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Preduals of semigroup algebras

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Abstract

For a locally compact group G, the measure convolution algebra $M(G)$ carries a natural coproduct. In previous work, we showed that the canonical predual $C_0(G)$ of $M(G)$ is the unique predual which makes both the product and the coproduct on $M(G)$ weak*-continuous. Given a discrete semigroup S, the convolution algebra $\ell^1(S)$ also carries a coproduct. In this paper we examine preduals for $\ell^1(S)$ making both the product and the coproduct weak*-continuous. Under certain conditions on S, we show that $\ell^1(S)$ has a unique such predual. Such S include the free semigroup on finitely many generators. In general, however, this need not be the case even for quite simple semigroups and we construct uncountably many such preduals on $\ell^1(S)$ when S is either $\mathbb{Z}_+ \times \mathbb{Z}$ or $(\mathbb{N}, \cdot)$.
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1 Introduction

A *dual Banach algebra* is a Banach algebra $\mathcal{A}$ which is the dual of a Banach space $\mathcal{A}_*$, such that the product on $\mathcal{A}$ is separately weak*-continuous. The motivating example is a von Neumann algebra, where the predual is isometrically unique. This need not be true for Banach algebras: consider ℓ^1 with the zero product. In [6], we considered the measure algebra of a locally compact group, $M(G)$, which is a dual Banach algebra with respect to the predual $C_0(G)$. We can define a natural coproduct on $M(G)$, dualising the multiplication on $C_0(G)$. In [6, Theorem 3.6], we showed that $C_0(G)$ is the unique predual making both the product and the coproduct on $M(G)$ weak*-continuous. The proof makes use of results which are only true for group algebras. In this paper we consider preduals of ℓ^1-semigroup algebras which make both the product and coproduct weak*-continuous. Such preduals are termed *Hopf algebra preduals*. Perhaps surprisingly, we show that quite simple semigroups, such as $\mathbb{Z}_+ \times \mathbb{Z}$, give rise to algebras with uncountably many Hopf algebra preduals.

Given a predual $\mathcal{A}_*$ of $\mathcal{A}$, we can naturally identify $\mathcal{A}_*$ with a closed subspace of $\mathcal{A}^*$. Then the product on $\mathcal{A}$ becomes separately weak*-continuous if and only if $\mathcal{A}_*$ is a submodule of $\mathcal{A}^*$, for the usual action of $\mathcal{A}$ on its dual. It is easy to see that isomorphic preduals will induce the same subspace of $\mathcal{A}^*$. Henceforth, by a *predual* $\mathcal{A}_*$ of a Banach algebra $\mathcal{A}$, we shall mean a closed submodule of $\mathcal{A}^*$ which is a Banach space predual.

We shall consider convolution algebras $\ell^1(S)$ for a (countable) semigroup S. We write $(\delta_s)_{s\in S}$ for the standard unit vector basis of $\ell^1(S)$, so each $a \in \ell^1(S)$ can be uniquely expressed as a norm-convergent sum

$$a = \sum_{s\in S} a_s \delta_s \quad \text{where} \quad \|a\| = \sum_{s\in S} |a_s|.$$

The coproduct on $\ell^1(S)$ is the map $\Gamma : \ell^1(S) \to \ell^1(S \times S)$ defined by

$$\Gamma(\delta_s) = \delta_{(s,s)} \qquad (s \in G).$$

Let $E \subseteq \ell^\infty(S) = \ell^1(S)^*$ be a predual for $\ell^1(S)$. Then it is easy to see (compare with [6, Lemma 3.3]) that Γ is weak*-continuous if and only if E is a subalgebra of $\ell^\infty(S)$. As in [6], we shall term such a predual a *Hopf algebra predual* of $\ell^1(S)$.

In the next section, we show that the study of Hopf algebra preduals of $\ell^1(S)$ is equivalent to the study of certain semigroup topologies on S. For certain cancellative semigroups S, including a finite direct sum of copies of $\mathbb{Z}_+$ and the free semigroup on finitely many generators, we show that $\ell^1(S)$ has a unique Hopf algebra predual. In Section 3, we exhibit semigroups admitting uncountably many Hopf algebra preduals. These semigroups are quite simple in nature, including $\mathbb{Z}_+ \times \mathbb{Z}$ and $(\mathbb{N}, \cdot)$. Since this last semigroup is isomorphic to the direct sum of countably many copies of $\mathbb{Z}_+$, it is not possible to extend the uniqueness result for Hopf algebra preduals of finite direct sums of $\mathbb{Z}_+$ to infinite direct sums. In Section 4, we exhibit a semigroup S for which $\ell^1(S)$ admits no Hopf algebra preduals and we end in Section 5 by showing that $\ell^1(\mathbb{N}, \max)$ has a unique predual in full generality.

Finally, some words about notation. For a Banach space E, we write E^* for its dual, and use the dual-pairing notation $\langle \mu, x\rangle = \mu(x)$ for $\mu \in E^*$ and $x \in E$.

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2 General theory

Let us recall the following, which is [6, Lemma 3.4].

Lemma 2.1. *Let K be a locally compact Hausdorff space, let L be a compact Hausdorff space, and let $\mathcal{A}$ be a closed subalgebra of $C_0(K)$ such that $\mathcal{A}^{**}$ is Banach algebra isomorphic to $C(L)$. Then $\mathcal{A}$ is a C^*-subalgebra of $C_0(K)$.*

A semigroup K which carries a topology is said to be *semitopological* when the product is separately continuous.

Proposition 2.2. *Let S be a discrete semigroup, and let $E \subseteq \ell^\infty(S)$ be a Hopf algebra predual for $\ell^1(S)$. Then E is a C^*-subalgebra of $\ell^\infty(S)$. Let K be the character space of E. Then K is canonically bijective to S, and we use this bijection to turn K into a semigroup. Then:*

1. *K is a semitopological semigroup;*
2. *for each $s \in K$ and $L \subseteq K$ compact, the sets $\{t \in K : st \in L\}$ and $\{t \in K : ts \in L\}$ are compact.*

This structure on K completely determines E.

Proof. Arguing as in the proof of [6, Corollary 3.5], E^{**} is Banach algebra isomorphic to $\ell^\infty(S)$. It then follows immediately from Lemma 2.1 that E is a C*-subalgebra of $\ell^\infty(S)$. Let K be the character space, and let $\mathcal{G} : E \to C_0(K)$ be the Gelfand transform. Then following the proof of [6, Theorem 3.6], we get a natural bijection $\theta : S \to K$ such that

$$f(k) = \langle \mathcal{G}^{-1}(f), \delta_{\theta^{-1}(k)} \rangle \qquad (f \in C_0(K), k \in K).$$

It also follows that K is semitopological.

The second condition follows as the left and right translations by elements of K will take $C_0(K)$ to $C_0(K)$. Let $L \subseteq K$ be compact, and let $s \in K$. For each $r \notin L$, we can find $f_r \in C_0(K)$ with $f_r(t) = 1$ for $t \in L$, and $f_r(r) = 0$. Then let $g = f_r \cdot \delta_s$, so that

$$g(t) = \langle \delta_t, g \rangle = \langle \delta_t, f_r \cdot \delta_s \rangle = \langle \delta_{st}, f_r \rangle = f_r(st) \qquad (t \in K).$$

Then as $E = C_0(K)$ is an $\ell^1(S)$-submodule, we have that $g \in C_0(K)$, so the set $L_r = \{t \in K : |f_r(st)| \geq 1\}$ is compact. As $f_r(r) = 0$, we see that $t \in L_r$ implies that $st \neq r$. Similarly, if $st \in L$, then certainly $t \in L_r$. So we conclude that

$$\bigcap_{r \notin L} L_r = \{t \in K : st \in L\}$$

is a compact set, as required. Similarly, $\{t \in K : ts \in L\}$ is compact.

Finally, suppose that $F \subseteq \ell^\infty(S)$ is another predual, inducing a semitopological semigroup L by a bijection $\phi : S \to L$. Suppose that $K = L$ in the sense that $\phi\theta^{-1} : K \to L$ is a bi-continuous homomorphism. Let $x \in E$, so that

$$\langle x, \delta_s \rangle = \mathcal{G}(x)(\theta(s)) \qquad (s \in S).$$

Let $f = \mathcal{G}(x) \in C_0(K)$, and let $g \in C_0(L)$ be defined by

$$g(l) = f\big(\theta\phi^{-1}(l)\big) \qquad (l \in L).$$

Let $\mathcal{H} : F \to C_0(L)$ be the Gelfand transform, and let $y = \mathcal{H}^{-1}(g) \in F$. Then

$$\langle y, \delta_s \rangle = \mathcal{H}(y)(\phi(s)) = g(\phi(s)) = f(\theta(s)) = \langle x, \delta_s \rangle,$$

so that $x = y$. As $x \in E$ was arbitrary, we conclude that $E \subseteq F$, and by symmetry, that actually $E = F$, as required. $\square$

Lemma 2.3. *Let S be a countable semigroup, and let σ be a locally compact, semitopological topology on S satisfying condition (2) above. Then $C_0(S, \sigma)$ is a predual for $\ell^1(S)$.*

Proof. As S is countable, it follows that $C_0(S,\sigma)^* = M(S,\sigma) = \ell^1(S)$. Let $f \in C_0(S,\sigma)$, so that

$$\langle \delta_s \cdot f, \delta_t \rangle = f(ts) \qquad (s,t \in S),$$

so $\delta_s \cdot f \in C_0(S,\sigma)$ if and only if $f_s : t \mapsto f(ts)$ is in $C_0(S,\sigma)$. As S is semitopological, f_s is continuous. For $\epsilon > 0$, we see that

$$X = \{s \in S : |f(s)| \geq \epsilon\}$$

is compact, so that

$$\{t \in S : |f_s(t)| \geq \epsilon\} = \{t \in S : ts \in X\},$$

is compact, by condition (2). So $f_s \in C_0(S,\sigma)$. Similarly, $f \cdot \delta_s \in C_0(S,\sigma)$ for $s \in S$, so $C_0(S,\sigma)$ is a submodule of $\ell^\infty(S)$. □

It hence follows that studying Hopf algebra preduals of $\ell^1(S)$ is completely equivalent to studying locally compact semitopological topologies on S which satisfy condition (2), a task we shall concern ourselves with in much of the rest of this paper.

Corollary 2.4. *Let S be a countable semigroup. Then a locally compact topology σ on S makes $C_0(S,\sigma)$ into a predual for $\ell^1(S)$ if and only if:*

1. *(S,σ) is a semitopological semigroup;*
2. *is $s \in S$ and $(t_n) \subseteq S$ is a sequence such that either $(t_n s)$ or (st_n) has a convergent subsequence, then (t_n) has a convergent subsequence.*

Proof. As S is countable, (S,σ) is metrisable. Hence the second condition is easily seen to be equivalent to the second condition in the result above. □

We say that a semigroup S is *weakly cancellative* when the left and right translation maps are all finite-to-one maps; S is *cancellative* when these maps are all injective. A simple calculation (see, for example, [4, Theorem 4.6]) shows that S is weakly cancellative if and only if $c_0(S)$ is a predual for $\ell^1(S)$.

However, even when S is not weakly-cancellative, $\ell^1(S)$ might still have a predual: for example, the one-point compactification of $\mathbb{N}$, denoted by $\mathbb{N}_\infty$, carries an obvious semigroup structure for which c, the space of convergence sequences, forms a predual for $\ell^1(\mathbb{N}_\infty)$. This fits into our framework, as $c = C_0(\mathbb{N}_\infty, \sigma)$ where σ is the one-point compactification topology on $\mathbb{N}_\infty$.

Proposition 2.5. *Let $E \subseteq \ell^\infty(\mathbb{N}_\infty)$ be a Hopf algebra predual for $\ell^1(\mathbb{N}_\infty)$. Then $E = c$.*

Proof. Let $(\mathbb{N}_\infty, \sigma)$ be the character space of E, so σ satisfies the conditions of Corollary 2.4. To show that $E = c$, we need to show that σ is the one-point compactification topology. That is, we need to show that ∞ is the only limit point of σ.

Towards a contradiction, suppose that (t_n) is a sequence in $\mathbb{N}$, converging to $t < \infty$. By moving to a subsequence, we may suppose that (t_n) is an increasing sequence, and that $t_n > t$ for all n. Then $(t_n) = (t_n - t + t)$ converges, so by the corollary, $(t_n - t)$ have a convergent subsequence, say (s_n). Let $s = \lim_n s_n$, so that as σ is semitopological,

$$s + t = \lim_n s_n + t = \lim_n (t_n - t) + t = \lim_n t_n = t.$$

However, this contradicts $s \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

□

2.1 Cancellation properties

We shall now consider cancellative semigroups S, so that $c_0(S)$ is a Hopf algebra predual for $\ell^1(S)$.

Lemma 2.6. *Let S be a countable cancellative semigroup. Then a locally compact topology σ on S makes $C_0(S, \sigma)$ into a predual for $\ell^1(S)$ if and only if:*

1. *(S, σ) is a semitopological semigroup;*
2. *is $s \in K$ and $(t_n) \subseteq S$ is a sequence such that either $(t_n s)$ or (st_n) converges, then (t_n) converges.*

Proof. We need only check that this new second condition is implied by the second condition in Corollary 2.4. So let $s \in S$ and let $(t_n) \subseteq S$ be a sequence such that $(t_n s)$ is convergent. Hence (t_n) has a convergent subsequence, converging to $t_0 \in S$ say. As (S, σ) is a semitopological semigroup, $(t_n s)$ converges to $t_0 s$. Suppose that (t_n) does not converge to t_0, so we can find a subsequence which is always distant from t_0. By applying the second condition from the preceding lemma again, we find a different limit point, say t_1, for some further subsequence. Again, we hence have that $(t_n s)$ converges to $t_1 s$. So $t_0 s = t_1 s$, and as S is cancellative, $t_0 = t_1$, a contradiction. Hence (t_n) converges, as required. □

We make the following temporary definition.

Definition 2.7. Let S be a semigroup. We say that S is *finitely left divisible* if for each $s \in S$, the set $\{t \in S : \exists\, r \in S, tr = s\}$ is finite.

Theorem 2.8. *Let S be a finitely left divisible, cancellative, finitely generated semigroup. Let $E \subseteq \ell^\infty(S)$ be a Hopf algebra predual for $\ell^1(S)$. Then $E = c_0(S)$.*

Proof. By the previous results, we need to show that if σ is a locally compact topology on S satisfying the conditions of Lemma 2.6, then σ is the discrete topology. Towards a contradiction, suppose that σ is not the discrete topology. Hence we can find a non-isolated point $t_0 \in S$. Let (t_n) be some sequence in $S \setminus \{t_0\}$ which converges to t_0.

Let C be a finite set which generates S. For each $t \in S$, define the *length* of t to be $l(t)$, the smallest n such that $t = c_1 \cdots c_n$ for $(c_i) \subseteq C$. As $\{t_n\}$ is infinite, we see that $\{l(t_n)\}$ is unbounded. By moving to a subsequence, we may suppose that $l(t_1) < l(t_2) < \cdots$. Write $t_n = c_{n,1} c_{n,2} \cdots c_{n,l(t_n)}$ where $c_{n,i} \in C$ for all n and i. By moving to a subsequence, we may suppose that $c_{n,1} = c_1$ for all n. By moving to a further subsequence, we may suppose that $c_{n,2} = c_2$ for all n. We can continue, and then by a diagonal argument, we may suppose that $c_{n,k} = c_k$ for all n and all $k \leq n$.

For each m, for $n \geq m$, we have that $t_n = c_1 c_2 \cdots c_n r_n$ for some $r_n \in S$. Let $u_{n,k} = c_k c_{k+1} \cdots c_n r_n$. As $t_n \to t_0$, we see that $\lim_n c_1 c_2 \cdots c_{k-1} u_{n,k} = t_0$, and so the corollary implies that $(u_{n,k})_{n=1}^\infty$ converges, to v_k say. Hence $t_0 = c_1 c_2 \cdots c_{k-1} v_k$, for all k.

As S is finitely left divisible, we see that the set $\{c_1 c_2 \cdots c_k : k \geq 1\}$ is a finite set in S. In particular, we can find $j < k$ with

$$c_1 c_2 \cdots c_j = c_1 c_2 \cdots c_k.$$

We have that $t_k = c_1 c_2 \cdots c_k c_{k,k+1} \cdots c_{k,l(t_k)}$, where $l(t_k)$ is minimal. However, we now also see that $t_k = c_1 c_2 \cdots c_j c_{k,k+1} \cdots c_{k,l(t_k)}$, which is a shorter expression, contradicting the minimality of $l(t_k)$. This contradiction shows that t_0 is indeed an isolated point, as required. $\square$

Corollary 2.9. *Let $k \geq 1$, and let $S = \mathbb{Z}_+^k$, or $S = \mathbb{S}_k$, the free semigroup on k generators. Then $c_0(S)$ is the unique Hopf algebra predual of $\ell^1(S)$.*

See Section 3 below for counter-examples showing that we cannot remove the "finitely left divisible" or "finitely generated" conditions from the above theorem.

2.2 Rees semigroups of matrix type

Rees matrix semigroups appear naturally in the study of when ℓ^1-semigroup algebras are amenable: see [4, Chapter 10] for further details. In general, they are a way of generating semigroups from other (semi)groups.

Let S be a semigroup with 0, let I and J be index sets, let P be an $J \times I$ matrix of entries from S, called the *sandwich matrix*. Then the *Rees semigroup* $\mathcal{M}(S; I, J; P)$ is the collection of all $I \times J$ matrices with entries from S, where exactly one entry is non-zero. Then we can define a product on $\mathcal{M}(S; I, J; P)$ by

$$A \cdot B = APB \qquad (A, B \in \mathcal{M}(S; I, J; P)).$$

Of course, we cannot multiply matrices with entries in S, as we have no concept of addition. However, a moment's though reveals that in all calculations, at most one entry will be non-zero, so there is no ambiguity in what we mean by "matrix multiplication" in this setting.

An alternative description is the following. Let $(\epsilon_{i,j})_{i \in I, j \in J}$ be the matrix units, so each element of $\mathcal{M}(S; I, J; P)$ can be written, formally, as $s\epsilon_{i,j}$ for some $s \in S$. Then

$$s\epsilon_{i,j} \cdot t\epsilon_{k,l} = sP_{j,k} t\epsilon_{i,l} \qquad (s, t \in S, i, k \in I, j, l \in J).$$

Lemma 2.10. *For a group G, we have that $\mathcal{M}(G; I, J; P)$ is weakly-cancellative if and only if I and J are finite.*

Proof. If J is infinite, then for $s \in G$, $i \in I$ and $j \in J$, fixed, notice that

$$s\epsilon_{i,j} \cdot t\epsilon_{k,l} = sP_{j,k}t\epsilon_{i,l} = sP_{j,a}P_{j,a}^{-1}P_{j,k}t\epsilon_{i,l} = s\epsilon_{i,j} \cdot P_{j,a}^{-1}P_{j,k}t\epsilon_{a,l},$$

for any $t \in G$, $k, a \in I$ and $l \in J$. So the map given by multiplication on the left by $s\epsilon_{i,j}$ is an infinite-to-one map. Hence $\mathcal{M}(G; I, J; P)$ weakly-cancellative implies that J is finite; similarly I must be finite. ☐

Proposition 2.11. *Let G be a countable group, let I and J be finite, and let $S = \mathcal{M}(G; I, J; P)$. Then $c_0(S)$ is the unique C^*-predual for $\ell^1(S)$.*

Proof. Let σ be a locally compact Hausdorff topology on S, making (S, σ) a semitopological semigroup. As (S, σ) is a countable Baire space, there exists $s_0 = g_0\epsilon_{i_0,j_0} \in S$ with $\{s_0\}$ open. Then, for $g \in G$, $i \in I$ and $j \in J$, notice that

$$\begin{aligned}\{(h,k,l) \in G \times I \times J : h\epsilon_{k,l} \cdot g\epsilon_{i,j} = s_0\} &= \{(h, i_0, l) : l \in J, hP_{l,i}g = g_0\} \\ &= \{(g_0 g^{-1} P_{l,i}^{-1}, i_0, l) : l \in J\}\end{aligned}$$

if $j = j_0$, or the empty set otherwise. Hence, for all $g \in G$ and $i \in I$, the set

$$U_{g,i} = \{gP_{l,i}^{-1}\epsilon_{i_0,l} : l \in J\}$$

is open. Similarly, we can check that

$$V_{g,j} = \{P_{j,k}^{-1} g\epsilon_{k,j_0} : k \in I\}$$

is open, for all $g \in G$ and $j \in J$.

Notice now that for $g, h \in G$, $I \in I$ and $j \in J$,

$$U_{g,i} \cap V_{h,j} = \begin{cases} \{gP_{j_0,i}^{-1}\epsilon_{i_0,j_0}\} & : gP_{j_0,i}^{-1} = P_{j,i_0}^{-1}h \\ \emptyset & : \text{otherwise.}\end{cases}$$

So we have that $\{g\epsilon_{i_0,j_0}\}$ is open, for any $g \in G$.

Let X be the collection of open singletons in (S, σ). Then we have just proved that if $g_0\epsilon_{i_0,j_0} \in X$ for some g_0, then $g\epsilon_{i_0,j_0} \in X$ for all $g \in G$. Also, X must be dense in (S, σ), for otherwise, the complement of the closure of X would be a non-empty open subset of a locally compact space, and hence locally compact itself. Repeating the Baire space argument would then yield an open singleton not in X, a contradiction.

Towards a contradiction, suppose that X is not all of S. Then there exists $s_0 = g_0\epsilon_{i_0,j_0} \notin X$ and a sequence $(g_n\epsilon_{i_n,j_n})$ in X converging to s_0. By moving to a subsequence, we may suppose that $i_n = i$ and $j_n = j$ for all n, so that

$$g_n\epsilon_{i,j} \longrightarrow s_0.$$

We then see that for $g, h \in G$, $k, p \in I$ and $l, q \in J$,

$$gP_{l,i}g_nP_{j,p}h\epsilon_{k,q} = g\epsilon_{k,l} \cdot g_n\epsilon_{i,j} \cdot h\epsilon_{p,q} \longrightarrow g\epsilon_{k,l} \cdot g_0\epsilon_{i_0,j_0} \cdot h\epsilon_{p,q} = gP_{l,i_0}g_0P_{j_0,p}h\epsilon_{k,q}.$$

Pick $(k,q) \in I \times J$ and $f \in G$ such that $f\epsilon_{k,q} \in X$. By the above, we see that

$$g_0^{-1}P_{l,i_0}^{-1}P_{l,i}g_nP_{j,p}P_{j_0,p}^{-1}f\epsilon_{k,q} \to g_0^{-1}P_{l,i_0}^{-1}P_{l,i_0}g_0P_{j_0,p}P_{j_0,p}^{-1}f\epsilon_{k,q} = f\epsilon_{k,q}.$$

As $\{f\epsilon_{k,q}\}$ is open, we must have that

$$P_{l,i}g_nP_{j,p} = P_{l,i_0}g_0P_{j_0,p} \qquad (n \in \mathbb{N}).$$

However, this would imply that the

$$P_{l,i}^{-1}P_{l,i_0}g_0P_{j_0,p}P_{j,p}\epsilon_{i,j} \to s_0 = g_0\epsilon_{i_0,j_0},$$

which is a contradiction, as $P_{l,i}^{-1}P_{l,i_0}g_0P_{j_0,p}P_{j,p}\epsilon_{i,j} \in X$, yet $s_0 \notin X$. □

3 Semigroup algebras with many preduals

In this section, we shall show that if $S = (\mathbb{N}, \cdot)$, the semigroup of natural numbers with multiplication product, or if $S = (\mathbb{Z}_+ \times \mathbb{Z}, +)$, then $\ell^1(S)$ admits a continuum of Hopf algebra preduals. Notice that both semigroups are cancellative, and the former is finitely (left) divisible but not finitely generated, whereas the latter is finitely generated but not finitely (left) divisible. We begin with a elementary technical observation.

Lemma 3.1. *Let $a, \alpha \in \mathbb{N}$, and define*

$$X_{a,\alpha} = \{0\} \ \cup \ \Big\{ \sum_{i=1}^{k} 2^{-m_i} : \ 1 \le k \le a, \ \alpha \le m_1 < \ldots < m_k \Big\} \subseteq [0,1],$$

equipped with the subspace topology. Let $\alpha \le m_1 < m_2 < \cdots < m_k$ for some $1 \le k \le a$, and let $x_0 = \sum_{i=1}^k 2^{-m_i}$. For $\beta > m_k$, let

$$Y = \Big\{ \sum_{i=1}^{k+l} 2^{-m_i} : 0 \le l \le a-k, \beta \le m_{k+1} < \cdots < m_{k+l} \Big\}.$$

Then Y is open in $X_{a,\alpha}$.

Proof. We claim that $(x_0 - 2^{-m_k-a}, x_0] \cap X_{a,\alpha} = \{x_0\}$. Given $x_1 \in X_{a,\alpha}$ with $x_0 - 2^{-m_k-a} < x_1 \le x_0$, write

$$x_1 = \sum_{i=1}^{l} 2^{-n_i}$$

for some $1 \le l \le a$ and $\alpha \le n_1 < \cdots < n_l$. Certainly $n_1 \ge m_1$, as otherwise $x_1 > x_0$. If $n_1 > m_1$, then

$$x_1 \le 2^{-n_1} + 2^{-n_1-1} + \cdots + 2^{-n_1-a+1} = 2^{1-n_1}\big(1-2^{-a}\big) < 2^{-m_1}(1-2^{-a}) \le x_0 - 2^{-m_k-a},$$

contrary to hypothesis. Thus $n_1 = m_1$. If $k = 1$, then $l = 1$ as $x_1 \le x_0$. Otherwise, we must have $n_2 \ge m_2$. Again if $n_2 > m_2$, then

$$x_1 \le 2^{-n_1} + 2^{-n_2} + 2^{-n_2-1} + \cdots + 2^{-n_2-a+2} \le 2^{-m_1} + 2^{-m_2}\left(1 - 2^{1-a}\right) \le x_0 - 2^{-m_k-a},$$

gives a contradiction. Therefore $n_2 = m_2$. Proceeding in this way shows that $l = k$ and $n_1 = m_1, \cdots, n_k = m_k$, establishing the claim.

A similar argument shows that any $x_1 \in X_{a,\alpha}$ with $x_0 \le x_1 < x_0 + 2^{-m_k-a}$ can be written in the form

$$x_1 = \sum_{i=1}^{k} 2^{-m_i} + \sum_{i=k+1}^{l} 2^{-m_i}$$

for some $k \le l \le a$ and $m_k < m_{k+1} < \cdots < m_l$. If in addition $x_1 < x_0 + 2^{-\beta}$, then $m_{k+1} \ge \beta$ so $x_1 \in Y$. Thus $(x_0 - 2^{-m_k-a}, x_0 + \min(2^{-m_k-a}, 2^{-\beta})) \cap X_{a,\alpha} = Y$ and so Y is open in $X_{a,\alpha}$. $\square$

Let S be a subsemigroup of a commutative group $(G, +)$, and let $S_1 = \mathbb{Z}_+ \times S$. Let $(w_n)_{n=1}^{\infty}$ be a sequence in S with the property that for each $s, t \in S$, $s - t \in G$ can be written in at most one way as

$$s - t = \sum_{i=1}^{k} w_{m_i} - \sum_{j=1}^{l} w_{n_j} \tag{*}$$

where $k, l \in \mathbb{Z}_+$ and $\{m_1, \ldots, m_k, n_1, \ldots, n_l\}$ is a collection of distinct natural numbers. As usual the empty sum takes the value $0 \in G$ so the condition (*) shows that if $k, l \ge 1$ and

$$\sum_{i=1}^{k} w_{m_i} = \sum_{j=1}^{l} w_{n_i}$$

for some sets (of distinct natural numbers) $\{m_1, \ldots, m_k\}$ and $\{n_1, \ldots, n_l\}$, then $k = l$ and $\{m_1, \ldots, m_k\} = \{n_1, \ldots, n_l\}$. For $(a, s) \in S_1$, and $\alpha \in \mathbb{N}$, define

$$U_{a,s,\alpha} = \left\{ \left(a - k, s + \sum_{i=1}^{k} w_{m_i}\right) : \ 0 \le k \le a, \ \alpha \le m_1 < \cdots < m_k \right\}.$$

Lemma 3.2. *With the notation and conditions as above, the collection $\{U_{a,s,\alpha}\}$ forms a base for a topology σ on S_1 making (S_1, σ) a locally compact semitopological semigroup.*

Proof. We first show that $\{U_{a,s,\alpha} : \ (a, s) \in S_1, \ \alpha \in \mathbb{N}\}$ is a base for a topology σ on S_1. To this end, note that if $(b, t) \in U_{a,s,\alpha} \setminus \{(a, s)\}$, say $(b, t) = (a - k, s + \sum_{i=1}^{k} w_{m_i})$, then $U_{b,t,\beta} \subset U_{a,s,\alpha}$ whenever $\beta \ge m_k + 1$. Now for $(a, s), (b, t) \in S_1$ and $\alpha, \beta \in \mathbb{N}$, take $(c, u) \in U_{a,s,\alpha} \cap U_{b,t,\beta}$. Then there exists γ_1, γ_2 such that $U_{c,u,\gamma_1} \subseteq U_{a,s,\alpha}$ and $U_{c,u,\gamma_2} \subseteq U_{b,t,\beta}$. Then

$$U_{c,u,\max(\gamma_1,\gamma_2)} \subseteq U_{c,u,\gamma_1} \cap U_{c,u,\gamma_2} \subseteq U_{a,s,\alpha} \cap U_{b,t,\beta},$$

so the $U_{a,s,\alpha}$ do form a base for a topology.

Next we show that σ is a Hausdorff topology. Take $(a,s) \neq (b,t) \in S_1$ and suppose first that $s = t$ so that $a \neq b$. If there exists α and β with $U_{a,s,\alpha} \cap U_{b,t,\beta} \neq \emptyset$, then we can find k, l with $a - k = b - l$, and sequences (m_i) and (n_j) with

$$s + \sum_{i=1}^{k} \omega_{m_i} = s + \sum_{j=1}^{l} \omega_{n_j}.$$

Condition (*) enables us to conclude that $k = l$, so that $a = b$, a contradiction. In particular $U_{a,s,1}$ and $U_{b,s,1}$ are disjoint neighbourhoods of (a,s) and (b,s) respectively. Now suppose that $t - s \neq 0$. If $t - s$ can be written as

$$t - s = \sum_{i=1}^{\tilde{k}} w_{\tilde{m}_i} - \sum_{j=1}^{\tilde{l}} w_{\tilde{n}_j}, \tag{1}$$

where $\tilde{k} + \tilde{l} \geq 1$ and where $\tilde{m}_i$ and $\tilde{n}_j$ are all distinct natural numbers, then set

$$\alpha = \beta = \max\{\tilde{m}_1, \ldots, \tilde{m}_{\tilde{k}}, \tilde{n}_1, \ldots, \tilde{n}_{\tilde{l}}\} + 1.$$

Otherwise, set $\alpha = \beta = 1$. Assume toward a contradiction that $U_{a,s,\alpha} \cap U_{b,t,\beta} \neq \emptyset$. So we can find k, l with $a - k = b - l$, and (m_i) and (n_j) with

$$s + \sum_{i=1}^{k} w_{m_i} = t + \sum_{j=1}^{l} w_{n_j} \quad \text{that is,} \quad t - s = \sum_{i=1}^{k} w_{m_i} - \sum_{j=1}^{l} w_{n_j}.$$

The uniqueness of the expression (1) implies that

$$\{\tilde{m}_1, \ldots, \tilde{m}_{\tilde{k}}, \tilde{n}_1, \ldots, \tilde{n}_{\tilde{l}}\} \subset \{m_1, \ldots, m_k, n_1, \ldots, n_l\}.$$

This contradicts either $\alpha \leq m_1 < \ldots < m_k$ or $\beta \leq n_1 < \ldots < n_l$. In conclusion σ is a Hausdorff topology.

Our next objective is to show that σ is locally compact. For $a \in \mathbb{Z}_+$ and $\alpha \in \mathbb{N}$ define

$$X_{a,\alpha} = \{0\} \ \cup \ \left\{ \sum_{i=1}^{k} 2^{-m_i} : \ 1 \leq k \leq a, \ \alpha \leq m_1 < \ldots < m_k \right\} \subseteq [0,1],$$

so that $X_{a,\alpha}$ is compact. Fix (a,s,α) and define a map

$$\psi : U_{a,s,\alpha} \to X_{a,\alpha}, \quad (a,s) \mapsto 0, \ \left(a - k, s + \sum_{i=1}^{k} w_{m_i}\right) \mapsto \sum_{i=1}^{k} 2^{-m_i}.$$

Condition (*) implies that ψ is well-defined; it is then obvious that ψ is a bijection. We claim that ψ is actually a homeomorphism. As $U_{a,s,\alpha}$ is Hausdorff and $X_{a,\alpha}$ is compact, it is enough to show that ψ^{-1} is continuous, or equivalently, that ψ is open.

To show this, as sets of the form $U_{b,t,\beta}$ form a basis for σ, it is enough to show that for each $(b,t) \in U_{a,s,\alpha}$ and β with $U_{b,t,\beta} \subseteq U_{a,s,\alpha}$, we have that $\psi(U_{b,t,\beta})$ is open in $X_{a,\alpha}$. For $(b,t) \in U_{a,s,\alpha} \setminus \{(a,s)\}$, say $(b,t) = (a-k, s + \sum_{i=1}^k w_{m_i})$ we have $U_{b,t,\beta} \subseteq U_{a,s,\alpha}$ if and only if $\beta > m_k$. Then

$$\psi(U_{b,t,\beta}) = \Big\{ \sum_{i=1}^{k+l} 2^{-m_i} : \ 0 \leq l \leq a-k, \ \beta \leq m_{k+1} < \ldots < m_{k+l} \Big\},$$

which is open in $X_{a,\alpha}$ by Lemma 3.1. Similarly $\psi(U_{a,s,\beta})$ is open in $X_{a,\alpha}$ for all $\beta \geq \alpha$, and so σ is locally compact.

Finally we show that σ makes the semigroup operation separately continuous. Let $(b,t) \in S$. We claim that the map

$$M_{b,t} : (x,u) \mapsto (x+b, u+t), \quad S_1 \to S_1,$$

is σ-continuous. Indeed, let $(a,s) \in S_1$ and let $\alpha \in \mathbb{N}$. Then $(x,u) \in M_{b,t}^{-1}(U_{a+b,s+t,\alpha})$ if and only if

$$x + b = a + b - k\,, \quad \text{and} \quad u + t = s + t + \sum_{i=1}^k w_{m_i};$$

where $0 \leq k \leq a+b$, and $\alpha \leq m_1 < \ldots < m_k$. This happens if and only if

$$x = a - k\,, \quad \text{and} \quad u = s + \sum_{i=1}^k w_{m_i};$$

where $0 \leq k \leq a$, and $\alpha \leq m_1 < \ldots < m_k$. So $M_{b,t}^{-1}(U_{a+b,s+t,\alpha}) = U_{a,s,\alpha}$. Since $(a,s) \in S_1$ and $\alpha \in \mathbb{N}$ are arbitrary, we deduce that $M_{b,t}$ is continuous. $\square$

We continue with the notation introduced prior to Lemma 3.2. We say (**) holds if for each $t \in S$, there exists an $\alpha_t \in \mathbb{N}$ such that whenever $s \in S$, $n \geq \alpha_t$, and $s - t + w_n \in S$, then $s - t \in S$.

Lemma 3.3. *With the notation above, if both (*) and (**) hold, then the topology σ satisfies the property that a sequence (x_n, u_n) in S_1 converges whenever $(x_n + a, u_n + s)$ converges, for some $(a,s) \in S_1$.*

Proof. Let $(a,s) \in S_1$ and let $(x_n, u_n) \subseteq S_1$ be such that $(x_n + a, u_n + s)$ converges to some $(b,t) \in S_1$. For each $\alpha \geq \alpha_s \in \mathbb{N}$, there exist n_α such that whenever $n \geq n_\alpha$ we have that $(x_n + a, u_n + s) \in U_{b,t,\alpha}$. That is, there exists $0 \leq l \leq b$ and $\alpha \leq n_1 < \cdots < n_l$ with

$$x_n + a = b - l, \quad u_n + s = t + \sum_{j=1}^l w_{n_j}.$$

So $b-l \geq a$. Also, as $n_l \geq \alpha \geq \alpha_s$, and $t-s+\sum_{i=1}^{l-1} w_{n_i} + w_{n_l} = u_n \in S$, (**) shows that $t-s+\sum_{i=1}^{l-1} w_{n_i} \in S$. By induction, $t-s \in S$. Thus

$$x_n = b-a-l, \quad u_n = t-s+\sum_{j=1}^{l} w_{n_j},$$

where $0 \leq l \leq b-a$ and $\alpha \leq n_1 < \ldots < n_l$; that is, $(x_n, u_n) \in U_{b-a,t-s,\alpha}$. We conclude that (x_n, u_n) converges to $(b-a, t-s)$, as required. ☐

Remark 3.4. When (w_n) satisfies the conditions (*) and (**), so too does any subsequence of (w_n). For a subsequence (w_{m_i}), denote by $\sigma_{(m_i)}$ the topology on S_1 constructed from (w_{m_i}). We see that $(0, w_{m_i}) \to (1,0)$ with respect to $\sigma_{(m_i)}$. If (w_{n_j}) is another subsequence such that $\{m_i\}\triangle\{n_j\}$ is infinite, then it is easy to see that $(0, w_{n_j}) \not\to (1,0)$ with respect to $\sigma_{(m_i)}$. Thus the topologies $\sigma_{(n_j)}$ and $\sigma_{(m_i)}$ differ. We can then deduce that there exists a continuum of different topologies σ on S_1 satisfying the conclusions of the previous two lemmas.

Corollary 2.9 above shows that $\ell^1(\mathbb{Z}_+^2)$ has a unique Hopf algebra predual. We shall now show that $\mathbb{Z}_+^2$ admits a continuum of distinct non-discrete locally compact topologies: of course, none can satisfy the 2nd condition of Corollary 2.4 or Lemma 2.6. We need the following easy fact.

Proposition 3.5. *Let $(m_i : 1 \leq i \leq k)$ and $(n_j : 1 \leq j \leq l)$ be sequences in $\mathbb{N}$ such that each $k \in \mathbb{N}$ occurs at most twice in each sequence. Suppose that*

$$\sum_{i=1}^{k} 2^{2^{m_i}} = \sum_{j=1}^{l} 2^{2^{n_j}}.$$

Then $k = l$, and (m_i) and (n_j) are rearrangements of each other.

Theorem 3.6. *There exist a continuum of locally compact topologies on $(\mathbb{Z}_+^2, +)$ making it a semitopological semigroup.*

Proof. Set $G = \mathbb{Z}$ and $S = \mathbb{Z}_+$, and let $w_n = 2^{2^n}$ for $n \in \mathbb{N}$. The previous proposition shows that this sequence satisfies (*) so Lemma 3.2 and Remark 3.4 apply. ☐

Theorem 3.7. *There exist a continuum of Hopf algebra preduals for $\ell^1(\mathbb{Z}_+ \times \mathbb{Z})$.*

Proof. Now we set $G = S = \mathbb{Z}$, and have (w_n) as above. As $S = G$, the condition (**) is obviously satisfied. The result then follows from Lemma 2.6. ☐

Theorem 3.8. *There exist a continuum of Hopf algebra preduals for $\ell^1(\mathbb{N}, \cdot)$.*

Proof. Let G be the multiplicative group of positive rational numbers, and let S be the sub-semigroup consisting of odd natural numbers. We see that $\mathbb{Z}_+ \times S \cong (\mathbb{N}, \cdot)$ by the isomorphism $(k, n) \mapsto 2^k n$. Finally, let (w_n) be any increasing sequence of odd prime numbers. We see that (w_n) satisfies the conditions (*) and (**). The result again follows from Lemma 2.6. ☐

4 Semigroup algebras with no Hopf algebra preduals

Recall that a semigroup S is said to be an *inverse semigroup* if, for each $s \in S$, there exists a unique $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. In this section, we shall exhibit an inverse semigroup S which admits no semitopological structure giving rise to a Hopf algebra predual for $\ell^1(S)$.

Let S be the collection of maps $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that there exists a finite (and possibly empty) set $F \subset \mathbb{N}$ such that f maps F injectively into $\mathbb{N}$ and $f(n) = 0$ for $n \notin F$. We call F the *injective domain* of f. Under composition of functions S is a countable inverse semigroup.

Lemma 4.1. *Let σ be any locally compact Hausdorff topology on S making S into a semitopological semigroup. Then there exists $f \in S$, with injective domain $F \subset \mathbb{N}$ such that, for any finite set $F' \subset \mathbb{N}$ disjoint from F, the set*

$$\mathcal{O}(f, F') = \{h \in S : h(n) = f(n)\ (n \in F),\ h(n) = 0\ (n \in F')\}.$$

is open.

Proof. As S is countable, and (S, σ) is a Baire space, there exists $f \in S$ with $\{f\}$ open. As (S, σ) is semitopological, for each $g \in S$, the set $\{h \in S : hg = f\}$ is open.

Fix $F' \subset \mathbb{N}$ finite and disjoint from F. Define $g \in S$ with injective domain G so that $g(n) = n$ for $n \in F$ and $g(\mathbb{N} \cap (G \setminus F)) = F'$. Then $h \in S$ has $hg = f$ if and only if $h(n) = f(n)$ for $n \in F$ and $h(n) = 0$ for $n \in F'$, that is, if and only if $h \in \mathcal{O}(f, F')$. $\square$

Theorem 4.2. *There is no locally compact Hausdorff topology σ on S such that $C_0(S, \sigma)$ is a predual for $\ell^1(S)$.*

Proof. Suppose towards a contradiction that $C_0(S, \sigma)$ is a predual for $\ell^1(S)$. By the previous lemma, there exists $f \in S$ with injective domain F such that $\mathcal{O}(f, F')$ is open for any finite $F' \subset \mathbb{N}$ which is disjoint from F. Fix such F' and fix $n_0 \in F'$. For each n, define $f_n : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$f_n(k) = \begin{cases} f(k) & : k \in F, \\ n & : k = n_0, \\ 0 & : \text{otherwise}. \end{cases}$$

Then $f_n \in S$ for sufficiently large n. Let $p \in S$ be the function $p(k) = k$ for $k \in F$, and $p(k) = 0$ otherwise.

Suppose, towards a contradiction, that (f_n) does not converge to f. Then there exists an open set $U \in \sigma$ with $f \in U$, and a sequence $n_1 < n_2 < \cdots$ with $f_{n_i} \notin U$ for all i. Now, $f_n p = f$ for all n, so $f_{n_i} p \to f$. By Corollary 2.4, there exists a subsequence of (f_{n_i}) which converges to g, say. As σ is Hausdorff, $g \neq f$. However, as $f_{n_i} p = f$ for all i, it follows that $gp = f$. That is $g(k) = f(k)$ for all $k \in F$. As $g \neq f$ there exists $k_0 \notin F$ with $g(k_0) \neq 0$.

Let $h_1 \in S$ be the function $h_1(k_0) = k_0$ and $h_1(k) = 0$ otherwise, and let $h_2 \in S$ be the function $h_2(g(k_0)) = g(k_0)$ and $h_2(k) = 0$ otherwise. By construction $h_2 g h_1 \neq 0$, yet

$h_2 f_{n_i} h_1 = 0$ unless $k_0 = n_0$ and $n_i = g(n_0)$. In particular $h_2 f_{n_i} h_1 = 0$ for sufficiently large i. Thus

$$0 \neq h_2 g h_1 = \lim_i h_2 f_{n_i} h_1 = 0,$$

a contradiction, as required.

Therefore $f_n \to f$. However, $f \in \mathcal{O}(f, F')$, yet $f_n \notin \mathcal{O}(f, F')$ for all n, giving the required contradiction to finish the proof. $\square$

5 A unique algebraic predual

We end with an example of an infinite semigroup S for which $c_0(S)$ is the unique Banach algebra predual on $\ell^1(S)$. Other examples of Banach algebras for which the predual is uniquely determined include von Neumann algebras (Sakai's classical result shows that von Neumann algebras have a unique *isometric* predual. The extension to the non-isometric case can be found in [6]) and $\mathcal{B}(E)$ for a reflexive Banach space E with the approximation property, [5, Theorem 4.4].

We first need a little machinery. Let $\mathcal{A}$ be a Banach algebra. We turn $\mathcal{A}^*$ into an $\mathcal{A}$-bimodule in the usual way,

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle \qquad (a, b \in \mathcal{A}, \mu \in \mathcal{A}^*).$$

Define bilinear maps from $\mathcal{A}^{**} \times \mathcal{A}^*$ and $\mathcal{A}^* \times \mathcal{A}^{**}$ to $\mathcal{A}^*$ by

$$\langle \Phi \cdot \mu, a \rangle = \langle \Phi, \mu \cdot a \rangle, \quad \langle \mu \cdot \Phi, a \rangle = \langle \Phi, a \cdot \mu \rangle \qquad (a \in \mathcal{A}, \mu \in \mathcal{A}^*, \Phi \in \mathcal{A}^{**}).$$

Finally, define bilinear maps $\Box, \Diamond : \mathcal{A}^{**} \times \mathcal{A}^{**} \to \mathcal{A}^{**}$ by

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle, \quad \langle \Phi \Diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle \qquad (\Phi, \Psi \in \mathcal{A}^{**}, \mu \in \mathcal{A}^*).$$

These are the *Arens products*; they are contractive Banach algebra products on $\mathcal{A}^{**}$. For further details, see [3, Section 2.6] or [7, Section 1.4]. When $\mathcal{A}$ is commutative, $\Phi \Box \Psi = \Psi \Diamond \Phi$ for $\Phi, \Psi \in \mathcal{A}^{**}$.

Define $\mathrm{WAP}(\mathcal{A}^*) \subseteq \mathcal{A}^*$ to be those functionals $\mu \in \mathcal{A}^*$ such that

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi \Diamond \Psi, \mu \rangle \qquad (\Phi, \Psi \in \mathcal{A}^{**}).$$

Then $\mathrm{WAP}(\mathcal{A}^*)$ is an $\mathcal{A}$-submodule of $\mathcal{A}^*$. It is a simple calculation (see [5, Section 2] or [8, Section 4]) that if $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_* \subseteq \mathcal{A}^*$, then $\mathcal{A}_* \subseteq \mathrm{WAP}(\mathcal{A}^*)$.

Now let $S = (\mathbb{N}, \max)$. Consider the character space of $\ell^\infty(S)$, which as S is discrete, is equal to the space of ultrafilters on S, written βS. For $\omega \in \beta S$, let $\delta_\omega \in \ell^\infty(S)^*$ be the character induced by ω, so that

$$\langle \delta_\omega, f \rangle = \lim_{s \to \omega} f(s) \qquad (f \in \ell^\infty(S)).$$

So for $S = (\mathbb{N}, \max)$, let $\omega, \upsilon \in \beta S$ be non-principal, so that for $f \in \ell^\infty(S)$, we have that

$$\langle \delta_\omega \Box \delta_\upsilon, f \rangle = \lim_{s \to \omega} \lim_{t \to \upsilon} f(\max(s,t)) = \lim_{t \to \upsilon} f(t) = \langle \delta_\upsilon, f \rangle.$$

Hence, if $f \in \mathrm{WAP}(\ell^\infty(S)) \subseteq \ell^\infty(S)$, then

$$\langle \delta_\upsilon, f \rangle = \langle \delta_\omega \Box \delta_\upsilon, f \rangle = \langle \delta_\upsilon \Box \delta_\omega, f \rangle = \langle \delta_\omega, f \rangle.$$

It follows easily that

$$\mathrm{WAP}(\ell^\infty(\mathbb{N}, \max)) = c_0(\mathbb{N}) \oplus \mathbb{C}1.$$

Lemma 5.1. *Let $\mathcal{A}_*$ be a Banach space, let $\mathcal{A} = \mathcal{A}_*^*$, and let $F \subseteq \mathcal{A}^*$ be a closed subspace such that $\mathcal{A}_* \subseteq F$ and $F/\mathcal{A}_*$ is one-dimensional. Let $E \subseteq F$ be a spacial predual for $\mathcal{A}$. Then*

$$E^\perp := \big\{ M \in F^* : \langle M, \mu \rangle = 0 \ (\mu \in E) \big\}$$

is also one-dimensional.

Proof. We can find $\mu_0 \in \mathcal{A}^* \setminus \mathcal{A}_*$ with F being the span of $\mathcal{A}_*$ and μ_0. Pick $M_0 \in \mathcal{A}^{**}$ with $\langle M_0, \mu_0 \rangle = 1$ and $\langle M_0, \mu \rangle = 0$ for all $\mu \in \mathcal{A}_*$. By restriction, we shall regard M_0 as a member of F^*. If $E^\perp = \{0\}$ then $E = F$, which is a contradiction, as F strictly contains $\mathcal{A}_*$ and so cannot be a predual for $\mathcal{A}$. So, towards a contradiction, suppose that we can find linearly independent vectors $M_1, M_2 \in E^\perp$.

For $i = 1, 2$, if we restrict M_i to $\mathcal{A}_* \subseteq F$, then we induce a member of $\mathcal{A}_*^* = \mathcal{A}$, say $a_i \in \mathcal{A}$, which satisfies $\langle M_i, \mu \rangle = \langle \mu, a_i \rangle$ for $\mu \in \mathcal{A}_*$. Then $M_i - a_i$ annihilates $\mathcal{A}_*$, so as F is the linear span of $\mathcal{A}_*$ and μ_0, we can find $\alpha_i \in \mathbb{C}$ with $M_i - a_i = \alpha_i M_0$. We hence have that

$$0 = \langle M_i, \mu \rangle = \langle \mu, a_i \rangle + \alpha_i \langle M_0, \mu \rangle \qquad (\mu \in E, i = 1, 2).$$

If $\alpha_1 = 0$, then for each $\mu \in E$, we have that $\langle \mu, a_1 \rangle = 0$. As E is a predual for $\mathcal{A}$, this means that $a_1 = 0$, so that $M_1 = 0$, a contradiction. Similarly, $\alpha_2 \neq 0$.

We hence see that

$$\langle \mu, \alpha_1^{-1} a_1 \rangle = -\langle M_0, \mu \rangle = \langle \mu, \alpha_2^{-1} a_2 \rangle \qquad (\mu \in E).$$

As E is a predual, this shows that $\alpha_1^{-1} a_1 = \alpha_1^{-2} a_2$. Thus

$$\begin{aligned} M_1 &= a_1 + \alpha_1 M_0 = \alpha_1 \big(\alpha_1^{-1} a_1 + M_0 \big) = \alpha_1 \big(\alpha_2^{-1} a_2 + M_0 \big) \\ &= \alpha_1 \alpha_2^{-1} \big(a_2 + \alpha_2 M_0 \big) = \alpha_1 \alpha_2^{-1} M_2, \end{aligned}$$

a contradiction, as required. $\square$

Theorem 5.2. *Let $S = (\mathbb{N}, \max)$, and let $E \subseteq \ell^\infty(S)$ be a predual for $\ell^1(S)$. Then $E = c_0(S)$.*

Proof. We have that

$$E \subseteq \mathrm{WAP}(\ell^\infty(S)) = c_0(\mathbb{N}) \oplus \mathbb{C}1.$$

We identify the dual of $c_0(\mathbb{N}) \oplus \mathbb{C}1$ with $\ell^1(\mathbb{N}) \oplus \mathbb{C}1$. By the previous lemma, $E^\perp$ is one dimensional, so there exists $a \in \ell^1(\mathbb{N})$ and $\alpha \in \mathbb{C}$, not both zero, such that

$$E^\perp = \{\Phi \in \mathrm{WAP}(\ell^\infty(S))^* : \langle \Phi, \mu \rangle = 0 \ (\mu \in E)\} = \mathbb{C}(a + \alpha 1).$$

It hence follows that

$$E = \big\{(x, \beta) \in c_0 \oplus \mathbb{C}1 : \langle a, x \rangle = -\alpha\beta\big\}.$$

Then $E = c_0$ if and only if $a = 0$.

So, towards a contradiction, suppose that $a \neq 0$. Pick $(x, \beta) \in E$. As $1 \in \ell^\infty(S)$ is clearly invariant for the $\ell^1(S)$ module action, we see that $(\delta_s \cdot x, \beta) \in E$ for $s \in S$, and so

$$\langle a, x \rangle = -\alpha\beta = \langle a, \delta_s \cdot x \rangle = \langle \delta_s \cdot a, x \rangle \qquad (s \in S).$$

Let $a = \sum_{n \in \mathbb{N}} a_n \delta_n$, so for $s \in \mathbb{N}$,

$$\delta_s \cdot a = \sum_n a_n \delta_{\max(s,n)} = \Big(\sum_{n=1}^{s} a_n\Big)\delta_s + \sum_{n=s+1}^{\infty} a_n \delta_n.$$

Let $x = \sum_n x_n \delta_n$, so we see that

$$\sum_{n=1}^{\infty} a_n x_n = \langle a, x \rangle = \langle \delta_s \cdot a, x \rangle = \sum_{n=1}^{s} a_n x_s + \sum_{n=s+1}^{\infty} a_n x_n \qquad (s \in \mathbb{N}).$$

That is,

$$\sum_{n=1}^{s} a_n x_n = x_s \sum_{n=1}^{s} a_n \qquad (s \in \mathbb{N}).$$

Letting $s \to \infty$, we conclude that $\langle a, x \rangle = \sum_n a_n x_n = 0$. As $(x, \beta) \in E$, we see that $\langle a, x \rangle = -\alpha\beta$, so either $\alpha = 0$, or $\beta = 0$. If $(x, \beta) \in E$ implies that $\beta = 0$, then $E \subseteq c_0$, which as E is a predual means that $E = c_0$ as required.

Otherwise, we have that $\alpha = 0$, so that $(x, \beta) \in E$ if and only if $\langle a, x \rangle = 0$. If $\langle 1, a \rangle = 0$, then for $(x, \beta) \in E$, $\langle x + \beta 1, a \rangle = 0$, so a annihilates E. As E is a predual, $a = 0$, contradiction. So $\sum_n a_n \neq 0$. As E is an $\ell^1(S)$-module, we have that $\langle a, x \rangle = 0$ implies that $\langle \delta_s \cdot a, x \rangle = 0$. If $a = a_{s_0}\delta_{s_0}$ for some $s_0 \in S$, then $\langle x, a \rangle = 0$ if and only if $\langle x, \delta_{s_0} \rangle = 0$, which clearly does not imply that $\langle x, \delta_s \cdot \delta_{s_0} \rangle$ is zero for all s. Otherwise, choose $s_0 < s_1$ minimal with $a_{s_0} \neq 0$ and $a_{s_1} \neq 0$. Let s be greater than s_0 and s_1 chosen such that $\sum_{n=1}^{s} a_n \neq 0$, which is possible, as $\sum_n a_n \neq 0$. Let $x = a_{s_0}\delta_{s_1} - a_{s_1}\delta_{s_0} + \delta_s$, so that $\langle x, a \rangle = 0$, but

$$\langle x, \delta_s \cdot a \rangle = \sum_{n=1}^{s} a_n \neq 0.$$

This final contradiction completes the proof. $\square$

The underlying fact which allows this proof to work is that for $S = (\mathbb{N}, \max)$, we have that $\mathrm{WAP}(\ell^\infty(S))$ is very small. In [2], Chou shows that when G is an infinite discrete group, then $\mathrm{WAP}(\ell^\infty(G))/c_0(G)$ contains an isometric copy of ℓ^∞. So there is no hope of a generalisation of the above proof to group algebras.

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