

*promoting access to White Rose research papers*



**Universities of Leeds, Sheffield and York**  
**<http://eprints.whiterose.ac.uk/>**

---

White Rose Research Online URL for this paper:

<http://eprints.whiterose.ac.uk/77323/>

---

**Paper:**

Deckelnick, K, Elliott, CM and Ranner, T (2013) *Unfitted finite element methods using bulk meshes for surface partial differential equations*. (Unpublished)

<http://arxiv.org/abs/1312.2905>

---

# UNFITTED FINITE ELEMENT METHODS USING BULK MESHES FOR SURFACE PARTIAL DIFFERENTIAL EQUATIONS

KLAUS DECKELNICK <sup>\*</sup>, CHARLES M. ELLIOTT <sup>†</sup>, AND THOMAS RANNER <sup>‡</sup>

**Abstract.** In this paper, we define new unfitted finite element methods for numerically approximating the solution of surface partial differential equations using bulk finite elements. The key idea is that the  $n$ -dimensional hypersurface,  $\Gamma \subset \mathbb{R}^{n+1}$ , is embedded in a polyhedral domain in  $\mathbb{R}^{n+1}$  consisting of a union,  $\mathcal{T}_h$ , of  $(n+1)$ -simplices. The finite element approximating space is based on continuous piece-wise linear finite element functions on  $\mathcal{T}_h$ . Our first method is a sharp interface method, *SIF*, which uses the bulk finite element space in an approximating weak formulation obtained from integration on a polygonal approximation,  $\Gamma_h$ , of  $\Gamma$ . The full gradient is used rather than the projected tangential gradient and it is this which distinguishes *SIF* from the method of [42]. The second method, *NBM*, is a narrow band method in which the region of integration is a narrow band of width  $O(h)$ . *NBM* is similar to the method of [13] but again the full gradient is used in the discrete weak formulation. The a priori error analysis in this paper shows that the methods are of optimal order in the surface  $L^2$  and  $H^1$  norms and have the advantage that the normal derivative of the discrete solution is small and converges to zero. Our third method combines bulk finite elements, discrete sharp interfaces and narrow bands in order to give an unfitted finite element method for parabolic equations on evolving surfaces. We show that our method is conservative so that it preserves mass in the case of an advection diffusion conservation law. Numerical results are given which illustrate the rates of convergence.

**Key words.** Unfitted finite elements; cut cells; error analysis; narrow band; sharp interface; elliptic and parabolic surface equations

## 1. Introduction.

**1.1. Model equations and motivation.** In this article we propose and analyse numerical methods based on bulk finite element meshes for the following model elliptic equation on a stationary surface.

*Model elliptic equation on stationary surface:* Let  $\Gamma$  be a smooth hypersurface in  $\mathbb{R}^{n+1}$  and  $f \in L^2(\Gamma)$ . We seek solutions  $u: \Gamma \rightarrow \mathbb{R}$  of

$$(1.1) \quad -\Delta_\Gamma u + u = f \quad \text{on } \Gamma.$$

The methods can be extended in natural ways to deal with variable coefficients and nonlinearities. The approach may be extended to the following advection diffusion equation on a moving surface.

*Model parabolic equation on evolving surface:* Let  $\{\Gamma(t)\}$  be a family of smooth hypersurfaces in  $\mathbb{R}^{n+1}$  for  $t \in [0, T]$ . We seek solutions  $u: \bigcup_t \Gamma(t) \times \{t\}$  of the advection diffusion equation

$$(1.2a) \quad \partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u = f \quad \text{on } \bigcup_{t \in (0, T)} \Gamma(t) \times \{t\},$$

$$(1.2b) \quad u(\cdot, 0) = u_0 \quad \text{on } \Gamma(0).$$

---

<sup>\*</sup>The work of C. M. Elliott was supported by the Warwick Impact Fund. The work of T. Ranner was supported by an EPSRC Ph.D. studentship (Grant EP/P504333/1 and EP/P50516X/1) and the Warwick Impact Fund.

<sup>†</sup>Otto-von-Guericke-Universität Magdeburg, Institut für Analysis und Numerik, Universitätsplatz 2, 39106 Magdeburg, Germany (klaus.deckelnick@ovgu.de)

<sup>‡</sup>Mathematics Institute, Zeeman Building, University of Warwick, Coventry. CV4 7AL. UK (C.M.Elliott@warwick.ac.uk, T.Ranner@warwick.ac.uk)

Here,  $\partial^\bullet u$  denotes the material derivative of  $u$  and  $v$  is the velocity vector. See §5 for notation.

Surface partial differential equations or partial differential equations (PDEs) on manifolds arise in a wide variety of applications in materials science, fluid dynamics and biology, [54, 39, 49, 5, 40, 37, 27, 33, 28, 2, 32, 30, 46, 38]. Often the surface on which the PDE holds is unknown and has to be found as part of the solution process. Thus complex applications involving surfaces and interfaces frequently require the formulation and approximation of equations on unknown stationary and moving surfaces and are not only coupled to equations for the surface but also to equations holding in the bulk. A number of computational approaches have been developed in recent years, see [24]. They are often designed in the context of solving the surface equations in a more complex application. In particular we mention:-

*Surface finite elements on triangulated surfaces:* This approach was pioneered in [18] for computing solutions to the Poisson equation using piece-wise linear elements on triangulated surfaces. This was extended to nonlinear and fourth order surface parabolic equations in [20]. Using an appropriate weak formulation and the transport property of finite element basis functions an evolving surface finite element method was devised in [19] in order to treat conservation laws on moving surfaces. The key idea is to use the Leibniz (or transport) formula for the time derivative of integrals over moving surfaces in order to derive weak and variational formulations. Further numerical analysis of surface finite element methods may be found in [16, 15, 20, 25, 23, 26].

*Bulk finite element or finite difference meshes for the approximation of implicit surface formulations:* The idea here is to use implicit surface formulations. The starting point is to use a level set function  $\Phi$  to define a degenerate partial differential equation whose solution solves the surface equation on all level sets of  $\Phi$ . Such methods are formulated in [9, 35, 10, 21, 22]. Approaches to obtaining a non-degenerate level set equation may be found in [34, 11].

*Bulk finite element or finite difference meshes on narrow bands:* A natural disadvantage of the approach 1.1 is the fact that formulation is in the ambient space rather than just on the surface. This leads to solving PDEs in one space dimension higher than the hypersurface. Narrow band methods confine the use of bulk finite elements to a narrow band around the surface and the region of integration is the narrow band. In particular the level set approach in [13] is confined to an  $O(h)$ -narrow band. Another direction is to solve the bulk PDE in a narrow band of width  $\epsilon$ , say. The solution of this problem converges to the solution of the surface PDE. This is the basis of the finite difference method in [50].

*Bulk finite element methods and phase field diffuse interfaces:* This approach is motivated by the diffuse interfaces that arise in phase field approximations of interface problems, see e.g. [12]. The idea is to exploit the methodology to generate methods for solving partial differential equations on the interfaces, [47], [29].

*Bulk finite element meshes and sharp interface weak forms* If one takes the width of the narrow band to be zero in the approach of [13] one obtains the appealing method of [42] for equations on stationary surfaces. The authors prove that for piece-wise linear elements one obtains second order convergence in the  $L^2$  norm. There is an issue about the dimension of the resulting linear algebraic equations and their conditioning. This is addressed in [41]. For other developments we refer to [17] which concerns an adaptive version and [44] which concerns the surface meshes induced by the bulk mesh on level sets. This approach has been extended recently to a novel Eulerian space-time formulation using space-time bulk finite element meshes, [43].

An important feature of the methods described above is the avoidance of *charts* both in the problem formulation and the numerical methods. For example, the surface finite element method is based simply on triangulated surfaces and requires the geometry solely through the knowledge of the vertices of the triangulation whereas methods based on implicit surfaces require only the level set function  $\Phi$  which encodes all the necessary geometry.

Another feature of some of these methods is the use of unfitted bulk meshes. Here we use the terminology *unfitted finite element methods* (sometimes called cut cell methods) when the underlying meshes that form the computational domain are not fitted to the domain in which the PDE holds. The motivation for using finite element spaces on meshes not fitting to the domain came from the desire to solve free or moving boundary problems. Such methods were introduced in [3, 4] for elliptic equations in curved domains. See also [36, 8, 31]. In this setting we are concerned with bulk meshes independent of the surface.

**1.2. The new methods.** The new unfitted finite element methods for surface elliptic equations proposed in this paper are variants of the bulk finite element approaches using a sharp interface or a narrow band. The new scheme for advection diffusion on an evolving surface is a hybrid of these. In the following we sketch the main ideas of these methods describing the details in §3-§5.

**1.2.1. Sharp interface method (SIF).** Given an interpolation  $\Gamma_h$  of  $\Gamma$ , we use a bulk finite element space  $V_h^I$  of the form

$$V_h^I = \{\phi_h \in C^0(U_h^I) \mid \phi_h|_T \in P_1(T) \text{ for each } T \in \mathcal{T}_h^I\},$$

where  $\mathcal{T}_h^I$  is a set of elements which intersect  $\Gamma_h$  and  $U_h^I = \bigcup_{T \in \mathcal{T}_h^I} T$ , see §3. The discrete scheme approximating the model elliptic equation (1.1) is:- Find  $u_h \in V_h^I$  such that

$$(1.3) \quad \int_{\Gamma_h} (\nabla u_h \cdot \nabla \phi_h + u_h \phi_h) d\sigma_h = \int_{\Gamma_h} f^e \phi_h d\sigma_h \quad \text{for all } \phi_h \in V_h^I,$$

where  $f^e$  is an extension of  $f$ .

The method is related to the following method of Olshanskii et. al., introduced in [42]:- Find  $u_h \in V_h^\Gamma$  such that

$$\int_{\Gamma_h} (\nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \phi_h + u_h \phi_h) d\sigma_h = \int_{\Gamma_h} f^e \phi_h d\sigma_h \quad \text{for all } \phi_h \in V_h^\Gamma.$$

Apart from the use of the full gradient in (1.3) as opposed to the tangential gradient, another difference relates to the use of the finite element space  $V_h^\Gamma$ , which essentially consists of the traces on  $\Gamma_h$  of elements in  $V_h^I$ . However, while  $V_h^I$  has a natural basis, this does not seem to be the case for  $V_h^\Gamma$ .

**1.2.2. Narrow band method (NBM).** We use the bulk finite element space  $V_h^B$  on the triangulation  $\mathcal{T}_h^B$

$$V_h^B = \{\phi_h \in C^0(U_h^B) \mid \phi_h|_T \in P_1(T) \text{ for each } T \in \mathcal{T}_h^B\}.$$

Here  $\mathcal{T}_h^B$  consists of those triangles intersecting a narrow band domain  $D_h$  defined by the  $\pm h$  level sets of an interpolated level set function  $I_h\Phi$  and  $U_h^B = \bigcup_{T \in \mathcal{T}_h^B} T$ . The

discrete scheme approximating the model elliptic equation (1.1) is:- Find  $u_h \in V_h^B$  such that

$$(1.4) \quad \int_{D_h} (\nabla u_h \cdot \nabla \phi_h + u_h \phi_h) |\nabla I_h \Phi| \, dx = \int_{D_h} f^e \phi_h |\nabla I_h \Phi| \, dx \quad \text{for all } \phi_h \in V_h^B.$$

This is similar to the method in [13] except that *NBM* uses the full instead of projected gradients thus avoiding the resulting degeneracy. As a result we are able to prove an optimal  $L^2$ -error bound which was not obtained for the method in [13]. It is also the case that the normal derivative of the discrete solution converges to zero.

**1.2.3. Hybrid unfitted evolving surface method.** The discrete problem approximating (1.2) is:- Given  $u_h^m \in V_h^m$ ,  $m = 0, \dots, N-1$ , find  $u_h^{m+1} \in V_h^{m+1}$  such that

$$(1.5) \quad \begin{aligned} & \int_{\Gamma_h^{m+1}} u_h^{m+1} \phi_h \, d\sigma_h - \int_{\Gamma_h^m} u_h^m \phi_h (\cdot + \tau_m v^{e,m+1}) \, d\sigma_h \\ & + \frac{\tau_m}{2h} \int_{D_h^{m+1}} \nabla u_h^{m+1} \cdot \nabla \phi_h |\nabla I_h \Phi^{m+1}| \, dx = \tau_m \int_{\Gamma_h^{m+1}} f^{e,m+1} \phi_h \, d\sigma_h \end{aligned}$$

for all  $\phi_h \in V_h^{m+1}$ . Here  $v^{e,m}$  denotes an extension of the surface velocity at time level  $m$ . We use time step labelled analogues of the notation for the narrow band method, see §5 for the details. Here,  $u_h^0$  is appropriate initial data. An important property of solutions of (1.2) is conservation of mass in the case that  $f \equiv 0$  and our numerical scheme preserves this property under some mild constraints on the discretization parameters, see §5.

**1.3. Outline.** The paper is organized as follows: in §2 we introduce our notation and collect some auxiliary results. In §3 and §4 we present and analyse unfitted methods for the model elliptic equation (1.1). In §5 we describe how a combination of these two approaches can be used to calculate solutions of the advection-diffusion equation on evolving hypersurfaces, (1.2). Details of the implementation and several numerical examples illustrating the orders of convergence are presented in §6.

## 2. Preliminaries.

**2.1. Surface calculus.** Let  $\Gamma$  be a connected compact smooth hypersurface embedded in  $\mathbb{R}^{n+1}$  ( $n = 1, 2$ ). We assume that there exists a smooth function  $\Phi : U \rightarrow \mathbb{R}$  such that

$$\Gamma = \{x \in U \mid \Phi(x) = 0\} \quad \text{and} \quad \nabla \Phi(x) \neq 0, x \in U,$$

where  $U$  is an open neighbourhood of  $\Gamma$ . For a function  $z : \Gamma \rightarrow \mathbb{R}$  we define its tangential gradient by

$$(2.1) \quad \nabla_\Gamma z(p) := \nabla \tilde{z}(p) - (\nabla \tilde{z}(p) \cdot \nu(p)) \nu(p), \quad p \in \Gamma,$$

where  $\tilde{z} : U \rightarrow \mathbb{R}$  is an arbitrary extension of  $z$  to  $U$  and

$$\nu(x) = \frac{\nabla \Phi(x)}{|\nabla \Phi(x)|}$$

is a unit vector to the level sets of  $\Phi$ . It can be shown that  $\nabla_\Gamma z(p)$  is independent of the particular choice of  $\tilde{z}$ . We denote by  $\underline{D}_i z$ ,  $1 \leq i \leq n+1$  the components of  $\nabla_\Gamma z$ .

Furthermore, we let

$$\Delta_\Gamma z = \nabla_\Gamma \cdot \nabla_\Gamma z = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i z$$

be the Laplace-Beltrami operator of  $z$ .

In what follows it will be convenient to use special coordinates which are adapted to  $\Phi$ . Consider for  $p \in \Gamma$  the system of ODEs

$$(2.2) \quad \gamma'_p(s) = \frac{\nabla \Phi(\gamma_p(s))}{|\nabla \Phi(\gamma_p(s))|^2}, \quad \gamma_p(0) = p.$$

It can be shown that there exists  $\delta > 0$  so that the solution  $\gamma_p$  of (2.2) exists uniquely on  $(-\delta, \delta)$  uniformly in  $p \in \Gamma$ , so that we can define the mapping  $F : \Gamma \times (-\delta, \delta) \rightarrow \mathbb{R}^{n+1}$  by

$$(2.3) \quad F(p, s) := \gamma_p(s), \quad p \in \Gamma, |s| < \delta.$$

Since  $\frac{d}{ds} \Phi(\gamma_p(s)) = 1$  and  $\gamma_p(0) = p \in \Gamma$ , we infer that  $\Phi(\gamma_p(s)) = s, |s| < \delta$  and hence that  $x = F(p, s)$  implies that  $|\Phi(x)| < \delta$ . As a result, we deduce that  $F$  is a diffeomorphism of  $\Gamma \times (-\delta, \delta)$  onto  $U_\delta := \{x \in U \mid |\Phi(x)| < \delta\}$  with inverse

$$(2.4) \quad F^{-1}(x) = (p(x), \Phi(x)), \quad x \in U_\delta,$$

where  $p : U_\delta \rightarrow \mathbb{R}^{n+1}$  satisfies  $p(x) \in \Gamma, x \in U_\delta$ . For later purposes it is convenient to expand  $p$  and its derivatives in terms of  $\Phi$ . Let us fix  $x \in U_\delta$  and define the function

$$\eta(\tau) := F(p(x), (1 - \tau)\Phi(x)), \tau \in [0, 1].$$

Since  $\frac{\partial F}{\partial s}(p, s) = \gamma'_p(s)$  we have

$$\eta'(\tau) = -\Phi(x) \gamma'_{p(x)}((1 - \tau)\Phi(x)) = -\Phi(x) \frac{\nabla \Phi(\gamma_{p(x)}((1 - \tau)\Phi(x)))}{|\nabla \Phi(\gamma_{p(x)}((1 - \tau)\Phi(x)))|^2}.$$

Observing that  $\gamma_{p(x)}(\Phi(x)) = F(p(x), \Phi(x)) = x$  and using similar arguments to calculate  $\eta''(\tau)$  we find that

$$\eta'_k(0) = -\Phi(x) \frac{\Phi_{x_k}(x)}{|\nabla \Phi(x)|^2}, \quad \eta''_k(0) = \Phi(x)^2 \sum_{l,r=1}^{n+1} \left( \delta_{kr} - \frac{2\Phi_{x_k}(x)\Phi_{x_r}(x)}{|\nabla \Phi(x)|^2} \right) \frac{\Phi_{x_l}(x)\Phi_{x_l x_r}(x)}{|\nabla \Phi(x)|^4},$$

$k = 1, \dots, n+1$ . Since  $\eta(1) = p(x), \eta(0) = x$  we deduce with the help of Taylor's theorem that

$$(2.5) \quad p_k(x) = x_k - \Phi(x) \frac{\Phi_{x_k}(x)}{|\nabla \Phi(x)|^2} + \frac{1}{2} \Phi(x)^2 \sum_{l,r=1}^{n+1} \left( \delta_{kr} - \frac{2\Phi_{x_k}(x)\Phi_{x_r}(x)}{|\nabla \Phi(x)|^2} \right) \frac{\Phi_{x_l}(x)\Phi_{x_l x_r}(x)}{|\nabla \Phi(x)|^4} + \Phi(x)^3 r_k(x), \quad k = 1, \dots, n+1,$$

where  $r_k$  are smooth functions. In a similar way we may write

$$(2.6) \quad \nabla \Phi(x) = \nabla \Phi(p(x)) + \Phi(x)G(x),$$

where  $G(x) = \int_0^1 D^2\Phi(p(x), \tau\Phi(x)) \frac{\partial F}{\partial s}(p(x), \tau\Phi(x)) d\tau$ .

Let us next use the function  $p$  in order to define a particular extension of  $z : \Gamma \rightarrow \mathbb{R}$  by

$$(2.7) \quad z^e(x) := z(p(x)), \quad x \in U_\delta.$$

Since  $p(F(p(x), s)) = p(x)$  we deduce that  $s \mapsto z^e(F(p(x), s))$  is independent of  $s$  and therefore

$$(2.8) \quad \nabla z^e(x) \cdot \nu(x) = 0, \quad x \in U_\delta.$$

In order to express the derivatives of  $z^e$  in terms of the tangential derivatives of  $z$  we first deduce from (2.5) that

$$\begin{aligned} p_{k,x_i}(x) = & \delta_{ik} - \frac{\Phi_{x_k}(x)\Phi_{x_i}(x)}{|\nabla\Phi(x)|^2} - \frac{\Phi(x)\Phi_{x_k x_i}(x)}{|\nabla\Phi(x)|^2} + 2\Phi(x)\Phi_{x_k}(x) \sum_{l=1}^{n+1} \frac{\Phi_{x_l}(x)\Phi_{x_l x_i}(x)}{|\nabla\Phi(x)|^4} \\ & + \Phi(x)\Phi_{x_i}(x) \sum_{l,r=1}^{n+1} \left( \delta_{kr} - \frac{2\Phi_{x_k}(x)\Phi_{x_r}(x)}{|\nabla\Phi(x)|^2} \right) \frac{\Phi_{x_l}(x)\Phi_{x_l x_r}(x)}{|\nabla\Phi(x)|^4} + \Phi(x)^2 \alpha_k^i(x). \end{aligned}$$

Combining this relation with (2.6) we deduce that

$$(2.9) \quad p_{k,x_i}(x) = \delta_{ik} - \nu_i(p(x))\nu_k(p(x)) + a_{ik}(x)\Phi(x),$$

$$(2.10) \quad \begin{aligned} p_{k,x_i x_j}(x) = & -\frac{\Phi_{x_i}(x)\Phi_{x_k x_j}(x)}{|\nabla\Phi(x)|^2} - \frac{\Phi_{x_j}(x)\Phi_{x_k x_i}(x)}{|\nabla\Phi(x)|^2} \\ & + \frac{\Phi_{x_i}(x)\Phi_{x_j}(x)}{|\nabla\Phi(x)|^2} \sum_{l=1}^{n+1} \frac{\Phi_{x_l}(x)\Phi_{x_k x_l}(x)}{|\nabla\Phi(x)|^2} + \beta_k^{ij}(x)\nu_k(p(x)) + \gamma_k^{ij}(x)\Phi(x) \end{aligned}$$

where  $a_{ik}, \beta_k^{ij}, \gamma_k^{ij}$  are smooth functions. Differentiating (2.7) and using (2.9), (2.10) as well as the fact that  $\sum_{k=1}^{n+1} \underline{D}_k z(p(x))\nu_k(p(x)) = 0$  we obtain

$$(2.11) \quad \nabla z^e(x) = (I + \Phi(x)A(x))\nabla_\Gamma z(p(x)),$$

$$(2.12) \quad \begin{aligned} & \frac{1}{|\nabla\Phi(x)|} \nabla \cdot (|\nabla\Phi(x)|\nabla z^e(x)) \\ & = (\Delta_\Gamma z)(p(x)) + \Phi(x) \left( \sum_{k,l=1}^{n+1} b_{lk}(x)\underline{D}_l \underline{D}_k z(p(x)) + \sum_{k=1}^{n+1} c_k(x)\underline{D}_k z(p(x)) \right), \end{aligned}$$

where  $A = (a_{ik}), b_{lk}$  and  $c_k$  are again smooth.

**2.2. Bulk finite element space and inequalities.** In what follows we assume that the set  $U$  is polyhedral. Let  $(\mathcal{T}_h)_{0 < h \leq h_0}$  be a family of triangulations consisting of closed simplices  $T$  with maximum mesh size  $h := \max_{T \in \mathcal{T}_h} h(T)$ , where  $h(T) = \text{diam}(T)$ . We assume that  $(\mathcal{T}_h)_{0 < h \leq h_0}$  is regular in the sense that there exists  $\rho > 0$  such that

$$(2.13) \quad \text{diam } B_T \geq \rho h(T), \quad \text{for all } T \in \mathcal{T}_h, 0 < h \leq h_0,$$

where  $B_T$  is the largest ball contained in  $T$ . Let us denote by  $X_h$  the space of linear finite elements

$$X_h = \{\phi_h \in C^0(\bar{U}) \mid \phi_h|_T \in P_1(T), T \in \mathcal{T}_h\},$$

and by  $I_h : C^0(\bar{U}) \rightarrow X_h$  the usual Lagrange interpolation operator. We have, for  $\eta \in W^{2,p}(U)$

$$(2.14) \quad \|\eta - I_h \eta\|_{W^{k,p}(T)} \leq Ch(T)^{2-k} \|\eta\|_{W^{2,p}(T)}, \quad T \in \mathcal{T}_h,$$

for  $k = 0, 1$  and  $1 < p \leq \infty$  with  $2 - \frac{n+1}{p} > 0$ . As a consequence,

$$(2.15) \quad \|\Phi - I_h \Phi\|_{L^\infty(U)} + h \|\nabla(\Phi - I_h \Phi)\|_{L^\infty(U)} \leq Ch^2,$$

so that we may assume that there exist constants  $c_0, c_1$  such that

$$(2.16) \quad c_0 \leq |\nabla I_h \Phi(x)| \leq c_1, \quad x \in U, 0 < h \leq h_0.$$

Let us next define

$$\begin{aligned} \Gamma_h &:= \{x \in U \mid I_h \Phi(x) = 0\} \\ D_h &:= \{x \in U \mid |I_h \Phi(x)| < h\}, \end{aligned}$$

as approximations of the given hypersurface  $\Gamma$  and the neighbourhood  $D^h := \{x \in U \mid |\Phi(x)| < h\}$ ; see Figure 2.1 for example. Note that  $\Gamma_h$  is a polygon whose facets are line segments if  $n = 1$  and a polyhedral surface whose facets consist of triangles or quadrilaterals if  $n = 2$ . The corresponding decomposition of  $\Gamma_h$  is in general not shape regular and can have arbitrary small elements.

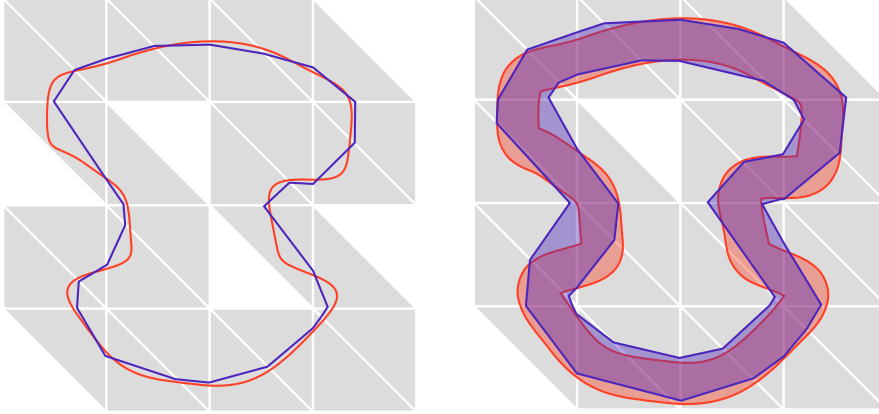


Fig. 2.1: A cartoon of the domains of the sharp interface (left) and the narrow band (right) method. The surface  $\Gamma$  resp. the set  $D^h$  is displayed in red, the approximations  $\Gamma_h$  resp.  $D_h$  in blue and the domains  $U_h^I, U_h^B$  in grey.

Furthermore, we introduce  $F_h : U \rightarrow \mathbb{R}^{n+1}$  by

$$F_h(x) := F(p(x), I_h \Phi(x)), \quad x \in U,$$

where  $F$  was defined in (2.3). From the properties of  $F$  we infer that

$$(2.17) \quad p(F_h(x)) = p(x) \quad \text{and} \quad \Phi(F_h(x)) = I_h \Phi(x), \quad \text{if } F_h(x) \in U_\delta.$$

The following lemma collects the relevant properties of  $F_h$ .



LEMMA 2.1. *There exists  $0 < h_1 \leq h_0$  such that for  $0 < h \leq h_1$  the mapping  $F_h : D_h \rightarrow D^h := \{x \in U \mid |\Phi(x)| < h\}$  is bilipschitz with  $F_h(\Gamma_h) = \Gamma$ . Furthermore,*

$$(2.18) \quad \|F_h - \text{Id}\|_{L^\infty(U)} + h \|DF_h - I\|_{L^\infty(U)} \leq ch^2,$$

$$(2.19) \quad \left\| \left| \det DF_h \right| - \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \right\|_{L^\infty(U)} \leq ch^2.$$

*Proof.* Since  $F(p(x), \Phi(x)) = x$  we deduce with the help of (2.15)

$$|F_h(x) - x| = |F(p(x), I_h \Phi(x)) - F(p(x), \Phi(x))| \leq c |I_h \Phi(x) - \Phi(x)| \leq ch^2.$$

Differentiating the relation  $F_i(p(x), \Phi(x)) = x_i$ ,  $i = 1, \dots, n+1$  we obtain

$$\sum_{k=1}^{n+1} \underline{D}_k F_i(p(x), \Phi(x)) p_{k,x_j}(x) + \frac{\partial F_i}{\partial s}(p(x), \Phi(x)) \Phi_{x_j}(x) = \delta_{ij}, \quad i, j = 1, \dots, n+1,$$

and hence

$$(2.20) \quad \begin{aligned} F_{hi,x_j}(x) &= \sum_{k=1}^{n+1} \underline{D}_k F_i(p(x), I_h \Phi(x)) p_{k,x_j}(x) + \frac{\partial F_i}{\partial s}(p(x), I_h \Phi(x)) (I_h \Phi)_{x_j}(x) \\ &= \delta_{ij} + \frac{\partial F_i}{\partial s}(p(x), \Phi(x)) (I_h \Phi - \Phi)_{x_j}(x) \\ &\quad + \sum_{k=1}^{n+1} \left( \underline{D}_k F_i(p(x), I_h \Phi(x)) - \underline{D}_k F_i(p(x), \Phi(x)) \right) p_{k,x_j}(x) \\ &\quad + \left( \frac{\partial F_i}{\partial s}(p(x), I_h \Phi(x)) - \frac{\partial F_i}{\partial s}(p(x), \Phi(x)) \right) (I_h \Phi)_{x_j}(x) \\ &= \delta_{ij} + \frac{\Phi_{x_i}(x)}{|\nabla \Phi(x)|^2} (I_h \Phi - \Phi)_{x_j}(x) + r_{ij}(x), \end{aligned}$$

where  $|r_{ij}(x)| \leq ch^2$  in view of (2.15). This implies (2.18). In particular we deduce that  $F_h$  is bilipschitz provided that  $h$  is sufficiently small, whereas the properties  $F_h(D_h) = D^h$  and  $F_h(\Gamma_h) = \Gamma$  follow from (2.17). Finally we deduce from (2.20) that

$$\begin{aligned} |\det DF_h| &= 1 + \frac{\nabla \Phi}{|\nabla \Phi|^2} \cdot \nabla (I_h \Phi - \Phi) + c_h = \frac{\nabla \Phi \cdot \nabla I_h \Phi}{|\nabla \Phi|^2} + c_h \\ &= \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} - \frac{1}{2} \left| \frac{\nabla I_h \Phi}{|\nabla I_h \Phi|} - \frac{\nabla \Phi}{|\nabla \Phi|} \right|^2 \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} + c_h = \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} + d_h, \end{aligned}$$

where  $|c_h|, |d_h| \leq ch^2$  proving (2.19).  $\square$

Given a function  $\phi : \Gamma_h \rightarrow \mathbb{R}$  we denote by  $\phi^\ell : \Gamma \rightarrow \mathbb{R}$ ,  $\phi^\ell(x) := \phi(F_h^{-1}(x))$  the lift of  $\phi$ . Note that

$$(2.21) \quad (\phi^\ell)^e(x) = \phi^\ell(p(x)) = \phi(F_h^{-1}(p(x))) = \phi(x), \quad x \in \Gamma_h,$$

since  $F_h(x) = F(p(x), 0) = p(x)$  for  $x \in \Gamma_h$ . Finally, let us introduce  $\mu_h : \Gamma_h \rightarrow \mathbb{R}$  via  $d\sigma(p(x)) = \mu_h(x) d\sigma_h(x)$ . It is well-known that

$$(2.22) \quad |1 - \mu_h| \leq ch^2 \quad \text{on } \Gamma_h.$$

Using the properties of  $F_h$  together with the coarea formula and (2.9),(2.10), (2.11), (2.22) one can prove the following result on the equivalence of certain norms.

LEMMA 2.2. *There exist constants  $c_1, c_2 > 0$  which are independent of  $h$ , such that for all  $z \in H^1(\Gamma)$*

$$\begin{aligned} c_1 \|z^e\|_{L^2(\Gamma_h)} &\leq \|z\|_{L^2(\Gamma)} \leq c_2 \|z^e\|_{L^2(\Gamma_h)} \\ c_1 \frac{1}{\sqrt{h}} \|z^e\|_{L^2(D_h)} &\leq \|z\|_{L^2(\Gamma)} \leq c_2 \frac{1}{\sqrt{h}} \|z^e\|_{L^2(D_h)} \\ c_1 \|\nabla z^e\|_{L^2(\Gamma_h)} &\leq \|\nabla_\Gamma z\|_{L^2(\Gamma)} \leq c_2 \|\nabla z^e\|_{L^2(\Gamma_h)} \\ c_1 \frac{1}{\sqrt{h}} \|\nabla z^e\|_{L^2(D_h)} &\leq \|\nabla_\Gamma z\|_{L^2(\Gamma)} \leq c_2 \frac{1}{\sqrt{h}} \|\nabla z^e\|_{L^2(D_h)}. \end{aligned}$$

If in addition  $z \in H^2(\Gamma)$  then

$$c_1 \frac{1}{\sqrt{h}} \|D^2 z^e\|_{L^2(D_h)} \leq \|z\|_{H^2(\Gamma)}.$$

### 2.3. Variational form of elliptic equation and Strang's second lemma.

It is well-known [1] that for every  $f \in L^2(\Gamma)$  there exists a unique solution  $u \in H^2(\Gamma)$  of (1.1) which satisfies

$$(2.23) \quad \|u\|_{H^2(\Gamma)} \leq c \|f\|_{L^2(\Gamma)}.$$

Let us write (1.1) in weak form

$$(2.24) \quad a(u, \varphi) = l(\varphi) \quad \text{for all } \varphi \in H^1(\Gamma),$$

where

$$a(w, \varphi) = \int_\Gamma (\nabla_\Gamma w \cdot \nabla_\Gamma \varphi + w\varphi) \, d\sigma, \quad l(\varphi) = \int_\Gamma f\varphi \, d\sigma.$$

Next, suppose that  $V_h$  is a finite-dimensional space and  $V^e := \{v^e \mid v \in H^1(\Gamma)\}$ . Assume that  $a_h : (V_h + V^e) \times (V_h + V^e) \rightarrow \mathbb{R}$  is a symmetric, positive semidefinite bilinear form which is in addition positive definite on  $V_h \times V_h$ . Furthermore, let  $l_h : V_h \rightarrow \mathbb{R}$  be linear. Then the approximate problem

$$(2.25) \quad a_h(u_h, v_h) = l_h(v_h) \quad \text{for all } v_h \in V_h$$

has a unique solution  $u_h \in V_h$ . Introducing

$$\|v\|_h := \sqrt{a_h(v, v)}, \quad v \in V_h + V^e$$

we have by Strang's Second Lemma

$$(2.26) \quad \|u^e - u_h\|_h \leq 2 \inf_{v_h \in V_h} \|u^e - v_h\|_h + \sup_{\phi_h \in V_h} \frac{|a_h(u^e, \phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h}.$$

In the following two sections we shall present two different choices of  $a_h$  and  $l_h$  along with the corresponding analysis of the resulting schemes.

### 3. Sharp interface method (SIF).

**3.1. Setting up the method.** Let us begin by observing that if  $T \in \mathcal{T}_h$  satisfies  $\mathcal{H}^n(T \cap \Gamma_h) > 0$ , then the following two cases can occur:

1.  $\Gamma_h \cap \text{int}(T) \neq \emptyset$ , in which case  $\mathcal{H}^n(\partial T \cap \Gamma_h) = 0$ ;
2.  $T \cap \Gamma_h = \partial T \cap \Gamma_h$  in which case  $T \cap \Gamma_h$  is the face between two elements.

We may now define a unique subset  $\mathcal{T}_h^I \subset \mathcal{T}_h$  by taking all elements satisfying case 1 and in case 2 taking just one of the two elements  $T$ . The numerical method does not depend on which element is chosen. We may therefore conclude that there exists  $N \subset \Gamma_h$  with  $\mathcal{H}^n(N) = 0$  and a subset  $\mathcal{T}_h^I \subset \mathcal{T}_h$  such that every  $x \in \Gamma_h \setminus N$  belongs to exactly one  $T \in \mathcal{T}_h^I$ . We then define

$$U_h^I = \bigcup_{T \in \mathcal{T}_h^I} T.$$

Clearly  $U_h^I \subseteq U_\delta$  provided that  $h$  is small enough. We define the finite element space  $V_h^I$  by

$$V_h^I = \{\phi_h \in C^0(U_h^I) \mid \phi_h|_T \in P_1(T) \text{ for each } T \in \mathcal{T}_h^I\}.$$

Note that  $\nabla \phi_h$  is defined on  $\Gamma_h \setminus N$  in view of the definition of  $\mathcal{T}_h^I$ . In particular the unit normal  $\nu_h$  to  $\Gamma_h$  is given by

$$(3.1) \quad \nu_h = \frac{\nabla I_h \Phi}{|\nabla I_h \Phi|} \quad \text{on } \Gamma_h \setminus N,$$

and we use (3.1) in order to extend  $\nu_h$  to  $U_h^I$ . Let us next turn to the approximation error for the space  $V_h^I$ . Note that for a function  $z \in H^2(\Gamma)$  we have  $z^e \in C^0(\bar{U}_\delta)$  so that  $I_h z^e$  is well-defined.

LEMMA 3.1. *Let  $z \in H^2(\Gamma)$ . Then*

$$(3.2) \quad \|z^e - I_h z^e\|_{L^2(\Gamma_h)} + h \|\nabla(z^e - I_h z^e)\|_{L^2(\Gamma_h)} \leq ch^2 \|z\|_{H^2(\Gamma)}.$$

*Proof.* We first observe that Theorem 3.7 in [42] yields

$$(3.3) \quad \|z^e - I_h z^e\|_{L^2(\Gamma_h)} + h \|\nabla_{\Gamma_h}(z^e - I_h z^e)\|_{L^2(\Gamma_h)} \leq ch^2 \|z\|_{H^2(\Gamma)}.$$

Hence, it remains to bound  $\|\nabla(z^e - I_h z^e) \cdot \nu_h\|_{L^2(\Gamma_h)}$ . To do so, we start by considering an element  $T \in \mathcal{T}_h^I$ . Then we see that

$$\begin{aligned} & \int_{T \cap \Gamma_h} |\nabla(z^e - I_h z^e) \cdot \nu_h|^2 \, d\sigma_h \\ & \leq 2 \int_{T \cap \Gamma_h} |\nabla z^e \cdot \nu_h|^2 \, d\sigma_h + 2 \int_{T \cap \Gamma_h} |\nabla(I_h z^e) \cdot \nu_h|^2 \, d\sigma_h \\ & \leq 2 \int_{T \cap \Gamma_h} |\nabla z^e \cdot (\nu_h - \nu)|^2 \, d\sigma_h + ch(T)^{-1} \int_T |\nabla(I_h z^e) \cdot \nu_h|^2 \, dx =: I_1 + I_2, \end{aligned}$$

in view of (2.8) and the fact that  $\mathcal{H}^n(T \cap \Gamma_h) \leq ch(T)^{-1} \mathcal{H}^{n+1}(T)$ . Note that by (3.1) and (2.15)

$$(3.4) \quad \|\nu - \nu_h\|_{L^\infty(T)} = \left\| \frac{\nabla \Phi}{|\nabla \Phi|} - \frac{\nabla I_h \Phi}{|\nabla I_h \Phi|} \right\|_{L^\infty(T)} \leq ch(T)$$

so that

$$I_1 \leq ch^2 \int_{T \cap \Gamma_h} |\nabla z^e|^2 d\sigma_h.$$

Furthermore, recalling (2.14) and using again (3.4)

$$I_2 \leq ch(T)^{-1} \int_T (|\nabla z^e \cdot (\nu_h - \nu)|^2 + |\nabla(z^e - I_h z^e)|^2) dx \leq ch \|z^e\|_{H^2(T)}^2.$$

We use the bounds for  $I_1, I_2$  and sum over all elements  $T \in \mathcal{T}_h^I$ , then apply Lemma 2.2 to see

$$\int_{\Gamma_h} |\nabla(z^e - I_h z^e) \cdot \nu_h|^2 d\sigma_h \leq ch^2 \|\nabla z^e\|_{L^2(\Gamma_h)}^2 + ch \|z^e\|_{H^2(D_{c_1 h})}^2 \leq ch^2 \|z\|_{H^2(\Gamma)}^2,$$

since  $T \subset D_{c_1 h}$  for all  $T \in \mathcal{T}_h^I$  in view of (2.16).  $\square$

**3.2. The method.** Let us write (1.3) in the form:- Find  $u_h \in V_h^I$  such that

$$(3.5) \quad a_h(u_h, \phi_h) = l_h(\phi_h) \quad \text{for all } \phi_h \in V_h^I,$$

where

$$a_h(w_h, \phi_h) = \int_{\Gamma_h} (\nabla w_h \cdot \nabla \phi_h + w_h \phi_h) d\sigma_h, \quad l_h(\phi_h) = \int_{\Gamma_h} f^e \phi_h d\sigma_h.$$

In order to verify that the symmetric bilinear form  $a_h$  is positive definite on  $V_h^I \times V_h^I$  we note that  $a_h(\phi_h, \phi_h) = 0$  implies that

$$\int_{\Gamma_h \cap T} (|\nabla \phi_h|^2 + \phi_h^2) d\sigma_h = 0 \quad \text{for all } T \in \mathcal{T}_h^I.$$

Since  $\mathcal{H}^n(T \cap \Gamma_h) > 0$  for  $T \in \mathcal{T}_h^I$  we infer that  $\nabla \phi_h = 0$  and hence  $\phi_h$  is constant on these elements. Using again that  $\mathcal{H}^n(T \cap \Gamma_h) > 0$  we deduce that  $\phi_h = 0$  on each  $T \in \mathcal{T}_h^I$  so that  $\phi_h \equiv 0$  in  $V_h^I$ . Hence (3.5) has a unique solution  $u_h \in V_h^I$  which satisfies

$$(3.6) \quad \|u_h\|_h = (\|\nabla u_h\|_{L^2(\Gamma_h)}^2 + \|u_h\|_{L^2(\Gamma_h)}^2)^{\frac{1}{2}} \leq c \|f^e\|_{L^2(\Gamma_h)} \leq c \|f\|_{L^2(\Gamma)}.$$

**REMARK 3.2.** *The right hand side  $l_h(\cdot)$  may be defined using other sufficiently accurate extensions of  $f$ .*

**3.3. Error analysis.** **THEOREM 3.3.** *Let  $u$  be the solution of (1.1) and  $u_h$  the solution of the finite element scheme (3.5). Then*

$$(3.7) \quad \|u^e - u_h\|_{L^2(\Gamma_h)} + h \|\nabla(u^e - u_h)\|_{L^2(\Gamma_h)} \leq ch^2 \|f\|_{L^2(\Gamma)}.$$

*Proof.* In view of the definition of  $\|\cdot\|_h$ , (2.26) and Lemma 3.1 we have for  $e_h := u^e - u_h$

$$\begin{aligned} & (\|e_h\|_{L^2(\Gamma_h)}^2 + \|\nabla e_h\|_{L^2(\Gamma_h)}^2)^{\frac{1}{2}} \\ & \leq 2(\|u^e - I_h u^e\|_{L^2(\Gamma_h)}^2 + \|\nabla(u^e - I_h u^e)\|_{L^2(\Gamma_h)}^2)^{\frac{1}{2}} + \sup_{\phi_h \in V_h^I} \frac{|a_h(u^e, \phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h} \\ & \leq ch \|u\|_{H^2(\Gamma)} + \sup_{\phi_h \in V_h^I} \frac{|a_h(u^e, \phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h}. \end{aligned}$$

In order to estimate the second term we let  $\phi_h \in V_h^I$  be arbitrary and denote by  $\varphi_h = \phi_h^\ell \in H^1(\Gamma)$  the corresponding lift. Then

$$a_h(u^e, \phi_h) - l_h(\phi_h) = (a_h(u^e, \phi_h) - a(u, \varphi_h)) + (l(\varphi_h) - l_h(\phi_h)) \equiv I + II.$$

Using the transformation rule, (2.11) and (2.21) we obtain

$$\begin{aligned} \int_{\Gamma} (\nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varphi_h + u \varphi_h) d\sigma &= \int_{\Gamma_h} ((\nabla_{\Gamma} u) \circ p \cdot (\nabla_{\Gamma} \varphi_h) \circ p + (u \circ p) (\varphi_h \circ p)) \mu_h d\sigma_h \\ &= \int_{\Gamma_h} ((I + \Phi A)^{-1} \nabla u^e \cdot (I + \Phi A)^{-1} \nabla \phi_h + u^e \phi_h) \mu_h d\sigma_h, \end{aligned}$$

so that by (2.15) and (2.22)

$$\begin{aligned} |I| &\leq \int_{\Gamma_h} |(\mu_h (I + \Phi A)^{-T} (I + \Phi A)^{-1} - I) \nabla u^e \cdot \nabla \phi_h| d\sigma_h + \int_{\Gamma_h} |(\mu_h - 1) u^e \phi_h| d\sigma_h \\ &\leq ch^2 \|u^e\|_h \|\phi_h\|_h \leq ch^2 \|u\|_{H^1(\Gamma)} \|\phi_h\|_h \end{aligned}$$

by Lemma 2.2. Similarly,

$$\begin{aligned} |II| &= \left| \int_{\Gamma} f \varphi_h d\sigma - \int_{\Gamma_h} f^e \phi_h d\sigma_h \right| \leq \int_{\Gamma_h} |1 - \mu_h| |f^e| |\phi_h| d\sigma_h \\ &\leq ch^2 \|f^e\|_{L^2(\Gamma_h)} \|\phi_h\|_h \leq ch^2 \|f\|_{L^2(\Gamma)} \|\phi_h\|_h. \end{aligned}$$

Combining these estimates with (2.23) we obtain

$$(3.8) \quad \sup_{\phi_h \in V_h^I} \frac{|a_h(u^e, \phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h} \leq ch^2 \|f\|_{L^2(\Gamma)},$$

so that we have in conclusion

$$(3.9) \quad \|e_h\|_{L^2(\Gamma_h)} + \|\nabla e_h\|_{L^2(\Gamma_h)} \leq ch \|f\|_{L^2(\Gamma)}.$$

In order to improve the  $L^2$ -error bound we employ the usual Aubin-Nitsche argument. Denote by  $w \in H^2(\Gamma)$  the solution of the dual problem

$$a(\varphi, w) = \int_{\Gamma} e_h^\ell \varphi d\sigma \quad \text{for all } \varphi \in H^1(\Gamma),$$

which satisfies

$$(3.10) \quad \|w\|_{H^2(\Gamma)} \leq c \|e_h^\ell\|_{L^2(\Gamma)}.$$

We have in view of (1.3)

$$\begin{aligned} \|e_h^\ell\|_{L^2(\Gamma)}^2 &= a(e_h^\ell, w) = (a(e_h^\ell, w) - a_h(e_h, w^e)) \\ (3.11) \quad &+ a_h(e_h, w^e - I_h w^e) + (a_h(u^e, I_h w^e) - l_h(I_h w^e)) \\ &\equiv I + II + III. \end{aligned}$$

Similarly as above we deduce with the help of (3.9) and Lemma 2.2

$$|I| \leq ch^2 \|e_h\|_h \|w^e\|_h \leq ch^3 \|f\|_{L^2(\Gamma)} \|w\|_{H^1(\Gamma)}.$$

Next, Lemma 3.1 and (3.9) imply

$$|II| \leq \|e_h\|_h \|w^e - I_h w^e\|_h \leq ch^2 \|f\|_{L^2(\Gamma)} \|w\|_{H^2(\Gamma)}.$$

Finally, (3.8) and Lemma 3.1 yield

$$|III| \leq ch^2 \|f\|_{L^2(\Gamma)} \|I_h w^e\|_h \leq ch^2 \|f\|_{L^2(\Gamma)} \|w\|_{H^2(\Gamma)}.$$

Inserting the above estimates into (3.11) and recalling (3.10) we obtain

$$\|e_h^\ell\|_{L^2(\Gamma)} \leq ch^2 \|f\|_{L^2(\Gamma)},$$

which together with Lemma 2.2 and the fact that  $(e_h^\ell)^e = e_h$  yields the  $L^2$ -error bound.  $\square$

#### 4. Narrow band method.

**4.1. Setting up the method.** Let us consider for  $D_h = \{x \in U_\delta \mid |I_h \Phi(x)| < h\}$  the set

$$\mathcal{T}_h^B = \{T \in \mathcal{T}_h \mid \mathcal{H}^{n+1}(T \cap D_h) > 0\},$$

along with

$$U_h^B = \bigcup_{T \in \mathcal{T}_h^B} T.$$

We define the finite element space  $V_h^B$  on the triangulation  $\mathcal{T}_h^B$  by

$$V_h^B = \{\phi_h \in C^0(U_h^B) \mid \phi_h|_T \in P_1(T) \text{ for each } T \in \mathcal{T}_h^B\}.$$

Let us first examine the approximation error for the space  $V_h^B$ .

LEMMA 4.1. *We have for each function  $z \in H^2(\Gamma)$ :*

$$(4.1) \quad \frac{1}{\sqrt{h}} \|z^e - I_h z^e\|_{L^2(D_h)} + \sqrt{h} \|\nabla(z^e - I_h z^e)\|_{L^2(D_h)} \leq ch^2 \|z\|_{H^2(\Gamma)}.$$

*Proof.* We infer from (2.14) and Lemma 2.2 that

$$\begin{aligned} & \frac{1}{h} \|z^e - I_h z^e\|_{L^2(D_h)}^2 + h \|\nabla(z^e - I_h z^e)\|_{L^2(D_h)}^2 \\ & \leq \sum_{T \cap D_h \neq \emptyset} \left( \frac{1}{h} \|z^e - I_h z^e\|_{L^2(T)}^2 + h \|\nabla(z^e - I_h z^e)\|_{L^2(T)}^2 \right) \\ & \leq ch^3 \sum_{T \cap D_h \neq \emptyset} \|z^e\|_{H^2(T)}^2 \leq ch^3 \|z^e\|_{H^2(D_{(1+c_1)h})}^2 \leq ch^4 \|z\|_{H^2(\Gamma)}^2, \end{aligned}$$

since  $T \subset D_{(1+c_1)h}$  for all  $T \cap D_h \neq \emptyset$  in view of (2.16).  $\square$

**4.2. The method.** Let us write (1.4) in the form:- Find  $u_h \in V_h^B$  such that

$$(4.2) \quad a_h(u_h, \phi_h) = l_h(\phi_h) \quad \text{for all } \phi_h \in V_h^B,$$

where

$$\begin{aligned} a_h(w_h, \phi_h) &= \frac{1}{2h} \int_{D_h} (\nabla w_h \cdot \nabla \phi_h + w_h \phi_h) |\nabla I_h \Phi| \, dx, \\ l_h(\phi_h) &= \frac{1}{2h} \int_{D_h} f^e \phi_h |\nabla I_h \Phi| \, dx. \end{aligned}$$

Note that the factors  $\frac{1}{h}$  in each of the above terms are there to aid the notation for the error analysis. In a similar way as for the sharp interface method one can verify that  $a_h$  is positive definite on  $V_h^B \times V_h^B$ . Hence, the finite element scheme (4.2) has a unique solution  $u_h \in V_h^B$  which satisfies

$$(4.3) \quad \|u_h\|_h = \left( \frac{1}{2h} \int_{D_h} (|\nabla u_h|^2 + u_h^2) |\nabla I_h \Phi| \, dx \right)^{\frac{1}{2}} \leq c \|f\|_{L^2(\Gamma)}.$$

REMARK 4.2. *The right hand side  $l_h(\cdot)$  may be defined using other sufficiently accurate extensions of  $f$ .*

**4.3. Error analysis.** Before we prove our main error bound we formulate a technical lemma which will be helpful in the error analysis.

LEMMA 4.3. *Suppose that  $u \in H^2(\Gamma)$  is a solution of (1.1). Then,*

$$(4.4) \quad a_h(u^e, \phi) = \frac{1}{2h} \int_{D_h} f^e \phi \circ F_h^{-1} |\nabla \Phi| \, dx + \frac{1}{2h} \int_{D_h} (\nabla u^e \cdot \nabla \eta_h) (\nabla \phi \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \, dx + \langle S, \phi \rangle,$$

for all  $\phi \in H^1(D_h)$ , where  $\eta_h = I_h \Phi - \Phi$  and

$$|\langle S, \phi \rangle| \leq Ch^2 \|u\|_{H^2(\Gamma)} \|\phi\|_h.$$

*Proof.* To begin, we derive from (2.12) and (1.1) that

$$(4.5) \quad -\frac{1}{|\nabla \Phi|} \nabla \cdot (|\nabla \Phi| \nabla u^e) + u^e = f^e + R \quad \text{in } U_\delta,$$

where

$$(4.6) \quad R(x) = -\Phi(x) \left( \sum_{k,l=1}^{n+1} b_{lk}(x) \underline{D}_l \underline{D}_k u(p(x)) + \sum_{k=1}^{n+1} c_k(x) \underline{D}_k u(p(x)) \right).$$

We multiply (4.5) by  $\phi \circ F_h^{-1} |\nabla \Phi|$ ,  $\phi \in H^1(D_h)$  and integrate over  $D^h$ . Since  $\frac{\partial u^e}{\partial \nu} = 0$  on  $\partial D^h$  we obtain after integration by parts

$$(4.7) \quad \begin{aligned} & \int_{D^h} \nabla u^e \cdot \nabla (\phi \circ F_h^{-1}) |\nabla \Phi| \, dx + \int_{D^h} u^e \phi \circ F_h^{-1} |\nabla \Phi| \, dx \\ &= \int_{D^h} f^e \phi \circ F_h^{-1} |\nabla \Phi| \, dx + \int_{D^h} R \phi \circ F_h^{-1} |\nabla \Phi| \, dx. \end{aligned}$$

Observing that  $\nabla(\phi \circ F_h^{-1}) = [(DF_h)^{-T} \circ F_h^{-1}] \nabla \phi \circ F_h^{-1}$  the transformation rule and Lemma 2.1 imply that

$$I := \int_{D_h} \nabla u^e \cdot \nabla(\phi \circ F_h^{-1}) |\nabla \Phi| \, dx = \int_{D_h} \nabla u^e \circ F_h \cdot (DF_h)^{-T} \nabla \phi |\nabla \Phi \circ F_h| |\det DF_h| \, dx.$$

Recalling (2.7) and (2.17) we have

$$(4.8) \quad z^e(x) = z(p(x)) = z(p(F_h(x))) = z^e(F_h(x)),$$

from which we deduce by differentiation

$$(4.9) \quad \nabla z^e \circ F_h = (DF_h)^{-T} \nabla z^e,$$

so that

$$I = \int_{D_h} (DF_h)^{-T} \nabla u^e \cdot (DF_h)^{-T} \nabla \phi |\nabla \Phi \circ F_h| |\det DF_h| \, dx.$$

We infer from (2.20) that

$$(DF_h)^{-T} = I - \frac{1}{|\nabla \Phi|} \nabla \eta_h \otimes \nu + B_h, \quad \text{with } |B_h| \leq ch^2,$$

and hence

$$(DF_h)^{-T} \nabla u^e = \nabla u^e + B_h \nabla u^e, \quad (DF_h)^{-T} \nabla \phi = \nabla \phi - \frac{1}{|\nabla \Phi|} (\nabla \phi \cdot \nu) \nabla \eta_h + B_h \nabla \phi,$$

since  $\nabla u^e \cdot \nu = 0$ . Furthermore, Lemma 2.1 implies that

$$|\nabla \Phi \circ F_h| |\det DF_h| = |\nabla I_h \Phi| + \gamma_h, \quad \text{where } |\gamma_h| \leq ch^2,$$

so that in conclusion

$$I = \int_{D_h} \nabla u^e \cdot \nabla \phi |\nabla I_h \Phi| \, dx - \int_{D_h} (\nabla u^e \cdot \nabla \eta_h) (\nabla \phi \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \, dx + \langle R_h^1, \phi \rangle,$$

where

$$|\langle R_h^1, \phi \rangle| \leq ch^2 \|\nabla u^e\|_{L^2(D_h)} \|\nabla \phi\|_{L^2(D_h)} \leq ch^3 \|u\|_{H^2(\Gamma)} \|\phi\|_h,$$

in view of Lemma 2.2 and the definition of  $\|\cdot\|_h$ . Similarly, (4.8) and (2.19) yield

$$\int_{D_h} u^e \phi \circ F_h^{-1} |\nabla \Phi| \, dx = \int_{D_h} u^e \phi |\nabla I_h \Phi| \, dx + \langle R_h^2, \phi \rangle$$

with  $|\langle R_h^2, \phi \rangle| \leq ch^3 \|u\|_{H^2(\Gamma)} \|\phi\|_h$ . Inserting the above identities into (4.7) and dividing by  $2h$  we derive

(4.10)

$$\begin{aligned} a_h(u^e, \phi) &= \frac{1}{2h} \int_{D_h} f^e \phi \circ F_h^{-1} |\nabla \Phi| \, dx + \frac{1}{2h} \int_{D_h} R \phi \circ F_h^{-1} |\nabla \Phi| \, dx \\ &\quad + \frac{1}{2h} \int_{D_h} (\nabla u^e \cdot \nabla \eta_h) (\nabla \phi \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \, dx - \frac{1}{2h} \langle R_h^1, \phi \rangle - \frac{1}{2h} \langle R_h^2, \phi \rangle. \end{aligned}$$



In order to rewrite the integral over  $D^h$  we recall that  $F$  is a diffeomorphism from  $\Gamma \times (-h, h)$  onto  $D^h$ . It is not difficult to see that

$$(4.11) \quad dx = \mu(p, s) d\sigma_p ds, \quad \text{where} \quad \left| \mu(p, s) - \frac{1}{|\nabla\Phi(F(p, s))|} \right| \leq C|s|, \quad |s| < h, p \in \Gamma.$$

Hence,

$$\begin{aligned} \int_{D^h} R \phi \circ F_h^{-1} |\nabla\Phi| dx &= \int_{-h}^h \int_{\Gamma} R \circ F \phi \circ F_h^{-1} \circ F |\nabla\Phi \circ F| \mu d\sigma_p ds \\ &= \int_{-h}^h \int_{\Gamma} R \circ F \phi \circ F_h^{-1} \circ F d\sigma_p ds + \int_{-h}^h \int_{\Gamma} \tilde{r} \phi \circ F_h^{-1} \circ F d\sigma_p ds \\ &\equiv T_1 + T_2, \end{aligned}$$

where  $\tilde{r}(p, s) = R(F(p, s))(\mu(p, s) |\nabla\Phi(F(p, s))| - 1)$ . In order to treat  $T_1$  we deduce from (4.6) and the fact that  $\Phi(F(p, s)) = s$  that

$$R(F(p, s)) = -s \left( \sum_{k,l=1}^{n+1} b_{lk}(F(p, s)) \underline{D}_l \underline{D}_k u(p) + \sum_{k=1}^{n+1} c_k(F(p, s)) \underline{D}_k u(p) \right).$$

Since  $\int_{-h}^h s ds = 0$ , the first term in  $T_1$  can be written as

$$- \int_{-h}^h \int_{\Gamma} \sum_{k,l=1}^{n+1} s \underline{D}_l \underline{D}_k u(p) \left\{ b_{lk}(F(p, s)) \phi \circ F_h^{-1}(F(p, s)) - b_{lk}(p) \phi \circ F_h^{-1}(p) \right\} d\sigma_p ds.$$

Treating the second term in  $T_1$  in the same way and observing that  $p = F(p, 0)$  we deduce with the help of the fundamental theorem of calculus that

$$|T_1| \leq ch^{\frac{5}{2}} \|u\|_{H^2(\Gamma)} \left( \int_{D^h} (|\nabla\phi \circ F_h^{-1}|^2 + |\phi \circ F_h^{-1}|^2) dx \right)^{\frac{1}{2}} \leq ch^3 \|u\|_{H^2(\Gamma)} \|\phi\|_h.$$

Next, we infer from (4.6) and (4.11) that

$$|\tilde{r}(p, s)| \leq cs^2 (|\nabla_{\Gamma} u(p)| + |D_{\Gamma}^2 u(p)|),$$

so that

$$|T_2| \leq Ch^{\frac{5}{2}} \|u\|_{H^2(\Gamma)} \left( \int_{D^h} |\phi \circ F_h^{-1}|^2 dx \right)^{\frac{1}{2}} \leq Ch^3 \|u\|_{H^2(\Gamma)} \|\phi\|_h.$$

The result now follows from (4.10) together with the bounds on  $R_h^1$  and  $R_h^2$ .  $\square$

We are now in position to prove optimal error bounds for our scheme.

**THEOREM 4.4.** *Let  $u$  be the solution of (1.1) and  $u_h$  the solution of the finite element scheme (4.2). Then*

$$(4.12) \quad \|u^e - u_h\|_{L^2(\Gamma_h)} + h \left( \frac{1}{2h} \int_{D^h} |\nabla(u^e - u_h)|^2 |\nabla I_h \Phi| dx \right)^{\frac{1}{2}} \leq ch^2 \|f\|_{L^2(\Gamma)}.$$

*Proof.* Let us write  $e_h := u^e - u_h$ . We infer from (2.26) and Lemma 4.1 that

$$(4.13) \quad \|e_h\|_h \leq ch \|u\|_{H^2(\Gamma)} + \sup_{\phi_h \in V_h^B} \frac{|a_h(u^e, \phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h}.$$

The second term on the right hand side can be estimated with the help of Lemma 4.3. The transformation rule together with (4.8) yields

$$\frac{1}{2h} \int_{D_h} f^e \phi_h \circ F_h^{-1} |\nabla \Phi| \, dx = \frac{1}{2h} \int_{D_h} f^e \circ F_h \phi_h |\nabla \Phi \circ F_h| |\det DF_h| \, dx,$$

so that we deduce from Lemma 4.3

$$\begin{aligned} a_h(u^e, \phi_h) - l_h(\phi_h) &= \frac{1}{2h} \int_{D_h} (\nabla u^e \cdot \nabla \eta_h) (\nabla \phi_h \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \\ &\quad + \frac{1}{2h} \int_{D_h} f^e \phi_h \left( |\nabla \Phi \circ F_h| |\det DF_h| - |\nabla I_h \Phi| \right) dx + \langle S, \phi_h \rangle. \end{aligned}$$

Using (2.19), (2.15), Lemma 2.1, (2.23) as well as Lemma 2.2 we infer that for  $\phi_h \in V_h^B$

$$(4.14) \quad \begin{aligned} |a_h(u^e, \phi_h) - l_h(\phi_h)| &\leq c \|\nabla u^e\|_{L^2(D_h)} \left( \int_{D_h} |\nabla \phi_h \cdot \nu|^2 \, dx \right)^{\frac{1}{2}} \\ &\quad + ch \|f^e\|_{L^2(D_h)} \|\phi_h\|_{L^2(D_h)} + ch^2 \|u\|_{H^2(\Gamma)} \|\phi_h\|_h \\ &\leq ch \|f\|_{L^2(\Gamma)} \left( \frac{1}{2h} \int_{D_h} |\nabla \phi_h \cdot \nu|^2 \, dx \right)^{\frac{1}{2}} + ch^2 \|f\|_{L^2(\Gamma)} \|\phi_h\|_h, \end{aligned}$$

so that (4.13) implies the following intermediate result:

$$(4.15) \quad \|e_h\|_h = \left( \frac{1}{2h} \int_{D_h} (|\nabla e_h|^2 + e_h^2) |\nabla I_h \Phi| \, dx \right)^{\frac{1}{2}} \leq ch \|f\|_{L^2(\Gamma)}.$$

In order to improve the  $L^2$ -error bound we define  $\tilde{e}_h := e_h \circ F_h^{-1}$  as well as

$$\tilde{E}_h(p) := \frac{1}{2h} \int_{-h}^h \tilde{e}_h(F(p, s)) \, ds, \quad p \in \Gamma$$

with  $F$  as above. We denote by  $w \in H^2(\Gamma)$  the unique solution of

$$-\Delta_\Gamma w + w = \tilde{E}_h \quad \text{on } \Gamma,$$

which satisfies

$$(4.16) \quad \|w\|_{H^2(\Gamma)} \leq c \left\| \tilde{E}_h \right\|_{L^2(\Gamma)}.$$

Similar to (4.5) the extension  $w^e$  solves

$$-\frac{1}{|\nabla \Phi|} \nabla \cdot (|\nabla \Phi| \nabla w^e) + w^e = \tilde{E}_h^e + \tilde{R} \quad \text{in } U_\delta,$$

where  $\tilde{R}$  is obtained from (4.6) by replacing  $u$  by  $w$ . Using the transformation rule together with (4.11) we obtain

$$\begin{aligned} \left\| \tilde{E}_h \right\|_{L^2(\Gamma)}^2 &= \frac{1}{2h} \int_{-h}^h \int_{\Gamma} \tilde{E}_h \tilde{e}_h \circ F \, d\sigma_p \, ds = \frac{1}{2h} \int_{-h}^h \int_{\Gamma} \tilde{E}_h^e \circ F \tilde{e}_h \circ F |\nabla \Phi \circ F| \mu \, d\sigma_p \, ds \\ &\quad + \frac{1}{2h} \int_{-h}^h \int_{\Gamma} \tilde{E}_h \tilde{e}_h \circ F (1 - |\nabla \Phi \circ F| \mu) \, d\sigma_p \, ds \\ &= \frac{1}{2h} \int_{D^h} \tilde{E}_h^e e_h \circ F_h^{-1} |\nabla \Phi| \, dx + \frac{1}{2h} \int_{-h}^h \int_{\Gamma} \tilde{E}_h \tilde{e}_h \circ F (1 - |\nabla \Phi \circ F| \mu) \, d\sigma_p \, ds. \end{aligned}$$

The first term can be rewritten with the help of Lemma 4.3 (applied to  $w$  instead of  $u$ ) to give

$$\begin{aligned} \left\| \tilde{E}_h \right\|_{L^2(\Gamma)}^2 &= a_h(w^e, e_h) - \langle \tilde{S}, e_h \rangle - \frac{1}{2h} \int_{D^h} (\nabla w^e \cdot \nabla \eta_h) (\nabla e_h \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \, dx \\ &\quad + \frac{1}{2h} \int_{-h}^h \int_{\Gamma} \tilde{E}_h \tilde{e}_h \circ F (1 - |\nabla \Phi \circ F| \mu) \, d\sigma_p \, ds \\ &\equiv \sum_{k=1}^4 I_k. \end{aligned}$$

In view of Lemma 4.1, (4.14), the fact that  $\nabla w^e \cdot \nu = 0$  and (4.15) we have

$$\begin{aligned} |I_1| + |I_2| &\leq |a_h(w^e - I_h w^e, e_h)| + |a_h(w^e, I_h w^e) - l_h(I_h w^e)| + |\langle \tilde{S}, e_h \rangle| \\ &\leq ch \|w\|_{H^2(\Gamma)} \|e_h\|_h + ch^2 \|f\|_{L^2(\Gamma)} \|I_h w^e\|_h \\ &\quad + ch \|f\|_{L^2(\Gamma)} \left( \frac{1}{2h} \int_{D^h} |\nabla(I_h w^e - w^e) \cdot \nu|^2 \, dx \right)^{\frac{1}{2}} + ch^2 \|w\|_{H^2(\Gamma)} \|e_h\|_h \\ &\leq ch^2 \|f\|_{L^2(\Gamma)} \|w\|_{H^2(\Gamma)}. \end{aligned}$$

Furthermore, (2.15), (4.11) and (4.15) imply

$$|I_3| + |I_4| \leq ch \left( \|w^e\|_h + \left\| \tilde{E}_h \right\|_{L^2(\Gamma)} \right) \|e_h\|_h \leq ch^2 \|f\|_{L^2(\Gamma)} \left( \|w\|_{H^2(\Gamma)} + \left\| \tilde{E}_h \right\|_{L^2(\Gamma)} \right),$$

so that we obtain together with (4.16)

$$\left\| \tilde{E}_h \right\|_{L^2(\Gamma)} \leq ch^2 \|f\|_{L^2(\Gamma)}.$$

Next, since  $F(p, 0) = p$  we may write for  $p \in \Gamma$

$$\tilde{E}_h(p) - \tilde{e}_h(p) = \frac{1}{2h} \int_{-h}^h \int_0^s \nabla \tilde{e}_h(F(p, \tau)) \cdot \frac{\partial F}{\partial s}(p, \tau) \, d\tau \, ds,$$

and hence we obtain with the help of (4.15)

$$\left\| \tilde{E}_h - \tilde{e}_h \right\|_{L^2(\Gamma)} \leq c\sqrt{h} \left( \int_{D^h} |\nabla \tilde{e}_h|^2 \, dx \right)^{\frac{1}{2}} \leq ch \|\nabla e_h\|_h \leq ch^2 \|f\|_{L^2(\Gamma)}.$$

In conclusion we deduce that

$$\|e_h\|_{L^2(\Gamma_h)} \leq c \|\tilde{e}_h\|_{L^2(\Gamma)} \leq \left\| \tilde{E}_h - \tilde{e}_h \right\|_{L^2(\Gamma)} + \left\| \tilde{E}_h \right\|_{L^2(\Gamma)} \leq ch^2 \|f\|_{L^2(\Gamma)}$$

and the theorem is proved.  $\square$

### 5. A hybrid method for equations on evolving surfaces.

**5.1. The setting.** The aim of this section is to combine ideas employed in §3 and §4 for the stationary problem in order to develop a finite element method for an advection–diffusion equation on a family of evolving hypersurfaces. More precisely, let  $(\Gamma(t))_{t \in [0, T]}$  be a family of compact, connected smooth hypersurfaces embedded in  $\mathbb{R}^{n+1}$  for  $n = 1, 2$ . We suppose that

$$\Gamma(t) = \{x \in \mathcal{N}(t) \mid \Phi(x, t) = 0\}, \quad \text{where } \nabla \Phi(x, t) \neq 0, x \in \mathcal{N}(t)$$

and  $\mathcal{N}(t)$  is an open neighbourhood of  $\Gamma(t)$ . We assume that  $\mathcal{N}(t)$  is chosen so small that we can construct the function  $p(\cdot, t)$  as in §2.1.

Given a velocity field  $v(\cdot, t) : \Gamma(t) \rightarrow \mathbb{R}^{n+1}$  we then consider the following initial value problem

$$(5.1a) \quad \partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u = f \quad \text{on } \bigcup_{t \in (0, T)} \Gamma(t) \times \{t\},$$

$$(5.1b) \quad u(\cdot, 0) = u_0 \quad \text{on } \Gamma(0).$$

Here,  $\partial^\bullet \eta$  denotes the material derivative of a function  $\eta : \bigcup_{t \in (0, T)} \Gamma(t) \times \{t\} \rightarrow \mathbb{R}$  which is given by

$$\partial^\bullet \eta = \partial_t \eta + v \cdot \nabla \eta,$$

if  $\eta$  is extended into a neighbourhood of  $\bigcup_{t \in (0, T)} \Gamma(t) \times \{t\}$ .

**5.2. The method.** In order to discretize the above problem we choose a partition  $0 = t_0 < t_1 < \dots < t_N = T$  of  $[0, T]$  with  $\tau_m := t_{m+1} - t_m, m = 0, \dots, N-1$  and  $\tau := \max_{m=0, \dots, N-1} \tau_m$ . Also, let  $\mathcal{T}_h$  be an unfitted regular triangulation with mesh size  $h$  of a region containing  $\mathcal{N}(t), t \in [0, T]$ . For  $m = 0, 1, \dots, N$  we set

$$\begin{aligned} \Gamma_h^m &= \{x \in \mathcal{N}(t_m) \mid I_h \Phi(x, t_m) = 0\} \\ D_h^m &= \{x \in \mathcal{N}(t_m) \mid |I_h \Phi(x, t_m)| < h\}, \end{aligned}$$

as well as

$$\mathcal{T}_h^m := \{T \in \mathcal{T}_h \mid \mathcal{H}^{n+1}(T \cap D_h^m) > 0\} \quad \text{and} \quad U_h^m := \bigcup_{T \in \mathcal{T}_h^m} T.$$

Here we assume that  $0 < h \leq h_0$ , where  $h_0$  is chosen so small that there exists  $c_0, c_1 > 0$  such that

$$c_0 \leq |\nabla I_h \Phi(x, t)| \leq c_1, \quad (x, t) \in \bigcup_{t \in (0, T)} \mathcal{N}(t) \times \{t\}.$$

Finally, we introduce

$$V_h^m = \{\phi_h \in C^0(U_h^m) \mid \phi_h|_T \in P_1(T) \text{ for each } T \in \mathcal{T}_h^m\}.$$

In what follows we shall frequently use the abbreviation  $z^m(x) := z(x, t_m)$ .

In order to motivate our method we fix  $m \in \{0, 1, \dots, N-1\}$  and let  $\Psi$  be the solution of

$$\Psi_t(x, t) + D\Psi(x, t)v^e(x, t) = 0, \quad \Psi(x, t_{m+1}) = x,$$

where  $v^e(x, t) := v(p(x, t), t)$ . For a sufficiently smooth function  $\varphi : \mathcal{N}(t_{m+1}) \rightarrow \mathbb{R}$  we define  $\eta(x, t) := \varphi(\Psi(x, t))$ . Clearly,  $\eta(\cdot, t_{m+1}) = \varphi$  and a short calculation shows that  $\partial^\bullet \eta = 0$ . Assuming that  $u$  is a solution of (5.1a) we obtain with the help of the Leibniz formula and integration by parts

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} u \eta \, d\sigma|_{t=t_{m+1}} &= \int_{\Gamma(t_{m+1})} (\partial^\bullet(u\eta) + u\eta \nabla_\Gamma \cdot v) \, d\sigma \\ &= \int_{\Gamma(t_{m+1})} \varphi (\partial^\bullet u + u \nabla_\Gamma \cdot v) \, d\sigma = \int_{\Gamma(t_{m+1})} (\varphi \Delta_\Gamma u + \varphi f) \, d\sigma \\ &= - \int_{\Gamma(t_{m+1})} \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \, d\sigma + \int_{\Gamma(t_{m+1})} f \varphi \, d\sigma. \end{aligned}$$

Since  $\Psi(\cdot, t_{m+1}) \equiv \text{id}$ , a Taylor expansion shows that

$$\begin{aligned} \Psi(x, t_m) &= \Psi(x, t_{m+1} - \tau_m) \approx x - \tau_m \Psi_t(x, t_{m+1}) \\ &= x + \tau_m D\Psi(x, t_{m+1}) v^{e, m+1}(x) = x + \tau_m v^{e, m+1}(x). \end{aligned}$$

Thus we may approximate the left hand side of the above relation by

$$\frac{d}{dt} \int_{\Gamma(t)} u \eta \, d\sigma|_{t=t_{m+1}} \approx \frac{1}{\tau_m} \left\{ \int_{\Gamma(t_{m+1})} u^{m+1} \varphi \, d\sigma - \int_{\Gamma(t_m)} u^m \varphi(\cdot + \tau_m v^{e, m+1}) \, d\sigma \right\}.$$

The above calculations motivate the following scheme, in which we use the narrow band approach in order to discretize the elliptic part. Given  $u_h^m \in V_h^m$ ,  $m = 1, \dots, N-1$ , find  $u_h^{m+1} \in V_h^{m+1}$  such that

$$(5.2) \quad \begin{aligned} &\int_{\Gamma_h^{m+1}} u_h^{m+1} \phi_h \, d\sigma_h - \int_{\Gamma_h^m} u_h^m \phi_h(\cdot + \tau_m v^{e, m+1}) \, d\sigma_h \\ &+ \frac{\tau_m}{2h} \int_{D_h^{m+1}} \nabla u_h^{m+1} \cdot \nabla \phi_h |\nabla I_h \Phi^{m+1}| \, dx = \tau_m \int_{\Gamma_h^{m+1}} f^{e, m+1} \phi_h \, d\sigma_h \end{aligned}$$

for all  $\phi_h \in V_h^{m+1}$ . Here,  $u_h^0 = I_h u^0$ . Existence and uniqueness of  $u_h^{m+1}$  follows in a similar way as for the narrow band method in the elliptic case.

**5.3. Mass conservation.** An important property of solutions of (5.1a) is conservation of mass in the case that  $\int_{\Gamma(t)} f(\cdot, t) \, d\sigma = 0$ . The following lemma shows that our numerical scheme preserves this property under some mild constraints on the discretization parameters.

LEMMA 5.1. *Suppose that  $\int_{\Gamma_h^{m+1}} f^{e, m+1} \, d\sigma_h = 0$ ,  $m = 0, \dots, N-1$ . Let  $u_h^m \in V_h^m$ ,  $m = 1, \dots, N$  be the solutions of (5.2). Then*

$$(5.3) \quad \int_{\Gamma_h^m} u_h^m \, d\sigma_h = \int_{\Gamma_h^0} u_h^0 \, d\sigma_h,$$

provided that  $0 < h \leq h_1$  and  $\tau \leq \gamma \sqrt{h}$ .

*Proof.* Let us first observe that

$$(5.4) \quad \{x + \tau_m v^{e, m+1}(x) \mid x \in \Gamma_h^m\} \subset U_h^{m+1}, \quad m = 0, \dots, N-1$$

provided that  $h, \tau$  are sufficiently small. To see this, let  $x \in \Gamma_h^m$  and choose an element  $T \in \mathcal{T}_h$  such that  $x \in T$ . Then,

$$\Phi^{m+1}(x + \tau_m v^{e, m+1}(x)) = \Phi^m(x) + \tau_m \nabla \Phi^m(x) \cdot v^{e, m+1}(x) + \tau_m \Phi_t(x, t_m) + R_m(x),$$

where  $|R_m(x)| \leq c\tau_m^2$ . Observing that  $\Phi_t + \nabla\Phi \cdot v = 0$  on  $\bigcup_{t \in (0, T)} \Gamma(t) \times \{t\}$  we may write

$$\begin{aligned} \nabla\Phi^m(x) \cdot v^{e, m+1}(x) + \Phi_t(x, t_m) &= \nabla\Phi^m(x) \cdot v^{m+1}(p^{m+1}(x)) + \Phi_t(x, t_m) \\ &= (\nabla\Phi^m(x) - \nabla\Phi^{m+1}(p^{m+1}(x))) \cdot v^{m+1}(p^{m+1}(x)) + \Phi_t(x, t_m) - \Phi_t(p^{m+1}(x), t_{m+1}), \end{aligned}$$

so that

$$\begin{aligned} |\Phi^{m+1}(x + \tau_m v^{e, m+1}(x))| &\leq |\Phi^m(x)| + c\tau_m |x - p^{m+1}(x)| + c\tau_m^2 \\ &\leq |\Phi^m(x)| + c\tau_m |\Phi^{m+1}(x)| + c\tau_m^2 \leq c(h^2 + \tau_m^2), \end{aligned}$$

in view of (2.5) and since  $|\Phi^m(x)| = |\Phi^m(x) - I_h\Phi^m(x)| \leq ch^2$ . As a result,

$$|(I_h\Phi^{m+1})(x + \tau_m v^{e, m+1}(x))| \leq \|I_h\Phi^{m+1} - \Phi^{m+1}\|_{L^\infty} + c(h^2 + \tau_m^2) \leq c(h^2 + \tau_m^2) < h,$$

provided that  $0 < h \leq h_1$  and  $\tau \leq \gamma\sqrt{h}$ . Hence,  $x + \tau_m v^{e, m+1}(x) \in D_h^{m+1} \subset U_h^{m+1}$  proving (5.4). The result of the lemma now follows from inserting  $\phi_h \equiv 1 \in V_h^{m+1}$  into (5.2) and using (5.4) together with our assumption that  $\int_{\Gamma_h^{m+1}} f^{e, m+1} d\sigma_h = 0$ .  $\square$

## 6. Numerical Experiments.

**6.1. Notes on implementation.** The methods were implemented using the Distributed and Unified Numerics Environment (DUNE) [6, 7, 14]. Assembly of the matrices is non standard in that the method requires integration over partial elements. To do so we subdivide the integration areas in simplices using the `Triangle` [51, 52] and `Tetgen` [53] packages. In each case, the linear system is solved with the conjugate gradient method until the residual is reduced by a factor of  $10^{-8}$  in comparison to its initial value in the  $\ell^2$  norm. Due to the lack of shape regularity of  $\Gamma_h$  and  $D_h$ , the matrix systems are ill conditioned and so we used a Jacobi preconditioner in order to speed up the convergence of our iterative solver. In practice, we will take  $U_h$  to be a subset of a cube shaped domain. The triangulation  $\mathcal{T}_h$  will be computed adaptively refining only those elements which intersect the computational domain, either  $\Gamma_h$  or  $D_h$ . Given errors  $E_i$  and  $E_{i-1}$  at two different mesh sizes  $h_i$  and  $h_{i-1}$ , we calculate the experimental order of convergence (eoc) by

$$(6.1) \quad (\text{eoc})_i = \frac{\log(E_i/E_{i-1})}{\log(h_i/h_{i-1})}.$$

**6.2. Poisson equation.** To test our methods, we present two numerical examples. The first is on a torus and is taken from [42] and the second is on a potato-like surface from [18].

We define the torus through the signed distance function:

$$\Gamma = \{x \in \mathbb{R}^3 \mid d(x) = 0\} \quad d(x) = \sqrt{\left(\sqrt{x_1^2 + x_2^2} - R\right)^2 + x_3^2} - r,$$

for  $R = 1, r = 0.6$ . To compute our exact solution, we parameterise the torus by

$$x_1 = (R + r \cos \theta) \cos \varphi, \quad x_2 = (R + r \cos \theta) \sin \varphi, \quad x_3 = r \sin \theta, \quad \text{for } \theta, \varphi \in (-\pi, \pi)$$

and take the exact solution

$$u(\theta, \varphi) = \cos(3\varphi) \sin(3\theta + \varphi).$$

$h$	$\ u^e - u_h^{SIF}\ _{L^2(\Gamma_h)}$	(eoc)	$\ u^e - u_h^{NBM}\ _{L^2(\Gamma_h)}$	(eoc)
$2^{-1}\sqrt{3}$	1.67739	—	1.61943	—
$2^{-2}\sqrt{3}$	$7.10825 \cdot 10^{-1}$	1.238650	$7.07220 \cdot 10^{-1}$	1.195260
$2^{-3}\sqrt{3}$	$1.90004 \cdot 10^{-1}$	1.903470	$2.32053 \cdot 10^{-1}$	1.607700
$2^{-4}\sqrt{3}$	$4.73865 \cdot 10^{-2}$	2.003480	$7.17605 \cdot 10^{-2}$	1.693190
$2^{-5}\sqrt{3}$	$1.19721 \cdot 10^{-2}$	1.984800	$1.97350 \cdot 10^{-2}$	1.862430
$2^{-6}\sqrt{3}$	$3.01376 \cdot 10^{-3}$	1.990040	$5.08158 \cdot 10^{-3}$	1.957410

Table 6.1: Error tables of sharp interface method (*SIF*) and narrow band method (*NBM*) for the first test problem. These calculations took successively 38, 69, 128, 240, 359, 641 conjugate gradient iterations for *SIF* and 33, 54, 97, 182, 392, 634 conjugate gradient iterations for *NBM*.

$h$	$\ u^e - u_h^{SIF}\ _{L^2(\Gamma_h)}$	(eoc)	$\ u^e - u_h^{NBM}\ _{L^2(\Gamma_h)}$	(eoc)
$2^{-1}\sqrt{3}$	$3.31237 \cdot 10^{-1}$	—	$2.31128 \cdot 10^{-1}$	—
$2^{-2}\sqrt{3}$	$9.97842 \cdot 10^{-2}$	1.730980	$9.06054 \cdot 10^{-2}$	1.351020
$2^{-3}\sqrt{3}$	$2.57329 \cdot 10^{-2}$	1.955200	$2.57213 \cdot 10^{-2}$	1.816630
$2^{-4}\sqrt{3}$	$6.59538 \cdot 10^{-3}$	1.964090	$7.43214 \cdot 10^{-3}$	1.791110
$2^{-5}\sqrt{3}$	$1.64586 \cdot 10^{-3}$	2.002610	$1.94710 \cdot 10^{-3}$	1.932450
$2^{-6}\sqrt{3}$	$4.10269 \cdot 10^{-4}$	2.004200	$4.99422 \cdot 10^{-4}$	1.962990
$2^{-7}\sqrt{3}$	$1.02735 \cdot 10^{-4}$	1.997640	$1.26086 \cdot 10^{-4}$	1.985850

Table 6.2: Error tables of sharp interface method (*SIF*) and narrow band method (*NBM*) for the second test problem. These calculations took successively 25, 53, 103, 196, 298, 585, 1151 conjugate gradient iterations for *SIF* and 37, 68, 116, 153, 297, 580, 1137 conjugate gradient iterations for *NBM*.

For this example, (2.4) can be calculated analytically.

For our second example, we set  $\Gamma = \{x \in \mathbb{R}^3 \mid \Phi(x) = 0\}$  using the level set function:

$$\Phi(x) = (x_1 - x_3^2)^2 + x_2^2 - 1.$$

From  $\Phi$ , we calculate the normal  $\nu = \frac{\nabla\Phi}{|\nabla\Phi|}$  and the mean curvature by

$$H = \nabla \cdot \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{1}{|\nabla\Phi|} \sum_{j,k=1}^3 \left( \delta_{jk} - \frac{\Phi_{x_j}\Phi_{x_k}}{|\nabla\Phi|^2} \right) \Phi_{x_j x_k}.$$

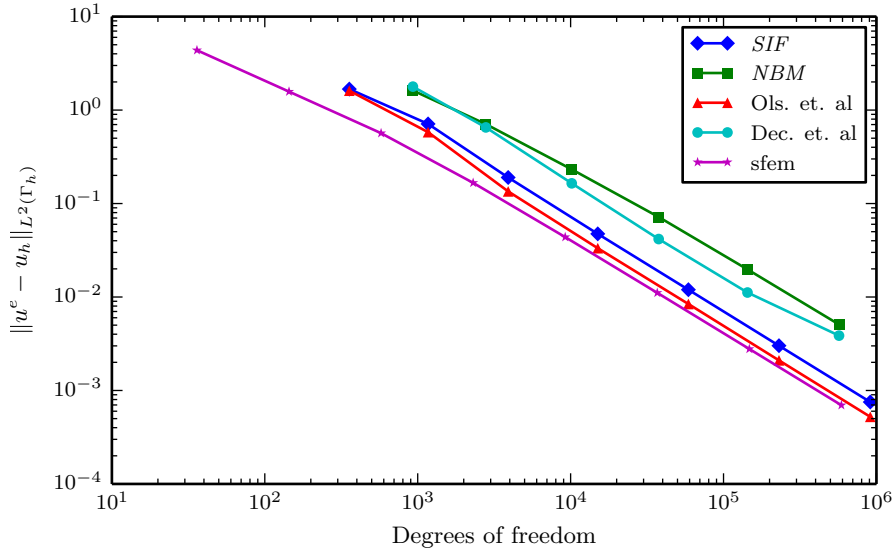
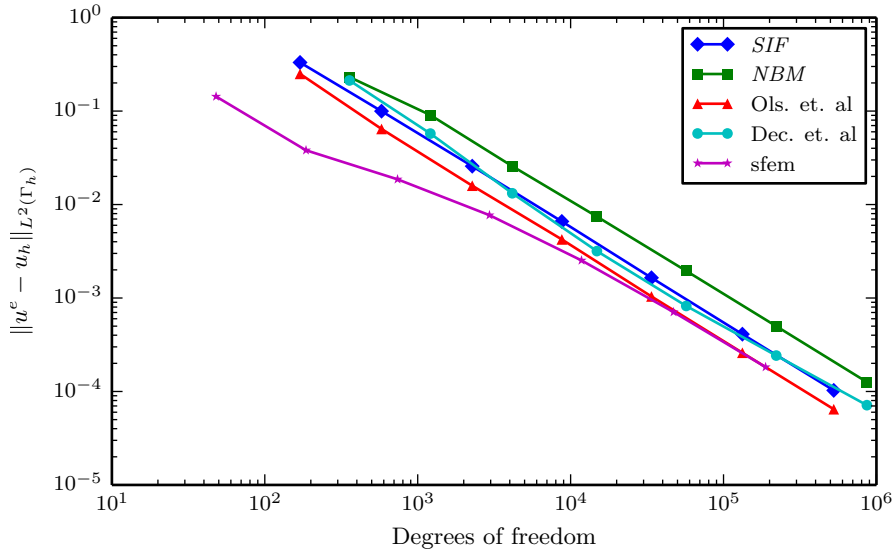
As exact solution, we take  $u(x) = x_1 x_2$  and calculate the right-hand side  $f$  as  $f = -\Delta_\Gamma u + u$  as

$$f(x) = 2\nu_1(x)\nu_2(x) + H(x)(x_2\nu_1(x) + x_1\nu_2(x)), \quad x \in \Gamma.$$

For this example, (2.4) can not be calculated exactly so we approximate using a gradient decent like iteration from [48] originally for the closest point operator.

The errors  $\|u^e - u_h\|_{L^2(\Gamma_h)}$  for *SIF* and *NBM* are shown in Tables 6.1 and 6.2. The numerical results confirm the theoretical bounds from Theorems 3.3 and 4.4. Results are also available for the  $H^1$ -semi-norm error.

To compare with other methods, we also include a plot of the error in the  $L^2$ -norm against  $h$  for *SIF* and *NBM* along with the unfitted finite element methods of

Fig. 6.1: Plot of the  $L^2$  error of various methods for the first test problem.Fig. 6.2: Plot of the  $L^2$  error of various methods for the second test problem.

[42, 13]. The plot shows that the error on  $\Gamma_h$  is almost the same for each of the four methods considered.

Further numerical examples are available in [45].

**6.3. Parabolic equation on an evolving curve.** For an example of an evolving curve we take  $\Gamma(t) = \{x \in \mathbb{R}^2 \mid \Phi(x, t) = 0\}$  for

$$\Phi(x, t) = \frac{x^2}{1 + \frac{1}{4} \sin(2\pi t)} + y^2 - 1,$$



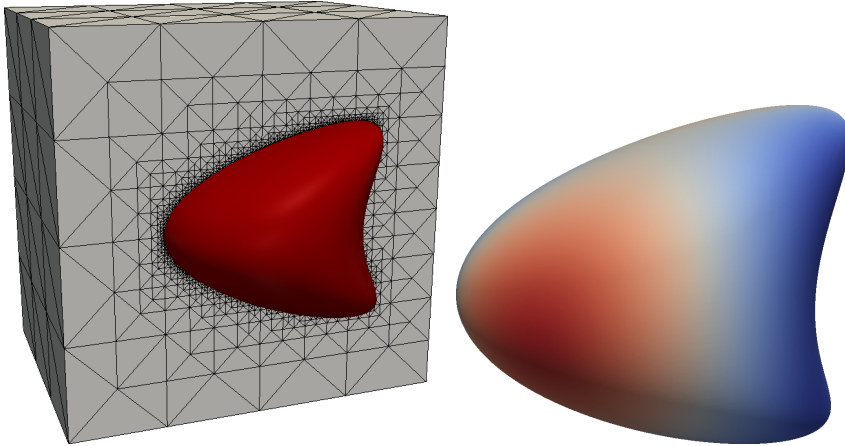


Fig. 6.3: Plots of the computation domain (left) and solution (right) with  $h = 2^{-6}\sqrt{3}$  for the second problem.

$h$	$\max_m \ u(t^m) - u_h^m\ _{L^2(\Gamma_h^m)}$	(eoc)
$2^{-1}\sqrt{2}$	$1.15457 \cdot 10^{-1}$	—
$2^{-2}\sqrt{2}$	$3.25344 \cdot 10^{-2}$	1.82732
$2^{-3}\sqrt{2}$	$8.64172 \cdot 10^{-3}$	1.91258
$2^{-4}\sqrt{2}$	$2.13241 \cdot 10^{-3}$	2.01883
$2^{-5}\sqrt{2}$	$5.42960 \cdot 10^{-4}$	1.97357

Table 6.3: Results of the hybrid scheme for a parabolic equation on an evolving curve.

for  $t \in [0, \frac{1}{2}]$ . We calculate a right-hand side  $f$  so that the exact solution is  $u(x, t) = \exp(-4t)x_1x_2$ . Taking  $\tau = 2h^2$ , the scheme demonstrates second order convergence in the  $L^2(\Gamma_h^m)$ -norm; see Table 6.3. Numerical experiments confirm the conservation of mass result.

#### REFERENCES

- [1] T. AUBIN, *Nonlinear analysis on manifolds. Monge-Ampère equations*, vol. 252 of Grundlehren der mathematischen Wissenschaften., Springer-Verlag, Berlin and New York., 1982.
- [2] R. BARREIRA, C. M. ELLIOTT, AND A. MADZVAMUSE, *The surface finite element method for pattern formation on evolving biological surfaces*, J. Math. Bio., 63 (2011), pp. 1095–1119.
- [3] J. W. BARRETT AND C. M. ELLIOTT, *A finite element method on a fixed mesh for the stefan problem with convection in a saturated porous medium*, in Numerical methods for fluid dynamics (Conference proceedings: University of Reading, 29-31 March, 1982), K. W. Morton and M. J. Baines, eds., Conference Series ( Institute of Mathematics and Its Applications), Academic Press, 1982, pp. 389–409.
- [4] J. W. BARRETT AND C. M. ELLIOTT, *A finite-element method for solving elliptic equations with Neumann data on a curved boundary using unfitted meshes*, IMA J. Numer. Anal., 4 (1984), pp. 309–325.
- [5] R. A. BARRIO, P. K. MAINI, P. PADILLA, R. G. PLAZA, AND SÁNCHEZ-GARDUNO, *The effect of growth and curvature on pattern formation*, J. Dyn. Diff. Eqn., 16 (2004), pp. 1093–1121.
- [6] P. BASTIAN, M. BLATT, A. DEDNER, C. ENGWER, R. KLOFKORN, M. OLDBERGER, AND O. SANDER, *A generic grid interface for parallel and adaptive scientific computing. part i: Abstract framework*, Comput., 82 (2008), pp. 103–119.
- [7] ———, *A generic grid interface for parallel and adaptive scientific computing. part ii: Impele-*

- mentation and tests in DUNE, *Comput.*, 82 (2008), pp. 121–138.
- [8] P. BASTIAN AND C. ENGWER, *An unfitted finite element method using discontinuous Galerkin*, *Int. J. Number. Meth. Eng.*, 79 (2009), pp. 1557–1576.
- [9] M. BERTALMIÓ, LI-TIEN CHENG, S. J. OSHER, AND G. SAPIRO, *Variational problems and partial differential equations on implicit surfaces*, *J. Comput. Phys.*, 174 (2001), pp. 759–780.
- [10] M. BURGER, *Finite element approximation of elliptic partial differential equations on implicit surfaces*, *Comput. Visual. Sci.*, 12 (2009), pp. 87–100.
- [11] A. Y. CHERNYSHENKO AND M. A. OLSHANSKII, *Non-degenerate Eulerian finite element method for solving pdes on surfaces*, *Russ. J. Numer. Anal. M.*, 28 (2013), pp. 101–124.
- [12] K. DECKELNICK, G. DZIUK, AND C. M. ELLIOTT, *Computation of geometric partial differential equations and mean curvature flow*, *Acta Numerica*, 14 (2005), pp. 139–232.
- [13] K. DECKELNICK, G. DZIUK, C. M. ELLIOTT, AND C.-J. HEINE, *An h-narrow band finite element method for implicit surfaces*, *IMA J. Numer. Anal.*, 30 (2010), pp. 351–376.
- [14] A. DEDNER, R. KLÖFKORN, M. NOLTE, AND M. OHLBERGER, *A generic interface for parallel and adaptive discretization schemes: abstraction principles and the DUNE-FEM module*, *Comput.*, 90 (2010), pp. 165–196.
- [15] A. DEMLOW, *Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces*, *SIAM J. Numer. Anal.*, 47 (2009), pp. 805–827.
- [16] A. DEMLOW AND G. DZIUK, *An adaptive finite element method for the Laplace-Beltrami operator on surfaces*, *SIAM J. Numer. Anal.*, 45 (2007), pp. 421–442.
- [17] A. DEMLOW AND M. A. OLSHANSKII, *An adaptive surface finite element method based on volume meshes*, *SIAM J. Numer. Anal.*, 50 (2012), pp. 1624–1647.
- [18] G. DZIUK, *Finite elements for the Beltrami operator on arbitrary surfaces*, in *Partial differential equations and calculus of variations*, S. Hildebrandt and R. Leis, eds., vol. 1357 of *Lecture Notes in Mathematics*, Springer Berlin Heidelberg New York London Paris Tokyo, 1988, pp. 142–155.
- [19] G. DZIUK AND C. M. ELLIOTT, *Finite elements on evolving surfaces*, *IMA J. Numer. Anal.*, 25 (2007), pp. 385–407.
- [20] ———, *Surface finite elements for parabolic equations*, *J. Comp. Math.*, 25 (2007), pp. 385–407.
- [21] ———, *Eulerian finite element method for parabolic pdes on implicit surfaces*, *Interface. Free Bound.*, 10 (2008), pp. 119–138.
- [22] ———, *An Eulerian approach to transport and diffusion on evolving implicit surfaces*, *Comput. Visual. Sci.*, 13 (2010), pp. 17–28.
- [23] ———, *Fully discrete evolving surface finite element method*, *SIAM J. Numer. Anal.*, 50 (2012), pp. 2677–2694.
- [24] ———, *Finite element methods for surface PDEs*, *Acta Numerica*, 22 (2013), pp. 289–396.
- [25] ———,  *$L^2$  estimates for the evolving surface finite element method*, *Math. Comp.*, 82 (2013), pp. 1–24.
- [26] G. DZIUK, C. LUBICH, AND D. MANSOUR, *Runge-Kutta time discretization of parabolic differential equations on evolving surfaces*, *IMA J. Numer. Anal.*, 32 (2012), pp. 394–416.
- [27] C. EILKS AND C. M. ELLIOTT, *Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method*, *J. Comput. Phys.*, 227 (2008), pp. 9727–9741.
- [28] C. M. ELLIOTT AND B. STINNER, *A surface phase field model for two-phase biological membranes*, *SIAM J. Appl. Math.*, 70 (2010), pp. 2904–2928.
- [29] C. M. ELLIOTT, B. STINNER, V. STYLES, AND R. WELFORD, *Numerical computation of advection and diffusion on evolving diffuse interfaces*, *IMA J. Numer. Anal.*, 31 (2011), pp. 786–812.
- [30] C. M. ELLIOTT, B. STINNER, AND C. VENKATARAMAN, *Modelling cell motility and chemotaxis with evolving surface finite elements*, *J. Roy. Soc. Interface*, 9 (2012), pp. 3027–3044.
- [31] C. ENGWER AND F. HEIMANN, *DUNE-UDG: a cut-cell framework for unfitted discontinuous Galerkin methods*, in *Advances in DUNE*, Springer, 2012, pp. 89–100.
- [32] J. ERLEBACHER AND I. MCCUE, *Geometric characterization of nanoporous metals*, *Acta Materialia*, 60 (2012), pp. 6164 – 6174.
- [33] S. GANESAN AND L. TOBISKA, *A coupled arbitrary Lagrangian-Eulerian and Lagrangian method for computation of free-surface flows with insoluble surfactants*, *J. Comput. Phys.*, 228 (2009), pp. 2859–2873.
- [34] JOHN B. GREER, *An improvement of a recent Eulerian method for solving PDEs on general geometries*, *J. Sci. Comput.*, 29 (2006), pp. 321–352.
- [35] J. B. GREER, A. L. BERTOZZI, AND G. SAPIRO, *Fourth order partial differential equations on general geometries*, *J. Comput. Phys.*, 216 (2006), pp. 216–246.
- [36] A. HANSBO AND P. HANSBO, *An unfitted finite element method, based on Nitsche’s method, for elliptic interface problems*, *Comput. Method. Appl. M.*, 191 (2002), pp. 5537–5552.

- [37] MING-CHIH LAI, YU-HAU TSENG, AND HUAXIONG HUANG, *An immersed boundary method for interfacial flows with insoluble surfactant*, J. Comput. Phys., 227 (2008), pp. 7279–7293.
- [38] D. MARENDUZZO AND E. ORLANDINI, *Phase separation dynamics on curved surfaces*, Soft Matter, 9 (2013), pp. 1178–1187.
- [39] W. W. MULLINS, *Mass transport at interfaces in single component systems*, Metall. Mater. Trans. A, 26 (1995), pp. 1917–1929.
- [40] I. L. NOVAK, F. GAO, Y.-S. CHOI, D. RESASCO, J. C. SCHAFF, AND B. M. SLEPCHENKO, *Diffusion on a curved surface coupled to diffusion in the volume: Application to cell biology*, J. Comput. Phys., 226 (2007), pp. 1271–1290.
- [41] M. A. OLSHANSKII AND A. REUSKEN, *A finite element method for surface PDEs: matrix properties*, Numer. Math., 114 (2010), pp. 491–520.
- [42] M. A. OLSHANSKII, A. REUSKEN, AND J. GRANDE, *A finite element method for elliptic equations on surfaces*, SIAM J. Numer. Anal., 47 (2009), pp. 3339–3358.
- [43] M. A. OLSHANSKII, A. REUSKEN, AND X. XU, *An Eulerian space-time finite element method for diffusion problems on evolving surfaces*. arXiv:1304.6155v1, 2013.
- [44] ———, *On surface meshes induced by level set functions*. arXiv:1301.3745v2, 2013.
- [45] T. RANNER, *Computational surface partial differential equations*, PhD thesis, University of Warwick, 2013.
- [46] A. RÄTZ AND M. ROGER, *Turing instabilities in a mathematical model for signaling networks*, J. Math. Bio., 65 (2012), pp. 1215–1244.
- [47] A. RÄTZ AND A. VOIGT, *PDEs on surfaces - a diffuse interface approach*, Commun. Math. Sci., 4 (2006), pp. 575–590.
- [48] A. REUSKEN, *A Finite Element Level Set Redistancing Method Based on Gradient Recovery*, SIAM J. Num. Anal., 51 (2013), pp. 2723–2745.
- [49] O. SCHONBORN AND R. C. DESAI, *Phase-ordering kinetics on curved surfaces*, Physica A, 239 (1997), pp. 412–419.
- [50] P. SCHWARTZ, D. ADALSTEINSSON, P. COLELLA, A. P. ARKIN, AND M. ONSUM, *Numerical computation of diffusion on a surface*, P. Natl. Acad. Sci. USA, 102 (2005), pp. 11151–11156.
- [51] J. R. SHEWCHUK, *Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator*, in Applied Computational Geometry Towards Geometric Engineering, Ming C. Lin and Dinesh Manocha, eds., vol. 1148 of Lecture Notes in Computer Science, Springer, Berlin / Heidelberg, 1996, pp. 203–222.
- [52] ———, *Triangle: A two-dimensional quality mesh generator and Delaunay triangulator (version 1.6)*. available at <http://www.cs.cmu.edu/quake/triangle.html>, 2005.
- [53] H. SI, *Tetgen: A quality tetrahedral mesh generator and 3D Delaunay triangulator (Version 1.4.1) [Computer software]*. Available at <http://tetgen.org>, 2006.
- [54] H. A. STONE, *A simple derivation of the time dependent convective diffusion equation for surfactant transport along a deforming interface*, Phys. Fluids A-Fluid, 2 (1990), pp. 111–112.