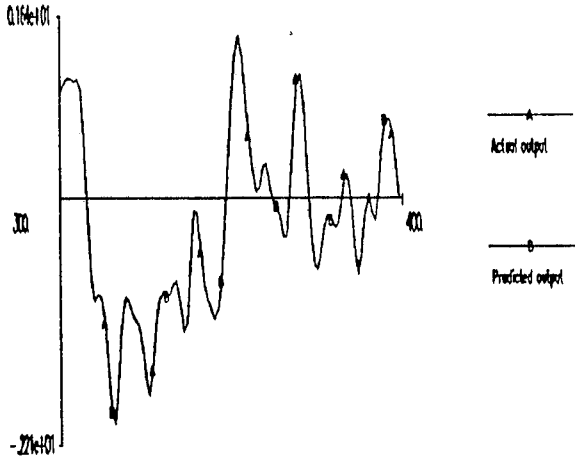
| Terms | Estimates | Stdev |
|---|---|---|
| y(t- 1)= | 0.17582e+00 | (0.33750e-01) |
| u(t- 0)= | 0.62366e-01 | (0.65280e-03) |
| u(t- 1)= | 0.16160e+00 | (0.33187e-02) |
| y(t- 1)*y(t- 1)= | -.38395e-01 | (0.94737e-03) |
| y(t- 2)= | 0.56907e+00 | (0.28441e-01) |
| u(t- 2)= | 0.31425e-01 | (0.24540e-02) |
| e(t- 2)= | -.38415e+00 | (0.46923e-01) |
| e(t- 3)*e(t- 3)= | -.39611e-01 | (0.14158e+00) |
| e(t- 3)= | -.86674e-01 | (0.45860e-01) |

(b) Optimised nonlinear estimate

Table 3



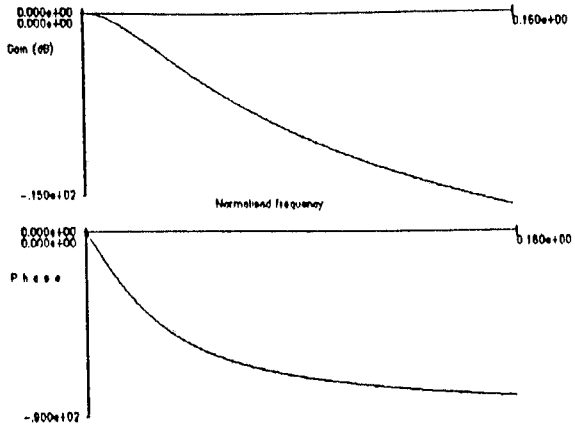
(a) Model validity tests



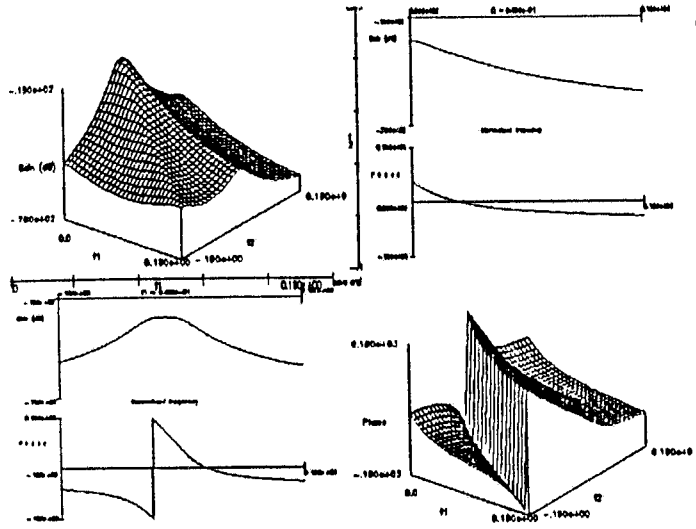

Predicted output superimposed on actual output

(b) Output prediction

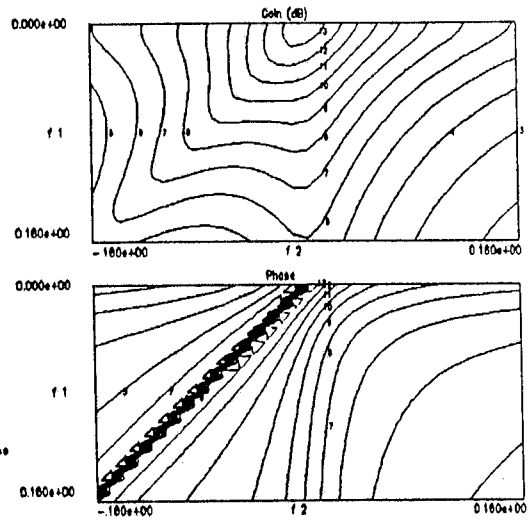
Figure 3



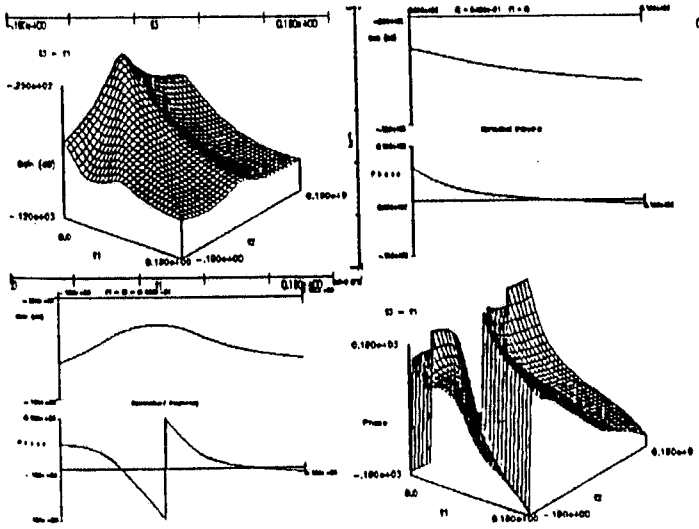

(a) $H_1(f)$



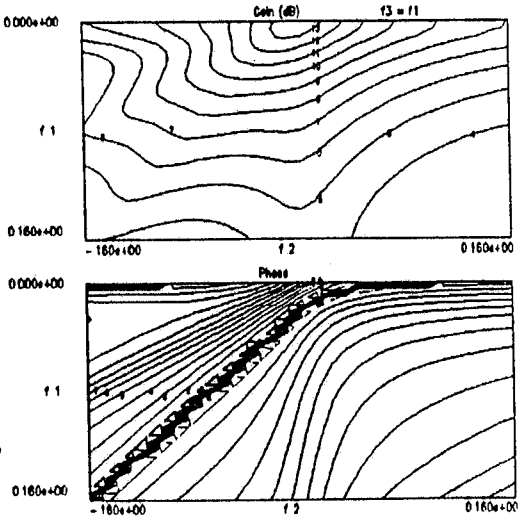
(b) $H_2(f_1,f_2)$




(c) $H_2(f_1,f_2)$



(d) $H_3(f_1,f_2,f_3)$, $f_1=f_3$




(e) $H_3(f_1,f_2,f_3)$, $f_1=f_3$

Figure 4 Estimated nonlinear frequency response functions

## 5. Conclusions

A new methodology for identifying the generalised frequency response functions of a wide class of nonlinear system has been presented. Although the derivation of the algorithms often involves complex mathematics the final procedures that are implemented are particularly simple, and are both easy to use and interpret. Whilst other more sophisticated algorithms have been developed to provide alternative solutions to the problems posed in the present paper [Leontaritis and Billings 1987,1988] the combination of the orthogonal estimator and the probing algorithm provide a powerful toolkit for the analysis of non-linear systems in the frequency domain. The methods are of course upwardly extendable to MIMO systems etc and these results will appear in forthcoming publications. The second part of this paper will consider the interpretation of the nonlinear frequency response functions and Part III will describe in detail the application of the methods to some case study examples.

## Acknowledgements

One of the authors (SAB) gratefully acknowledges that this work was supported by the Science and Engineering Research Council Ref GR/D/75809. Both authors wish to thank Prof G Tomlinson of Heriot-Watt University for numerous stimulating discussions on the topics in this paper.

## References

Bedrosian E, Rice SO (1971): The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputs; ProcIEEE, 59, pp1688-1707.

Billings SA (1980): Identification of nonlinear systems - a survey; Proc IEE, 127, pp272-285.

Billings SA, Chen S (1988): Identification of nonlinear rational systems using a prediction error estimation algorithm; Int J Systems Sci, (to appear).

Billings SA, Korenberg MJ, Chen S (1988): Identification of nonlinear output-affine systems using an orthogonal least squares algorithm; Int J Systems Sci, 19, pp1559-1568.

Billings SA, Leontaritis IJ (1981): Identification of nonlinear systems using parameter estimation techniques; IEE Conf Publ 194, Warwick Univ, pp183-187.

Billings SA, Tsang KM (1987): Estimating higher order spectra; 5th IMAC, London.

Billings SA, Tsang KM, Tomlinson GR (1988): Application of the NARMAX method to nonlinear frequency response estimation; 6th IMAC Orlando.

Billings SA, Tsang KM (1988): Spectral analysis of block structured systems; submitted for publication.

Billings SA, Voon WSF (1986): Correlation based model validity tests for nonlinear models; Int J Control, 44, pp235-244.

Brillinger DR, Rosenblatt M (1967): Asymptotic theory of estimates of k'th order spectra; in B Harris (ED): Spectral Analysis of Time Series, Wiley.

Bussgang JJ, Ehrman L, Graham JW (1974): Analysis of nonlinear systems with multiple inputs; Proc IEEE, 62, pp1088-1119.

Chen S, Billings SA (1988): Representations of nonlinear systems: the NARMAX model; Int J Control, (to appear).

Chua LO, Ng CY (1979): Frequency domain analysis of nonlinear systems: formulation of transfer functions: general theory; Electronic Cts and Systems, 3, pp165-185 and 257-269.

Leontaritis IJ, Billings SA (1985): Input-output parametric models for nonlinear systems. Part I - Deterministic nonlinear systems. Part II - Stochastic nonlinear systems; Int J Control, 41, pp303-344.

Leontaritis IJ, Billings SA (1987): Model selection and validation methods for nonlinear systems; Int J Control, 45, pp311-341.

Leontaritis IJ, Billings SA (1987b): Experiment design and indentifiability of nonlinear systems; Int J Systems Sci, 18, pp189-202.

Leontaritis IJ, Billings SA (1988): Prediction error estimation for nonlinear stochastic systems; Int J Systems Sci, 19, pp519-536.

Korenberg MJ, Billings SA, Liu YP (1988): An orthogonal parameter estimation algorithm for nonlinear stochastic systems; Int J Control, 48, pp193-210.

Marmarelis PZ, Marmarelis VZ (1978): Analysis of physiological systems - the white noise approach; Plenum Press.

Peyton Jones J, Billings SA (1988): Prediction error filtering to detect nonlinearity, (in preparation).

Schetzen M (1980): The Volterra and Wiener theories of nonlinear systems; Wiley.

Tomlinson GR (1987): Developments in the use of the Hilbert transform for detecting and quantifying nonlinearity associated with frequency response functions; Mechanical Syst Sig Proc, 1, pp151-171.

Vinh TT, Chonychai HL, Djouder M (1987): Second order transfer function computation and physical interpretation; 5th IMAC, London.