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Multipliers of locally compact quantum groups via Hilbert C^* -modules

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Abstract

A result of Gilbert shows that every completely bounded multiplier f of the Fourier algebra $A(G)$ arises from a pair of bounded continuous maps $\alpha, \beta : G \rightarrow K$, where K is a Hilbert space, and $f(s^{-1}t) = (\beta(t)|\alpha(s))$ for all $s, t \in G$. We recast this in terms of adjointable operators acting between certain Hilbert C^* -modules, and show that an analogous construction works for completely bounded left multipliers of a locally compact quantum group. We find various ways to deal with right multipliers: one of these involves looking at the opposite quantum group, and this leads to a proof that the (unbounded) antipode acts on the space of completely bounded multipliers, in a way which interacts naturally with our representation result. The dual of the universal quantum group (in the sense of Kustermans) can be identified with a subalgebra of the completely bounded multipliers, and we show how this fits into our framework. Finally, this motivates a certain way to deal with two-sided multipliers.

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1 Introduction

Let G be a locally compact group G , and let $A(G)$ be the Fourier algebra of G , the subalgebra of $C_0(G)$ given by coefficient functionals of the left regular representation λ of G on $L^2(G)$, see [9]. A *multiplier* of $A(G)$ is a continuous function $f \in C^b(G)$ such that $fa \in A(G)$ for each $a \in A(G)$. A multiplier f induces an automatically bounded map $A(G) \rightarrow A(G)$. As $A(G)$ is the predual of the group von Neumann algebra $VN(G)$, it carries a natural operator space structure, and so we can ask when the map induced by f is completely bounded. The collection of such f is the algebra of *completely bounded multipliers* of $A(G)$, written $M_{cb}A(G)$. A result of Gilbert (see [3], the short proof in [11], the introduction of [4], or the survey [23]) shows that $f \in M_{cb}A(G)$ if and only if there is a Hilbert space K and bounded continuous functions $\alpha, \beta : G \rightarrow K$ with

$$f(s^{-1}t) = (\beta(t)|\alpha(s)) \quad (s, t \in G),$$

where $(\cdot|\cdot)$ denotes the inner-product on K . (This formula has $s^{-1}t$ instead of $t^{-1}s$ as considered by Jolissaint in [11]; see Section 2.1 below for an explanation).

In this paper, we shall propose variations of this result for the convolution algebra $L^1(\mathbb{G})$ of a locally compact quantum group \mathbb{G} (see below for definitions). Clearly the space of continuous functions $G \rightarrow K$ will be important, and we start with a short discussion of this. Indeed, consider the C^* -algebra $A = C_0(G)$. Let $A \otimes K$ be the standard Hilbert C^* -module (see [19]) which in this case can be identified with $C_0(G, K)$. Then the “multiplier space” of $A \otimes K$ is identified with $C^b(G, K)$; abstractly, this is the space $\mathcal{L}(A, A \otimes K)$ of adjointable maps from A to $A \otimes K$. To induce a member of $M_{cb}A(G)$, we need that the pair (α, β) is “invariant” in the sense that $(\beta(t^{-1})|\alpha(t^{-1}s^{-1})) = f(s)$ for all $s, t \in G$.

In the quantum setting, we replace $C_0(G)$ by a possibly non-commutative C^* -algebra, denoted $C_0(\mathbb{G})$. The dual quantum group to $C_0(G)$ is $C_r^*(G)$, and the Fourier algebra is the predual of the $VN(G) = C_r^*(G)''$. Thus, by analogy, we will study completely bounded multipliers of the

convolution algebra of the dual quantum group, denoted $L^1(\hat{\mathbb{G}})$. Indeed, we work firstly by looking at completely bounded *left* multipliers of $L^1(\hat{\mathbb{G}})$. We restrict attention to those multipliers which are “represented” by some $x \in C^b(\mathbb{G})$ (so that under the regular representation $\hat{\lambda} : L^1(\hat{\mathbb{G}}) \rightarrow C_0(\mathbb{G})$, left multiplication by x induces our left multiplier). This is automatic for the left part of two-sided multipliers, see [6, Section 8.2]. In this setting, we get a complete analogy of Gilbert’s result. To study right multipliers, we can either use the unitary antipode, or study the opposite algebra $L^1(\hat{\mathbb{G}})^{\text{op}}$. These turn out *not* to be totally equivalent, and the study of $L^1(\hat{\mathbb{G}})^{\text{op}}$ leads us to study how the (unbounded, in general) antipode of \mathbb{G} acts on the space of multipliers. A corollary is that two-sided multipliers are invariant under the action of the antipode. Furthermore, this now puts us in a position to use the representation result of Junge, Neufang and Ruan proved in [12], which implies that in fact every completely bounded left multiplier is represented in our sense. By taking a different perspective on the space $\mathcal{L}(A, A \otimes K)$, we are lead to consider ideas very close to those studied by Vaes and Van Daele in [28].

In the final part of the paper, we look at the universal quantum group (in the sense of Kustermans, [15]) of $\hat{\mathbb{G}}$. This always induces completely bounded multipliers of $L^1(\hat{\mathbb{G}})$, and we show how this fits into our framework. Motivated by this construction, we end by giving one, reasonably symmetric, way to deal with two-sided multipliers.

We follow [19] for the theory of Hilbert C*-modules. In particular, all our inner-products will be linear in the *second variable*, and we consider *right* (Hilbert C*-)modules. We similarly often let scalars act on the right of a vector space.

Acknowledgements: We thank Martin Lindsay for suggesting the idea of viewing $\mathcal{L}(A, A \otimes K)$ as a “corner” or “slice” of $\mathcal{L}(A \otimes K)$; this both simplifies proofs in Section 3 and also provides motivation for our treatment of two-sided multipliers. While visiting Leeds on the EPSRC grant EP/I002316/1, Zhong-Jin Ruan and Matthias Neufang pointed the author in the direction of [12], and Nico Spronk suggested the comment about $\text{wap}(G)$ in Section 5.2. Finally, the anonymous referee provided many helpful comments which have substantially improved the paper.

2 Locally compact quantum groups and multipliers

In this section, we sketch (rather briefly) the theory of locally compact quantum groups; our main aim is to fix notation. For details on the von Neumann algebraic side of the theory, see [17], and for the C*-algebraic side, see [18] and [20]. The survey [14], and Vaes’s PhD thesis [26], are gentle, well-motivated introductions.

A *locally compact quantum group* is a von Neumann algebra M together with a coproduct $\Delta : M \rightarrow M \overline{\otimes} M$. This is a unital normal *-homomorphism with $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. Furthermore, we assume the existence of left and right invariant weights on M . The coproduct Δ turns the predual M_* into a completely contractive Banach algebra.

Associated to (M, Δ) is a *reduced* C*-algebraic quantum group (A, Δ) . Here A is a C*-subalgebra of M , and $\Delta : A \rightarrow M(A \otimes A)$, the multiplier algebra of $A \otimes A$, the minimal C*-algebra tensor product (which is the only tensor product of C*-algebras which we shall consider). Here we identify $M(A \otimes A)$ with a subalgebra of $M \overline{\otimes} M$. The dual space A^* becomes a completely contractive Banach algebra which contains M_* as a closed ideal.

We use the left invariant weight to build a Hilbert space H ; then M is in standard position on H . There is a privileged unitary operator W on $H \otimes H$ (the Hilbert space tensor product of H with itself) with $\Delta(x) = W^*(1 \otimes x)W$ for $x \in M$. Then W is a *multiplicative unitary*, and $W \in M(A \otimes \mathcal{B}_0(H))$, where $\mathcal{B}_0(H)$ is the algebra of compact operators on H . Define $\lambda : M_* \rightarrow$

$\mathcal{B}(H)$ by $\lambda(\omega) = (\omega \otimes \iota)(W)$. Let the closure of $\lambda(M_*)$ be \hat{A} , which is a C^* -algebra. Let \hat{M} be the σ -weak closure, which is a von Neumann algebra. We may define a coproduct $\hat{\Delta}$ on \hat{M} by $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, where $\hat{W} = \sigma W^* \sigma$, where σ is the flip map on $H \otimes H$. It is possible to construct left and right invariant weights on \hat{M} , turning this into a locally compact quantum group, whose C^* -algebraic counterpart is \hat{A} . We have the biduality theorem, that $\hat{\hat{M}} = M$ canonically.

As is becoming common, we write \mathbb{G} for an abstract object, to be thought of as a locally compact quantum group, and we write $L^1(\mathbb{G}), L^\infty(\mathbb{G}), C_0(\mathbb{G}), C^b(\mathbb{G})$ and $M(\mathbb{G})$ for, respectively, $M_*, M, A, M(A)$ and A^* . We shall then write $\hat{\mathbb{G}}$ for the abstract object corresponding to the dual quantum group, so that \hat{M} is denoted by $L^\infty(\hat{\mathbb{G}})$, and so forth. We shall use the hat notation to signify that an object should be thought of as corresponding to $\hat{\mathbb{G}}$. For example, for $\xi, \eta \in L^2(\mathbb{G})$, we have the vector functional $\omega_{\xi, \eta} : \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathbb{C}; x \mapsto (\xi | x \eta)$, and then the restriction of this to $L^\infty(\hat{\mathbb{G}})$ is denoted by $\hat{\omega}_{\xi, \eta} \in L^1(\hat{\mathbb{G}})$.

Locally compact quantum groups generalise Kac algebras (see [8] and [27, Page 7]). However, unlike for a Kac algebra, $L^1(\mathbb{G})$ need not be $*$ -algebra, as the antipode S is in general unbounded. However, $L^1(\mathbb{G})$ contains a dense $*$ -subalgebra $L^1_{\sharp}(\mathbb{G})$. This is the space of functionals $\omega \in L^1(\mathbb{G})$ such that there exists $\sigma \in L^1(\mathbb{G})$ with $\langle x, \sigma \rangle = \langle S(x), \omega^* \rangle$ for $x \in D(S)$, the domain of S . Here ω^* is the functional given by $\langle y, \omega^* \rangle = \overline{\langle y^*, \omega \rangle}$ for $y \in L^\infty(\mathbb{G})$. We write $\sigma = \omega^\sharp$ in this case, and then $\lambda(\omega^\sharp) = \lambda(\omega)^*$. See [15, Section 3] or [17, Section 2] for further details.

As we are working with right multipliers, to avoid a notational clash, we shall write κ (and not R) for the unitary antipode on $L^\infty(\mathbb{G})$. This is a normal anti- $*$ -homomorphism with $(\kappa \otimes \kappa)\sigma\Delta = \Delta\kappa$. Thus the pre-adjoint κ_* is an anti-homomorphism of $L^1(\mathbb{G})$. Furthermore, κ is spatially implemented, as $\kappa(x) = \hat{J}x^*\hat{J}$ for $x \in L^\infty(\mathbb{G})$, where \hat{J} is the modular conjugation for (the left weight of) $\hat{\mathbb{G}}$. The unitary antipodes interact well with duality, in that $\kappa\hat{\lambda} = \hat{\lambda}\kappa_*$.

There is a one-parameter group of automorphisms (τ_t) of $C_0(\mathbb{G})$ which links S and κ , by $S = R\tau_{-i/2}$. Then R commutes with (τ_t) , so also $S = \tau_{-i/2}R$, and we see that $D(S) = D(\tau_{-i/2})$. The group (τ_t) extends to a group of automorphisms, continuous for the σ -strong* topology, of $L^\infty(\mathbb{G})$.

As we are looking at the left regular representation, it is natural that things work best for us when looking at left multipliers. We shall later deal with right multipliers: these can be converted to left multipliers by looking at the *opposite algebra*. At the quantum group level, we define $\hat{\mathbb{G}}^{\text{op}}$ to be the *opposite* quantum group to $\hat{\mathbb{G}}$, see [17, Section 4]. That is, $L^\infty(\hat{\mathbb{G}}^{\text{op}}) = L^\infty(\hat{\mathbb{G}})$, but the multiplication in $L^1(\hat{\mathbb{G}}^{\text{op}})$ is reversed from that in $L^1(\hat{\mathbb{G}})$. This is equivalent to defining the comultiplication on $L^\infty(\hat{\mathbb{G}}^{\text{op}})$ to be $\sigma\hat{\Delta}$.

Then we have that $L^\infty((\hat{\mathbb{G}}^{\text{op}})) = L^\infty(\mathbb{G})'$, the commutant of $L^\infty(\mathbb{G})$ in $\mathcal{B}(L^2(\mathbb{G}))$. Let the resulting locally compact quantum group be denoted by \mathbb{G}' . The natural coproduct Δ' is defined as follows, where J is the modular conjugation on $L^\infty(\mathbb{G})$,

$$\Delta'(x) = (J \otimes J)\Delta(JxJ)(J \otimes J) \quad (x \in L^\infty(\mathbb{G}') = L^\infty(\mathbb{G}')),$$

The associated multiplicative unitary is $W' = (J \otimes J)W(J \otimes J)$. Then $C_0(\mathbb{G}')$ is the norm closure of $\{(\iota \otimes \omega)(W') : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}$, which is easily seen to be $J C_0(\mathbb{G}) J$. Consider the unitary map $\hat{J}J$, and for $x \in C_0(\mathbb{G}')$ define $\Phi(x) = \hat{J}JxJ\hat{J} = \kappa(JxJ)^* \in C_0(\mathbb{G})$, so that Φ is a C^* -isomorphism of $C_0(\mathbb{G}')$ to $C_0(\mathbb{G})$. We then get the *right regular representation* $\hat{\rho} : L^1(\hat{\mathbb{G}}) \rightarrow C_0(\mathbb{G}'); \hat{\omega} \mapsto \Phi(\hat{\lambda}(\hat{\omega}))$.

2.1 Multipliers and duality

For a Banach algebra \mathcal{A} , a (*two-sided*) *multiplier* (also called a (*double*) *centraliser*) is a pair of maps $L, R : \mathcal{A} \rightarrow \mathcal{A}$ such that $aL(b) = R(a)b$. We write $(L, R) \in M(\mathcal{A})$, and then $M(\mathcal{A})$ becomes

an algebra for the product $(L, R)(L', R') = (LL', R'R)$. We shall always suppose that \mathcal{A} is *faithful*, that is, if $bac = 0$ for all $b, c \in \mathcal{A}$, then $a = 0$. In this case, we can show that $L(ab) = L(a)b$ and $R(ab) = aR(b)$. A closed graph argument will show that L and R are automatically bounded. There is a natural map (injective, as \mathcal{A} is faithful) of \mathcal{A} into $M(\mathcal{A})$ given by $a \mapsto (L_a, R_a)$ where $L_a(b) = ab$ and $R_a(b) = ba$ for $b \in \mathcal{A}$. For further details, see [5], [22, Section 1.2] or [6].

When \mathcal{A} is a completely contractive Banach algebra, we can restrict attention to those $(L, R) \in M(\mathcal{A})$ such that L and R are completely bounded. We write $M_{cb}(\mathcal{A})$ for the algebra of *completely bounded multipliers*. If \mathcal{A} has a bounded approximate identity, then $M(\mathcal{A}) = M_{cb}(\mathcal{A})$ with equivalent norms, see [13, Proposition 3.1] or [6, Theorem 6.2]. Otherwise, there appears to be no general relationship between $M(\mathcal{A})$ and $M_{cb}(\mathcal{A})$.

We shall also work with *left multipliers*, that is, bounded maps $L : \mathcal{A} \rightarrow \mathcal{A}$ with $L(ab) = L(a)b$ for $a, b \in \mathcal{A}$. We write $L \in M^l(\mathcal{A})$. Similarly, we define the *right multipliers* $M^r(\mathcal{A})$, and the analogous completely bounded versions, $M_{cb}^l(\mathcal{A})$ and $M_{cb}^r(\mathcal{A})$.

Definition 2.1. Let \mathbb{G} be a locally compact quantum group. A multiplier $L \in M^l(L^1(\hat{\mathbb{G}}))$ is *represented* if there exists $a \in C^b(\mathbb{G})$ such that $\hat{\lambda}(L(\hat{\omega})) = a\hat{\lambda}(\hat{\omega})$ for each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Similarly, $R \in M^r(L^1(\hat{\mathbb{G}}))$ is *represented* if there exists $a \in C^b(\mathbb{G})$ such that $\hat{\lambda}(R(\hat{\omega})) = \hat{\lambda}(\hat{\omega})a$ for each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$.

Building on work of Kraus and Ruan in [13], we showed in [6, Theorem 8.9] that a two-sided multiplier $(L, R) \in M_{cb}(L^1(\hat{\mathbb{G}}))$ is represented by some $a \in C^b(\mathbb{G})$; that is, $a\hat{\lambda}(\omega) = \hat{\lambda}(L(\omega))$ and $\hat{\lambda}(\omega)a = \hat{\lambda}(R(\omega))$ for each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$ (compare with Proposition 2.4 below). The resulting map $\hat{\Lambda} : M_{cb}(L^1(\hat{\mathbb{G}})) \rightarrow C^b(\mathbb{G})$ is a completely contractive algebra homomorphism. We remark that we don't know if $\hat{\Lambda}$ can be extended (even just as an algebra homomorphism) to $M(L^1(\hat{\mathbb{G}}))$.

To illustrate this, let G be a locally compact group, and form the commutative quantum group $L^\infty(G)$. Here the coproduct is given by $\Delta(F)(s, t) = F(st)$ for $F \in L^\infty(G)$ and $s, t \in G$. The left and right invariant weights are given by integrating against the left and right Haar measures, respectively. Then the dual quantum group is $VN(G)$, which has predual $A(G)$, the Fourier algebra. The associated Hilbert space is simply $L^2(G)$, and as $VN(G)$ is in standard position, every normal functional $\omega \in A(G)$ is of the form $\omega_{\xi, \eta}$, where $\langle x, \omega_{\xi, \eta} \rangle = (\xi|x(\eta))$ for $x \in VN(G)$ and $\xi, \eta \in L^2(G)$. The multiplicative unitary is given by $W\xi(s, t) = \xi(s, s^{-1}t)$ for $\xi \in L^2(G \times G)$, $s, t \in G$. Let $\lambda : G \rightarrow \mathcal{B}(L^2(G))$ be the left regular representation, where

$$\lambda(s) : \xi \mapsto \eta, \quad \eta(t) = \xi(s^{-1}t) \quad (\xi \in L^2(G), s, t \in G).$$

This does integrate to give the expected map $\lambda : L^1(G) \rightarrow \mathcal{B}(L^2(G))$. Then $\hat{\lambda} : A(G) \rightarrow C_0(G)$, and we can check that

$$\hat{\lambda}(\omega)(s) = \langle \lambda(s^{-1}), \omega \rangle \quad (s \in G, \omega \in A(G)).$$

Thus $\hat{\lambda}$ gives the map considered by Takesaki in [25, Chapter VII, Section 3], and *not* the map considered by Eymard in [9] (where s^{-1} is replaced by s). This also explains why our formulas in the introduction were different to those considered Jolissaint in [11], as the embedding $\hat{\Lambda} : M_{cb}A(G) \rightarrow C^b(G)$ is consequently also different to that usually considered.

A representation result for completely bounded multipliers was shown by Junge, Neufang and Ruan in [12]. The principle result of that paper is [12, Theorem 4.5], which shows a completely isometric identification between $M_{cb}^r(L^1(\hat{\mathbb{G}}))$ and $\mathcal{CB}_{L^\infty(\mathbb{G})}^{\sigma, L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$. This latter space is the algebra of weak*-continuous, completely bounded maps $\mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ which are $L^\infty(\mathbb{G})$ -bimodule maps, and which map $L^\infty(\hat{\mathbb{G}})$ into itself. This space can be also be studied by using the

(extended) Haagerup tensor product, and it is possible to view our constructions using a viewpoint similar to the Haagerup tensor product— this is explored in Section 5.2 below; in some sense, our results are C^* -algebraic counterparts to the von Neumann algebra approach of [12]. Indeed, [12] was preceded by work of Neufang, Ruan and Spronk in [21] where the $L^1(G)$ and $A(G)$ cases are worked out. Links with the Haagerup tensor product, and Gilbert's theorem, are explicitly used in [21, Section 4].

For us, the importance of [12] is the following result (recall the discussion in the previous section about the right regular representation $\hat{\rho}$).

Theorem 2.2 ([12, Corollary 4.4]). *Let $R \in M_{cb}^r(L^1(\hat{\mathbb{G}}))$. There exists $x \in L^\infty(\mathbb{G}')$ such that $\hat{\rho}(\hat{\omega})x = \hat{\rho}(R(\hat{\omega}))$ for all $\hat{\omega} \in L^1(\hat{\mathbb{G}})$.*

Actually, the full power of the representation result of [12] is not needed to show this— see Section 5.2 below where a very brief sketch of the proof is given. However, undoubtedly the proof of this result is more ingenious than the two-sided multiplier case.

Using the unitary antipode, it's easy to transfer this result to completely bounded left multipliers. Notice that we only get $x \in L^\infty(\mathbb{G}')$, not $C^b(\mathbb{G}')$, which is slightly weaker than the requirement for R to be *represented* in our sense. However, we are able to boot-strap this result and show that actually x is in $C^b(\mathbb{G}')$ (see Theorem 4.2 and Proposition 5.3 below).

The following results extract a little bit more information than we found in [6]; they are also similar to, for example, [12, Theorem 4.10].

Proposition 2.3. *Let R be a normal completely bounded map $L^\infty(\hat{\mathbb{G}}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$, and let $a \in \mathcal{B}(L^2(\mathbb{G}))$ be such that $(R \otimes \iota)(\hat{W}) = \hat{W}(1 \otimes a)$. Then R maps into $L^\infty(\hat{\mathbb{G}})$, and the pre-adjoint R_* is a right multiplier of $L^1(\hat{\mathbb{G}})$ with $\hat{\lambda}(R_*(\hat{\omega})) = \hat{\lambda}(\hat{\omega})a$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Furthermore, $a \in L^\infty(\mathbb{G})$.*

Proof. Let $T(L^2(\mathbb{G}))$ be the trace-class operators on $L^2(\mathbb{G})$, and let $q : T(L^2(\mathbb{G})) \rightarrow L^1(\hat{\mathbb{G}})$ be the natural quotient map, which is actually a complete quotient map, see [7, Section 4.2]. Let $\xi_0, \eta_0, \xi, \eta \in L^2(\mathbb{G})$, so that

$$\begin{aligned} (\xi | \hat{\lambda}(\omega_{\xi_0, \eta_0}) a \eta) &= (\xi | (\omega_{\xi_0, \eta_0} \otimes \iota)(\hat{W}) a \eta) = (\xi_0 \otimes \xi | \hat{W}(\eta_0 \otimes a \eta)) \\ &= (\xi_0 \otimes \xi | (R \otimes \iota)(\hat{W})(\eta_0 \otimes \eta)) = \langle \hat{W}, R_*(\omega_{\xi_0, \eta_0}) \otimes \omega_{\xi, \eta} \rangle = (\xi | \hat{\lambda}(R_*(\omega_{\xi_0, \eta_0})) \eta) \end{aligned}$$

Thus $\hat{\lambda}(q(\omega))a = \hat{\lambda}(R_*(\omega))$ for $\omega \in T(L^2(\mathbb{G}))$.

In particular, $\hat{\lambda}(\hat{\omega})a \in \hat{\lambda}(L^1(\hat{\mathbb{G}}))$ for each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. As $\hat{\lambda}$ is injective, there exists some function $r : L^1(\hat{\mathbb{G}}) \rightarrow L^1(\hat{\mathbb{G}})$ with $\hat{\lambda}(\hat{\omega})a = \hat{\lambda}(r(\hat{\omega}))$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Using again that $\hat{\lambda}$ is an injective homomorphism, it is easy to check that r is linear and a right multiplier (but maybe not bounded). However, we then see that

$$\hat{\lambda}(q(\omega))a = \hat{\lambda}(r(q(\omega))) = \hat{\lambda}(R_*(\omega)) \quad (\omega \in T(L^2(\mathbb{G}))).$$

So $R_* = rq$ and hence R_* drops to a completely bounded map $L^1(\hat{\mathbb{G}}) \rightarrow L^1(\hat{\mathbb{G}})$, and then $r = R_*$, as required.

Finally, let $x \in L^\infty(\mathbb{G})'$, so for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, we have that $\hat{\lambda}(\hat{\omega})ax = \hat{\lambda}(R_*(\hat{\omega}))x = x\hat{\lambda}(R_*(\hat{\omega})) = x\hat{\lambda}(\hat{\omega})a = \hat{\lambda}(\hat{\omega})xa$. This is enough to imply that $ax = xa$ (compare with the proof of [6, Proposition 8.8]) and so $a \in L^\infty(\mathbb{G})'' = L^\infty(\mathbb{G})$. \square

It is easily checked that, similarly, when $L \in \mathcal{CB}(L^\infty(\hat{\mathbb{G}}), \mathcal{B}(L^2(\mathbb{G})))$ is normal with there existing $a \in \mathcal{B}(L^2(\mathbb{G}))$ with $(L \otimes \iota)(\hat{W}) = (1 \otimes a)\hat{W}$, then $L \in \mathcal{CB}(L^\infty(\hat{\mathbb{G}}))$, and the pre-adjoint L_* is a left multiplier of $L^1(\mathbb{G})$ with $\hat{\lambda}(L_*(\hat{\omega})) = a\hat{\lambda}(\hat{\omega})$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$.

Similarly, if (L, R) is a pair of maps, both associated to the same $a \in \mathcal{B}(L^2(\mathbb{G}))$, then the pre-adjoints form a multiplier $(L_*, R_*) \in M_{cb}(L^1(\hat{\mathbb{G}}))$. As $\hat{\lambda}(L^1(\hat{\mathbb{G}}))$ is dense in $C_0(\mathbb{G})$, it follows that automatically $a \in C^b(\mathbb{G})$. We shall later prove that this is true for one-sided multipliers as well, see Theorem 4.2 and Proposition 5.3 below.

Proposition 2.4. *Let $L_* \in M_{cb}^l(L^1(\hat{\mathbb{G}}))$ and $a \in L^\infty(\mathbb{G})$ be such that $a\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L_*(\hat{\omega}))$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Setting $L = L_*^*$, we have that $(L \otimes \iota)(\hat{W}) = (1 \otimes a)\hat{W}$. Similarly, if R is the adjoint of a completely bounded right multiplier associated to a , then $(R \otimes \iota)(\hat{W}) = \hat{W}(1 \otimes a)$. If L_* and R_* are both associated to the same $a \in L^\infty(\mathbb{G})$, then $(L_*, R_*) \in M_{cb}(L^1(\hat{\mathbb{G}}))$ and $a \in C^b(\mathbb{G})$.*

Proof. We simply reverse some previous calculations, where, for variety, we work with left multipliers. For $\xi_0, \eta_0, \xi, \eta \in L^2(\mathbb{G})$, we have

$$\begin{aligned} (\xi | a\hat{\lambda}(\hat{\omega}_{\xi_0, \eta_0})\eta) &= (\xi | \hat{\lambda}(L_*(\hat{\omega}_{\xi_0, \eta_0}))\eta) = \langle \hat{W}, L_*(\hat{\omega}_{\xi_0, \eta_0}) \otimes \omega_{\xi, \eta} \rangle = \langle (L \otimes \iota)(\hat{W}), \hat{\omega}_{\xi_0, \eta_0} \otimes \omega_{\xi, \eta} \rangle \\ &= (a^*\xi | (\hat{\omega}_{\xi_0, \eta_0} \otimes \iota)(\hat{W})\eta) = \langle \hat{W}, \hat{\omega}_{\xi_0, \eta_0} \otimes \omega_{a^*\xi, \eta} \rangle = \langle (1 \otimes a)\hat{W}, \hat{\omega}_{\xi_0, \eta_0} \otimes \omega_{\xi, \eta} \rangle. \end{aligned}$$

Thus $(L \otimes \iota)(\hat{W}) = (1 \otimes a)\hat{W}$. A similar calculation holds for right multipliers.

If L_* and R_* are associated to the same a , then for $\hat{\omega}, \hat{\sigma} \in L^1(\mathbb{G})$,

$$\hat{\lambda}(\hat{\omega}L_*(\hat{\sigma})) = \hat{\lambda}(\hat{\omega})a\hat{\lambda}(\hat{\sigma}) = \hat{\lambda}(R_*(\hat{\omega})\hat{\sigma}),$$

using that $\hat{\lambda}$ is a homomorphism. As $\hat{\lambda}$ injects, it follows that (L_*, R_*) is a multiplier, which is completely bounded by assumption. That now $a \in C^b(\mathbb{G})$ follows from the comment above. \square

3 Hilbert C^* -modules

We shall use the basic theory of Hilbert C^* -modules, following [19], for example. Let us develop a little of this theory. Given a C^* -algebra A and a Hilbert space K , we let $A \odot K$ be the algebraic tensor product of A with K , turned into a right A -module in the obvious way, and given the A -valued inner-product $(a \otimes \xi | b \otimes \eta) = a^*b(\xi | \eta)$. Let $A \otimes K$ be the completion.

Let E and F be Hilbert C^* -modules over A . We write $\mathcal{K}(E, F)$ for the ‘‘compact’’ operators from E to F , the closure of the linear span of maps $\theta_{x, y}$. Here $x \in F, y \in E$ and we have $\theta_{x, y}(z) = x(y|z)$ for $z \in E$. Let $\mathcal{L}(E, F)$ be the space of all adjointable operators from E to F . Recall that the unit ball of $\mathcal{K}(E, F)$ is strictly dense in the unit ball of $\mathcal{L}(E, F)$. When $E = F$, we can identify $\mathcal{L}(E)$ with the multiplier algebra $M(\mathcal{K}(E))$. Indeed, $\mathcal{K}(E)$ is an essential ideal in $\mathcal{L}(E)$, so we have an inclusion $\mathcal{L}(E) \rightarrow M(\mathcal{K}(E))$, which is actually surjective. When $E = F = A$, we have $\mathcal{K}(A) = A$ and $\mathcal{L}(A)$ is identified with the multiplier algebra $M(A)$.

We identify $\mathcal{K}(A \otimes K)$ with $A \otimes \mathcal{B}_0(K)$. The isomorphism sends $\theta_{a \otimes \xi, b \otimes \eta}$ to $ab^* \otimes \theta_{\xi, \eta}$. Here $\theta_{\xi, \eta} \in \mathcal{B}_0(K)$ is the finite-rank map $\phi \mapsto \xi(\eta|\phi)$. That this extends by continuity is a little subtle; see [19]. Notice that if $P \in \mathcal{B}(K)$, then $\iota \otimes P \in \mathcal{L}(A \otimes K)$.

More generally, let E and F be Hilbert C^* -modules over A and B , respectively. We let $E \otimes F$ be the exterior tensor product, which is a Hilbert C^* -module over $A \otimes B$, with the inner-product

$$(x \otimes y | w \otimes z) = (x | w) \otimes (y | z).$$

We then have an embedding $\mathcal{L}(E) \otimes \mathcal{L}(F) \rightarrow \mathcal{L}(E \otimes F)$, and more generally, an embedding of $\mathcal{L}(E_1, E_2) \otimes \mathcal{L}(F_1, F_2)$ into $\mathcal{L}(E_1 \otimes F_1, E_2 \otimes F_2)$.

As mentioned in the introduction, for a locally compact space G , we may identify $C_0(G) \otimes K$ with $C_0(G, K)$, the continuous functions from G to K which vanish at infinity. Given $\alpha \in C^b(G, K)$, a bounded continuous function from G to K , we define $\mathcal{T} \in \mathcal{L}(C_0(G), C_0(G) \otimes K)$ by

$$\mathcal{T}(a) = (a(s)\alpha(s))_{s \in G} \quad (a \in C_0(G)).$$

A calculation shows that \mathcal{T} is indeed adjointable: if $x \in C_0(G, K)$ then $\mathcal{T}^*(x)(s) = (\alpha(s)|x(s))$ for $s \in G$. Conversely, it is not too hard to show that any member of $\mathcal{L}(C_0(G), C_0(G) \otimes K)$ arises in this way.

This hence motivates the study of $\mathcal{L}(A, A \otimes K)$ for an arbitrary C^* -algebra A . Fix a unit vector $\xi_0 \in K$, and regard K as the ‘‘row space’’ $\mathcal{L}(K, \mathbb{C})$, where K is a module over \mathbb{C} . So ξ_0 is identified with the map $\eta \mapsto (\xi_0|\eta)$. This is adjointable, with adjoint $\xi_0^* : \mathbb{C} \rightarrow K; t \mapsto t\xi_0$. Let $\iota : A \rightarrow A$ be the identity, so, as above, we can form the tensor product $\iota \otimes \xi_0 \in \mathcal{L}(A \otimes K, A \otimes \mathbb{C}) = \mathcal{L}(A \otimes K, A)$. This is simply the map $a \otimes \eta \mapsto a(\xi_0|\eta)$, and the adjoint is $(\iota \otimes \xi_0)^* = \iota \otimes \xi_0^* : a \mapsto a \otimes \xi_0$. It is actually not particularly hard to show by direct calculation that these maps are contractive and are mutual adjoints.

Then we have an embedding and a quotient map, both of which are adjointable, and hence A -module maps:

$$\begin{aligned} \mathcal{L}(A, A \otimes K) &\rightarrow \mathcal{L}(A \otimes K) \cong M(A \otimes \mathcal{B}_0(K)); & \alpha &\mapsto \alpha(\iota \otimes \xi_0), \\ \mathcal{L}(A \otimes K) &\rightarrow \mathcal{L}(A, A \otimes K); & \mathcal{T} &\mapsto \mathcal{T}(\iota \otimes \xi_0)^*. \end{aligned}$$

Hence we can identify $\mathcal{L}(A, A \otimes K)$ as a complemented submodule of $\mathcal{L}(A \otimes K)$. This follows, as $(\iota \otimes \xi_0)(\iota \otimes \xi_0)^*$ is the identity on A , and so the map $\mathcal{T} \mapsto \mathcal{T}(\iota \otimes \xi_0)^*(\iota \otimes \xi_0)$ is a projection from $\mathcal{L}(A \otimes K)$ onto the image of $\mathcal{L}(A, A \otimes K)$.

We shall use the notation that $\mathcal{T} \in \mathcal{L}(A \otimes K)$ is identified with $T \in M(A \otimes \mathcal{B}_0(K))$. Suppose that A is faithfully and non-degenerately represented on H . Then we can identify $M(A \otimes \mathcal{B}_0(K))$ with a subalgebra of $\mathcal{B}(H \otimes K)$, and we shall continue to write T for the resulting operator in $\mathcal{B}(H \otimes K)$. Similarly, we identify $M(A)$ with $\{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\}$.

It will be useful to define some auxiliary maps. For $\xi \in H$, define $e_\xi : A \otimes K \rightarrow H \otimes K$ by $e_\xi(a \otimes \eta) = a(\xi) \otimes \eta$, and linearity and continuity. This makes sense, as given $\tau = \sum_n a_n \otimes \eta_n \in A \otimes K$, we have that

$$\|e_\xi(\tau)\|^2 = \sum_{n,m} (a_n(\xi)|a_m(\xi))(\eta_n|\eta_m) = \left(\xi \left| \sum_{n,m} a_n^* a_m (\eta_n|\eta_m) \xi \right. \right) = (\xi | (\tau|\tau)\xi) \leq \|\xi\|^2 \|\tau\|^2.$$

Thus e_ξ is bounded, with $\|e_\xi\| \leq \|\xi\|$. Notice that this calculation also shows that

$$(e_\xi(\tau)|e_\eta(\sigma)) = (\xi | (\tau|\sigma)\eta) \quad (\tau, \sigma \in A \otimes K, \xi, \eta \in H),$$

where here $(\tau|\sigma) \in A \subseteq \mathcal{B}(H)$.

The next two propositions show a tight connection between these ideas. In the following, we could have *defined* $\tilde{\alpha}$ using (iii). Notice that as A has a bounded approximate identity and is non-degenerately represented on H , it follows that $H = \{a(\xi) : a \in A, \xi \in H\}$; this uses the Cohen Factorisation Theorem (compare [5, Corollary 2.9.25] or [20, Theorem A.1]). However, we would still have to prove that $\tilde{\alpha}$ were well-defined.

Proposition 3.1. *Let A be a C^* -algebra faithfully and non-degenerately represented on H , and let K be a Hilbert space. Let $\alpha \in \mathcal{L}(A, A \otimes K)$ and $\mathcal{T} \in \mathcal{L}(A \otimes K)$ be related by $\alpha = \mathcal{T}(\iota \otimes \xi_0)^*$, where $\xi_0 \in K$ is a unit vector. (For example, if given α , we could define $\mathcal{T} = \alpha(\iota \otimes \xi_0)$). Let $\tilde{\alpha} : H \rightarrow H \otimes K$ be the operator given by $\tilde{\alpha}(\xi) = T(\xi \otimes \xi_0)$ for $\xi \in H$. Then:*

1. $\|\tilde{\alpha}\| = \|\alpha\|$;
2. $\tilde{\alpha}^*\tilde{\alpha} = \alpha^*\alpha \in \mathcal{L}(A) \cong M(A)$, where we identify $M(A)$ as a subalgebra of $\mathcal{B}(H)$.
3. $\tilde{\alpha}(a(\xi)) = e_\xi\alpha(a)$ for $a \in A$ and $\xi \in H$; so $\tilde{\alpha}$ depends only on α (and not ξ_0 or \mathcal{T}).

Proof. Let $\Gamma : \mathcal{K}(A \otimes K) \rightarrow A \otimes \mathcal{B}_0(K) \subseteq \mathcal{B}(H \otimes K)$ be the isomorphism, which satisfies $\Gamma(\theta_{a \otimes \xi, b \otimes \eta}) = ab^* \otimes \theta_{\xi, \eta}$ for $a, b \in A$ and $\xi, \eta \in K$. Thus, for $c \in A$, $\phi \in H$ and $\gamma \in K$,

$$\Gamma(\theta_{a \otimes \xi, b \otimes \eta})(c(\phi) \otimes \gamma) = ab^*c(\phi) \otimes \xi(\eta|\gamma).$$

Also, $e_\phi(\theta_{a \otimes \xi, b \otimes \eta}(c \otimes \gamma)) = ab^*c(\phi) \otimes \xi(\eta|\gamma)$. Let $\theta \in \mathcal{K}(A \otimes K)$, $\tau \in A \otimes K$ and $\phi \in H$. So we have shown that $e_\phi(\theta(\tau)) = \Gamma(\theta)(e_\phi(\tau))$. By definition, we have that $\Gamma(\mathcal{T}\theta) = T\Gamma(\theta)$, and so

$$e_\phi(\mathcal{T}\theta(\tau)) = \Gamma(\mathcal{T}\theta)(e_\phi(\tau)) = T\Gamma(\theta)(e_\phi(\tau)) = Te_\phi(\theta(\tau)).$$

By density, it follows that

$$e_\phi(\mathcal{T}(\tau)) = Te_\phi(\tau) \quad (\tau \in A \otimes K, \phi \in H).$$

So immediately we see that for $a \in A$ and $\xi \in H$,

$$\tilde{\alpha}(a(\xi)) = T(a(\xi) \otimes \xi_0) = Te_\xi(a \otimes \xi_0) = e_\xi(\mathcal{T}(a \otimes \xi_0)) = e_\xi\alpha(a),$$

as claimed. Then, for $a, b \in A$ and $\xi, \eta \in H$,

$$(\tilde{\alpha}(a(\xi))|\tilde{\alpha}(b(\eta))) = (e_\xi\alpha(a)|e_\eta\alpha(b)) = (\xi|(\alpha(a)|\alpha(b))\eta) = (\xi|a^*\alpha^*\alpha b\eta) = (a(\xi)|\alpha^*\alpha b(\eta)).$$

It follows that $\tilde{\alpha}^*\tilde{\alpha}$ agrees with $\alpha^*\alpha$ as operators on H . Then $\|\alpha\|^2 = \|\alpha^*\alpha\| = \|\tilde{\alpha}^*\tilde{\alpha}\| = \|\tilde{\alpha}\|^2$, finishing the proof. \square

Proposition 3.2. *Let B be a C^* -algebra and let $\phi : A \rightarrow M(B)$ be a non-degenerate $*$ -homomorphism. Let $\alpha \in \mathcal{L}(A, A \otimes K)$ and $\mathcal{T} \in \mathcal{L}(A \otimes K)$ be related by $\alpha = \mathcal{T}(\iota \otimes \xi_0)^*$, where $\xi_0 \in K$ is a unit vector. Let $S = (\phi \otimes \iota)T \in M(B \otimes \mathcal{B}_0(K))$, use this to induce $\mathcal{S} \in \mathcal{L}(B \otimes K)$, and then define $\phi * \alpha = \mathcal{S}(\iota \otimes \xi_0)^* \in \mathcal{L}(B, B \otimes K)$. Then:*

1. $(\iota \otimes \xi)(\phi * \alpha) = \phi((\iota \otimes \xi)\alpha)$ for each $\xi \in K$;
2. $\phi * \alpha$ depends only upon α ;

Proof. As before, let $\Gamma : \mathcal{K}(A \otimes K) \rightarrow A \otimes \mathcal{B}_0(K)$ be the isomorphism, with strict extension $\tilde{\Gamma}$; we use the same notation for the isomorphism $\mathcal{K}(B \otimes K) \rightarrow B \otimes \mathcal{B}_0(K)$. Let ϕ_0 be the following composition

$$\mathcal{K}(A \otimes K) \xrightarrow{\Gamma} A \otimes \mathcal{B}_0(K) \xrightarrow{\phi \otimes \iota} M(B) \otimes \mathcal{B}_0(K) \hookrightarrow M(B \otimes \mathcal{B}_0(K)) \xrightarrow{\tilde{\Gamma}^{-1}} \mathcal{L}(B \otimes K),$$

and let $\tilde{\phi}_0 : \mathcal{L}(A \otimes K) \rightarrow \mathcal{L}(B \otimes K)$ be the strict extension. Thus $\mathcal{S} = \tilde{\phi}_0(\mathcal{T})$. For $\xi \in K$, let

$$y = (\iota \otimes \xi)\tilde{\phi}_0(\mathcal{T})(\iota \otimes \xi_0)^* \in M(B), \quad x = (\iota \otimes \xi)\mathcal{T}(\iota \otimes \xi_0)^* \in M(A).$$

To show (i), we are required to show that $\phi(x) = y$. As ϕ is non-degenerate, this is equivalent to $\phi(xa)b = y\phi(a)b$ for $a \in A, b \in B$, that is,

$$\phi((\iota \otimes \xi)\mathcal{T}(a \otimes \xi_0))b = (\iota \otimes \xi)\tilde{\phi}_0(\mathcal{T})(\phi(a)b \otimes \xi_0) \quad (a \in A, b \in B).$$

Now, for $a, c \in A$, $b \in B$ and $\eta, \gamma \in K$,

$$\tilde{\phi}_0(\theta_{a \otimes \xi_0, c \otimes \eta})(b \otimes \gamma) = \Gamma^{-1}\left((\phi \otimes \iota)(ac^* \otimes \theta_{\xi_0, \eta})\right)(b \otimes \gamma) = \phi(ac^*)b \otimes \xi_0(\eta|\gamma).$$

So also

$$\tilde{\phi}_0(\mathcal{T})(\phi(ac^*)b \otimes \xi_0)(\eta|\gamma) = \tilde{\phi}_0(\mathcal{T})\tilde{\phi}_0(\theta_{a \otimes \xi_0, c \otimes \eta})(b \otimes \gamma) = \tilde{\phi}_0(\theta_{\mathcal{T}(a \otimes \xi_0), c \otimes \eta})(b \otimes \gamma).$$

It seems easier to use an approximation argument now. For $\epsilon > 0$, we can find $\tau \in A \odot K$ with

$$\tau = \sum_k a_k \otimes \xi_k, \quad \|\mathcal{T}(a \otimes \xi_0) - \tau\| \leq \epsilon.$$

Then $\|\theta_{\mathcal{T}(a \otimes \xi_0), c \otimes \eta} - \theta_{\tau, c \otimes \eta}\| \leq \epsilon\|c\|\|\eta\|$. Thus the previous paragraph shows that

$$\left\| \tilde{\phi}_0(\mathcal{T})(\phi(ac^*)b \otimes \xi_0)(\eta|\gamma) - \sum_k \phi(a_k c^*)b \otimes \xi_k(\eta|\gamma) \right\| \leq \epsilon\|c\|\|\eta\|\|b\|\|\gamma\|,$$

Letting c run through an approximate identity for A , and choosing $\eta = \gamma$ to be a unit vector shows that

$$\left\| \tilde{\phi}_0(\mathcal{T})(\phi(a)b \otimes \xi_0) - \sum_k \phi(a_k)b \otimes \xi_k \right\| \leq \epsilon\|b\|.$$

Thus also

$$\left\| (\iota \otimes \xi)\tilde{\phi}_0(\mathcal{T})(\phi(a)b \otimes \xi_0) - \sum_k \phi(a_k)b \otimes (\xi|\xi_k) \right\| \leq \epsilon\|b\|\|\xi\|.$$

However, similarly

$$\left\| \phi((\iota \otimes \xi)\mathcal{T}(a \otimes \xi_0))b - \sum_k \phi(a_k)b \otimes (\xi|\xi_k) \right\| \leq \epsilon\|b\|\|\xi\|.$$

As $\epsilon > 0$ was arbitrary, this completes the proof of (i). It is immediate that (i) implies (ii). \square

Proposition 3.3. *With the notation of the previous proposition, suppose that B is non-degenerately represented on $H \otimes H$, and that for some $V \in \mathcal{B}(H \otimes H)$, we have that $\phi(a) = V^*(1 \otimes a)V$ for $a \in A$. Then $(\phi * \alpha)^\sim = V_{12}^*(1 \otimes \tilde{\alpha})V$.*

Proof. Combining the two previous propositions, we see that $(\phi * \alpha)^\sim(\xi) = S(\xi \otimes \xi_0)$ for $\xi \in H \otimes H$. Now, clearly $S = V_{12}^*T_{23}V_{12} \in \mathcal{B}(H \otimes H \otimes K)$, and so

$$(\phi * \alpha)^\sim(\xi) = V_{12}^*T_{23}(V(\xi) \otimes \xi_0) = V_{12}^*(1 \otimes \tilde{\alpha})V(\xi) \quad (\xi \in H \otimes H),$$

as required. \square

4 Left-multipliers

Let \mathbb{G} be a locally compact quantum group. In this section, we prove a complete analogy of Gilbert's result, for represented, completely bounded left multipliers of $L^1(\hat{\mathbb{G}})$.

Let K be a Hilbert space, and consider the Hilbert C^* -module $C_0(\mathbb{G}) \otimes K$. We shall say that a pair (α, β) of maps in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ is *invariant* if

$$(1 \otimes \beta)^*(\Delta * \alpha) \in \mathcal{L}(C_0(\mathbb{G}) \otimes C_0(\mathbb{G})) = M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$$

is really in $C^b(\mathbb{G}) \otimes 1$. Here $\Delta : C_0(\mathbb{G}) \rightarrow M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ is non-degenerate, and so we can apply Proposition 3.2 to form $\Delta * \alpha \in \mathcal{L}(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{G}) \otimes K)$.

When $\mathbb{G} = G$ a locally compact group, then $\alpha, \beta \in C^b(G, K)$, and $\Delta * \alpha \in C^b(G \times G, K)$. For $\xi \in K$ and $s, t \in G$, we have

$$(\xi | (\Delta * \alpha)(s, t)) = (\iota \otimes \xi)(\Delta * \alpha)(s, t) = \Delta((\iota \otimes \xi)\alpha)(s, t) = ((\iota \otimes \xi)\alpha)(st) = (\xi | \alpha(st)).$$

So $(\Delta * \alpha)(s, t) = \alpha(st)$, as we might hope. Then (α, β) is an invariant pair if there exists $f \in C^b(G)$ with

$$(\beta(t) | \alpha(st)) = f(s) \quad (s, t \in G),$$

or equivalently, if $f(st^{-1}) = (\beta(t) | \alpha(s))$ for $s, t \in G$. This is clearly equivalent, though not identical, to Gilbert's condition, as outlined in the introduction. Proposition 4.1 below shows that it is no surprise that the $f \in C^b(G)$ appearing from $(1 \otimes \beta)^*(\Delta * \alpha) = f \otimes 1$ should be the multiplier given by the pair (α, β) .

By Proposition 3.3, we see that, equivalently, (α, β) is invariant if

$$(1 \otimes \tilde{\beta})^* W_{12}^* (1 \otimes \tilde{\alpha}) W \in C^b(\mathbb{G}) \otimes 1,$$

as operators on $\mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$. Here we use that $(1 \otimes \beta)^\sim = 1 \otimes \tilde{\beta}$.

Proposition 4.1. *Let $\alpha, \beta \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$, and for $x \in L^\infty(\hat{\mathbb{G}})$, define $L(x) = \tilde{\beta}^*(x \otimes 1)\tilde{\alpha}$. Let $a \in C^b(\mathbb{G})$. The following are equivalent:*

1. L is the adjoint a completely bounded left multiplier on $L^1(\hat{\mathbb{G}})$ represented by a ;
2. the pair (α, β) is invariant, with $(1 \otimes \beta)^*(\Delta * \alpha) = a \otimes 1$.

Proof. Clearly L is a normal completely bounded map $L^\infty(\hat{\mathbb{G}}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$. As $\hat{W} = \sigma W^* \sigma$, we see that

$$\begin{aligned} (L \otimes \iota)(\hat{W}) &= (\tilde{\beta}^* \otimes 1)\hat{W}_{13}(\tilde{\alpha} \otimes 1) = (\tilde{\beta}^* \otimes 1)\sigma_{13}W_{13}^*\sigma_{13}(\tilde{\alpha} \otimes 1) \\ &= \sigma(1 \otimes \tilde{\beta}^*\sigma)W_{13}^*(1 \otimes \sigma\tilde{\alpha})\sigma = \sigma(1 \otimes \tilde{\beta}^*)W_{12}^*(1 \otimes \tilde{\alpha})\sigma. \end{aligned}$$

So, if (2) holds, then

$$(L \otimes \iota)(\hat{W}) = \sigma(a \otimes 1)W^*\sigma = (1 \otimes a)\hat{W}.$$

By the (left) version of Proposition 2.3, it follows that (1) holds.

Conversely, if (1) holds, then by Proposition 2.4, we have that $(L \otimes \iota)(\hat{W}) = (1 \otimes a)\hat{W}$, which shows that (2) holds. \square

Notice that we here assume that $a \in C^b(\mathbb{G})$, while in Section 2.1 we could only ensure that $a \in L^\infty(\mathbb{G})$. The next result clarifies this.

Theorem 4.2. *Let $L_* \in \mathcal{CB}(L^1(\hat{\mathbb{G}}))$ and $a \in L^\infty(\mathbb{G})$ be such that $a\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L_*(\hat{\omega}))$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. There exists a Hilbert space K and an invariant pair (α, β) of maps in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ such that (α, β) induces $L = (L_*)^*$ as in Proposition 4.1, and with $\|\alpha\| \|\beta\| = \|L\|_{cb}$. Furthermore, automatically $a \in C^b(\mathbb{G})$, so L_* is represented.*

Proof. Let $L = L_*^* \in \mathcal{CB}(L^\infty(\hat{\mathbb{G}}))$. As L is normal, we can find a Hilbert space K , a normal $*$ -representation $\pi : L^\infty(\hat{\mathbb{G}}) \rightarrow \mathcal{B}(K)$ and maps $P, Q : L^2(\mathbb{G}) \rightarrow K$ with $\|P\| \|Q\| = \|L\|_{cb}$, and with

$$L(x) = Q^* \pi(x) P \quad (x \in L^\infty(\hat{\mathbb{G}})).$$

This is, of course, the usual representation result for completely bounded maps, but as L is normal, we can assume that π is normal: the details of this change are worked out in the proof of [10, Theorem 2.4], for example.

Kustermans showed in [15, Corollary 4.3] that if B is a C^* -algebra and $\phi : L^1_{\sharp}(\mathbb{G}) \rightarrow M(B)$ is a non-degenerate $*$ -homomorphism (in the sense that $\{\phi(\omega)b : \omega \in L^1_{\sharp}(\mathbb{G}), b \in B\}$ is linearly dense in B), then there is a unitary $U \in M(C_0(\mathbb{G}) \otimes B)$ such that

$$\phi(\omega) = (\omega \otimes \iota)(U) \quad (\omega \in L^1_{\sharp}(\mathbb{G})) \quad (\Delta \otimes \iota)(U) = U_{13}U_{23}.$$

The philosophy here is that ϕ extends to a $*$ -homomorphism from the enveloping C^* -algebra of $L^1_{\sharp}(\mathbb{G})$, and so ϕ can be thought of as a representation of the (universal) quantum group $\hat{\mathbb{G}}$, whereas U is a corepresentation of \mathbb{G} ; Kustermans's result is that there is a correspondence between representations of $\hat{\mathbb{G}}$ and corepresentations of \mathbb{G} .

As we may assume that $\pi : L^\infty(\hat{\mathbb{G}}) \rightarrow \mathcal{B}(K)$ is unital, and $L^1_{\sharp}(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, it follows that $\pi\lambda : L^1_{\sharp}(\mathbb{G}) \rightarrow \mathcal{B}(K) = M(\mathcal{B}_0(K))$ is non-degenerate, and so we can find a representing unitary $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(K))$. Notice that $C_0(\mathbb{G}) \otimes \mathcal{B}_0(K)$ acts non-degenerately on $L^2(\mathbb{G}) \otimes K$, and so we may identify U with an operator in the von Neumann algebra $L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(K)$.

Let $\omega \in L^1_{\sharp}(\mathbb{G})$ and let $\gamma, \delta \in K$. Then

$$\begin{aligned} \langle U, \omega \otimes \omega_{\gamma, \delta} \rangle &= (\gamma | (\omega \otimes \iota)(U) \delta) = (\gamma | \pi(\lambda(\omega)) \delta) = \langle \lambda(\omega), \pi_*(\omega_{\gamma, \delta}) \rangle \\ &= \langle (\omega \otimes \iota)(W), \pi_*(\omega_{\gamma, \delta}) \rangle = \langle \pi((\omega \otimes \iota)(W)), \omega_{\gamma, \delta} \rangle \\ &= \langle (\omega \otimes \iota)(\iota \otimes \pi)(W), \omega_{\gamma, \delta} \rangle = \langle (\iota \otimes \pi)(W), \omega \otimes \omega_{\gamma, \delta} \rangle. \end{aligned}$$

Here $\pi_* : \mathcal{B}(K)_* \rightarrow L^1(\hat{\mathbb{G}})$ is the pre-adjoint, which exists as π is normal. By density of $L^1_{\sharp}(\mathbb{G})$ in $L^1(\mathbb{G})$, we conclude that $U = (\iota \otimes \pi)(W) \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(K)$. Indeed, if we wished, we could define U this way, and avoid using [15].

Also, we identify $M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(K))$ with $\mathcal{L}(C_0(\mathbb{G}) \otimes K)$ and so U induces $\mathcal{U} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$. Similarly, $W \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(L^2(\mathbb{G})))$ is associated to $\mathcal{W} \in \mathcal{L}(C_0(\mathbb{G}) \otimes L^2(\mathbb{G}))$. Fix a unit vector $\xi_0 \in L^2(\mathbb{G})$ and define

$$\begin{aligned} \alpha &= \mathcal{U}^*(1 \otimes P)\mathcal{W}(\iota \otimes \xi_0)^* \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K), \\ \beta &= \mathcal{U}^*(1 \otimes Q)\mathcal{W}(\iota \otimes \xi_0)^* \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K). \end{aligned}$$

Notice that $\|\alpha\| \|\beta\| \leq \|P\| \|Q\| = \|L\|_{cb}$. By Proposition 3.1, α induces $\tilde{\alpha} \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$, and similarly β induces $\tilde{\beta}$, and in fact, we have that

$$\tilde{\alpha}(\xi) = U^*(1 \otimes P)W(\xi \otimes \xi_0), \quad \tilde{\beta}(\xi) = U^*(1 \otimes Q)W(\xi \otimes \xi_0) \quad (\xi \in L^2(\mathbb{G})).$$

We next show that (α, β) is invariant, for which we need to consider $(1 \otimes \tilde{\beta})^* W_{12}^*(1 \otimes \tilde{\alpha})W$. Let $\xi, \eta \in L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$, and we calculate that

$$\begin{aligned} &((1 \otimes \tilde{\beta})\xi | W_{12}^*(1 \otimes \tilde{\alpha})W\eta) \\ &= (U_{23}^*(1 \otimes 1 \otimes Q)W_{23}(\xi \otimes \xi_0) | W_{12}^*U_{23}^*(1 \otimes 1 \otimes P)W_{23}W_{12}(\eta \otimes \xi_0)) \\ &= (U_{23}^*(1 \otimes 1 \otimes Q)W_{23}(\xi \otimes \xi_0) | W_{12}^*U_{23}^*W_{12}(1 \otimes 1 \otimes P)W_{13}W_{23}(\eta \otimes \xi_0)) \end{aligned}$$

Here we used the Pentagonal relation $W_{12}W_{13}W_{23} = W_{23}W_{12}$. Now, if $X \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(K)$, then $W_{12}^*XW_{12} = (\Delta \otimes \iota)X$, so we find that $W_{12}^*U_{23}^*W_{12} = (\Delta \otimes \iota)(U^*) = U_{23}^*U_{13}$ as Δ is a $*$ -

homomorphism. Thus we get

$$\begin{aligned}
& ((1 \otimes \tilde{\beta})\xi | W_{12}^*(1 \otimes \tilde{\alpha})W\eta) \\
&= (U_{23}^*(1 \otimes 1 \otimes Q)W_{23}(\xi \otimes \xi_0) | U_{23}^*U_{13}^*(1 \otimes 1 \otimes P)W_{13}W_{23}(\eta \otimes \xi_0)) \\
&= (W_{23}(\xi \otimes \xi_0) | (1 \otimes 1 \otimes Q^*)(\iota \otimes \pi)(W^*)_{13}(1 \otimes 1 \otimes P)W_{13}W_{23}(\eta \otimes \xi_0)) \\
&= (W_{23}(\xi \otimes \xi_0) | (\iota \otimes L)(W^*)_{13}W_{13}W_{23}(\eta \otimes \xi_0))
\end{aligned}$$

By Proposition 2.4, $(L \otimes \iota)(\hat{W}) = (1 \otimes a)\hat{W}$. Equivalently, we have $(\iota \otimes L)(W^*) = (a \otimes 1)W^*$, so we get

$$((1 \otimes \tilde{\beta})\xi | W_{12}^*(1 \otimes \tilde{\alpha})W\eta) = (W_{23}(\xi \otimes \xi_0) | (a \otimes 1 \otimes 1)W_{13}^*W_{13}W_{23}(\eta \otimes \xi_0)) = (\xi | (a \otimes 1)\eta).$$

Thus (α, β) is invariant, and induces a ; in particular, we must have that $a \in C^b(\mathbb{G})$. So by Proposition 4.1, if $L_0^*(x) = \tilde{\beta}^*(x \otimes 1)\tilde{\alpha}$ for $x \in L^\infty(\hat{\mathbb{G}})$, then L_0^* is normal, maps into $L^\infty(\hat{\mathbb{G}})$, and the pre-adjoint L_0 satisfies $\hat{\lambda}(L_0(\hat{\omega})) = a\hat{\lambda}(\hat{\omega})$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. As $\hat{\lambda}$ injects, it follows that $L_0 = L_*$, as required. \square

5 Approaches to right multipliers

In the previous section, we studied represented completely bounded left multipliers. There are a number of ways to deal with right multipliers:

- Directly try to generalise the proof of Proposition 4.1. We do this in Proposition 5.1. However, there are no *a priori* links with $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$.
- Use the unitary antipode to convert right multipliers into left multipliers. We do this in Lemma 5.2, which gives formulas suggestive of those in Proposition 5.1. We are also now in a position to use [12, Corollary 4.4] to show that every completely bounded left multiplier is represented.
- Use the opposite algebra $L^1(\hat{\mathbb{G}}^{\text{op}})$, as a right multiplier of $L^1(\hat{\mathbb{G}})$ is a left multiplier of $L^1(\hat{\mathbb{G}}^{\text{op}})$. However, by the duality theory, this leads us to consider the algebra $C_0(\mathbb{G}')$. We find a way to move back to $C_0(\mathbb{G})$ which gives exactly the formulas we were led to consider by Lemma 5.2. Further, we find that a pair (α, β) in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ is invariant if and only if (β, α) is invariant. This “swap” operation $(\alpha, \beta) \mapsto (\beta, \alpha)$ induces a natural map $L_* \mapsto L_*^\dagger$ of left multipliers, see Proposition 5.8.
- To make links with [15], we consider a “coordinate” approach in Section 5.2 which leads to Theorem 5.9 which, in particular, allows us to show that the map $(\alpha, \beta) \mapsto (\beta, \alpha)$ is the antipode (in a technical sense).

Proposition 5.1. *Let $P, Q \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$, and define a map $R : L^\infty(\hat{\mathbb{G}}) \rightarrow \mathcal{B}(L^2(\mathbb{G}))$ by $R(x) = P^*(x \otimes 1)Q$ for $x \in L^\infty(\hat{\mathbb{G}})$. Let $a \in C^b(\mathbb{G})$. The following are equivalent:*

1. *R is the adjoint of a completely bounded right multiplier of $L^1(\hat{\mathbb{G}})$ which is represented by a ;*
2. $(1 \otimes Q^*)W_{12}(1 \otimes P)W^* = a^* \otimes 1$.

Proof. As in the proof of Proposition 4.1,

$$(R \otimes \iota)(\hat{W}) = \sigma(1 \otimes P^*)W_{12}^*(1 \otimes Q)\sigma.$$

Thus, if (i) holds, then by Proposition 2.4,

$$\sigma(1 \otimes P^*)W_{12}^*(1 \otimes Q)\sigma = \sigma W^*(a \otimes 1)\sigma.$$

Taking the adjoint gives (ii). The converse follows from Proposition 2.3. \square

Compared to Proposition 4.1, we have swapped W with W^* . As such, it's not immediately clear how to relate P and Q to maps in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$.

Another approach to right multipliers is to use the unitary antipode $\hat{\kappa}$ to convert the problem to studying left multipliers, which follows, as $\hat{\kappa}_*$ is anti-multiplicative on $L^1(\hat{\mathbb{G}})$.

Lemma 5.2. *Let $R_* : L^1(\hat{\mathbb{G}}) \rightarrow L^1(\hat{\mathbb{G}})$ be a right multiplier, and defined $L_* = \hat{\kappa}_* R_* \hat{\kappa}_*$, a left multiplier. Then:*

1. R_* is completely bounded if and only if L_* is;
2. if R_* is represented by $a \in C^b(\mathbb{G})$, then L_* is represented by $\kappa(a) \in C^b(\mathbb{G})$.

Proof. For (i), suppose first that R_* is completely bounded, so that $R \in \mathcal{CB}(L^\infty(\hat{\mathbb{G}}))$. For $x \in L^\infty(\hat{\mathbb{G}})$, we have that $\hat{\kappa}(x) = Jx^*J$, and so

$$L(x) = \hat{\kappa}R\hat{\kappa}(x) = JR(Jx^*J)^*J \quad (x \in L^\infty(\hat{\mathbb{G}})).$$

As R is completely bounded, it admits a dilation—compare with the proof of Theorem 4.2 above, but here we will assume that the normal representation π is an amplification (as we may, see [24, Chapter IV, Theorem 5.5] for example). So there exists a Hilbert space H and bounded maps $U, V : L^2(\mathbb{G}) \rightarrow H \otimes L^2(\mathbb{G})$ such that $R(x) = V^*(1 \otimes x)U$ for $x \in L^\infty(\hat{\mathbb{G}})$. Thus

$$L(x) = JU^*(1 \otimes J)(1 \otimes x)(1 \otimes J)VJ \quad (x \in L^\infty(\hat{\mathbb{G}})),$$

showing that L , and hence also L_* , are completely bounded. The converse follows similarly.

For (ii), let $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, so that

$$\hat{\lambda}(L_*(\hat{\omega})) = \kappa\hat{\lambda}(R_*\hat{\kappa}_*(\hat{\omega})) = \kappa(\hat{\lambda}(\hat{\kappa}_*(\hat{\omega}))a) = \kappa(a)\kappa\hat{\lambda}(\hat{\kappa}_*(\hat{\omega})) = \kappa(a)\hat{\lambda}(\hat{\omega}),$$

using that $\kappa\hat{\lambda} = \hat{\lambda}\hat{\kappa}_*$. Hence L_* is represented by $\kappa(a)$, as required. \square

Thus, if R_* is a completely bounded right multiplier which is represented, then $L_* = \hat{\kappa}_* R_* \hat{\kappa}_*$ is a represented left multiplier, and hence admits an invariant pair (α, β) in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. Indeed, for $x \in L^\infty(\hat{\mathbb{G}})$, we have that $L(x) = \tilde{\beta}^*(x \otimes 1)\tilde{\alpha}$, so that

$$\begin{aligned} R(x) &= \hat{\kappa}L\hat{\kappa}(x) = JL(\hat{\kappa}(x))^*J = J\tilde{\alpha}^*(\hat{\kappa}(x)^* \otimes 1)\tilde{\beta}J = J\tilde{\alpha}^*(JxJ \otimes 1)\tilde{\beta}J \\ &= J\tilde{\alpha}^*(J \otimes J_K)(x \otimes 1)(J \otimes J_K)\tilde{\beta}J. \end{aligned}$$

Here J_K is some involution on K : a conjugate linear isometry with $J_K^2 = 1$ (we can always find such a map: just write K as $\ell^2(I)$ for some index set I). This gives one way to link the maps P and Q appearing in Proposition 5.1 above to maps α, β in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. Furthermore, the map $(J \otimes J_K)\tilde{\alpha}J$ will appear (in slightly different context) below in Lemma 5.4.

The following is an improvement upon [12, Corollary 4.4], in that we can show that every left or right multiplier is represented by an element of $C^b(\mathbb{G})$, and not just $L^\infty(\mathbb{G})$.

Proposition 5.3. *Any left or right completely bounded multiplier of $L^1(\hat{\mathbb{G}})$ is represented by an element of $C^b(\mathbb{G})$.*

Proof. Let R_* be a completely bounded right multiplier of $L^1(\hat{\mathbb{G}})$, and choose $x \in L^\infty(\mathbb{G}')$ by Theorem 2.2 (that is, using [12]) so that $\hat{\rho}(\hat{\omega})x = \hat{\rho}(R_*(\hat{\omega}))$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. By the definition of $\hat{\rho}$, we see that $\hat{\lambda}(\hat{\omega})J\hat{J}x\hat{J}J = \hat{\lambda}(R_*(\hat{\omega}))$ for each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Set $b = J\hat{J}x\hat{J}J$, and let $L_* = \hat{\kappa}_*R_*\hat{\kappa}_*$ so by (the proof of) Lemma 5.2, L_* is a completely bounded left multiplier with $\kappa(b)\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L_*(\hat{\omega}))$ for each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. From Theorem 4.2, it follows that $\kappa(b) \in C^b(\mathbb{G})$, and so also $b \in C^b(\mathbb{G})$. Similarly, using the unitary antipode, a similar argument gives the result for completely bounded left multipliers. \square

5.1 Using the opposite algebra

Recall the definition of the opposite quantum group $\hat{\mathbb{G}}^{\text{op}}$ from Section 2. Given a completely bounded right multiplier R_* of $L^1(\hat{\mathbb{G}})$, write R_*^{op} for R_* considered as a map on $L^1(\hat{\mathbb{G}}^{\text{op}})$, so that R_*^{op} is a completely bounded left multiplier.

We now know that R_*^{op} is represented, say by $JbJ \in C^b(\mathbb{G}') = JC^b(\mathbb{G})J$. By Theorem 4.2, we can find $\alpha', \beta' \in \mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$ such that the pair (α', β') is invariant with respect to JbJ , that is, $(1 \otimes \beta')^*(\Delta' * \alpha') = JbJ \otimes 1$, and such that $R(x) = \tilde{\beta}'^*(x \otimes 1)\tilde{\alpha}'$ for $x \in L^\infty(\mathbb{G})$.

Recall the isomorphism $\Phi : C_0(\mathbb{G}') \rightarrow C_0(\mathbb{G}); a \mapsto \hat{J}JaJ\hat{J}$. Given $\alpha' \in \mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$, we notice that $(\Phi \otimes \iota)\alpha'\Phi^{-1}$ is in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. However, this isomorphism does not interact well with forming $\Delta * \alpha$ or $\tilde{\alpha}$ (for example, we get nothing like Lemma 5.5 below). Rather, we study another bijection between $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ and $\mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$ which comes at the cost of choosing an involution J_K on K , which the bijection will depend upon. However, the results below show that, as far as multipliers are concerned, there is no dependence upon J_K . From now on, fix some involution J_K on K .

Lemma 5.4. *Define an anti-linear isomorphism $\theta : C_0(\mathbb{G}') \rightarrow C_0(\mathbb{G}); a \mapsto JaJ$. For $\alpha' \in \mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$, the map $\alpha = (\theta \otimes J_K)\alpha'\theta^{-1}$ is in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. Furthermore, we have that $\tilde{\alpha} = (J \otimes J_K)\tilde{\alpha}'J$.*

Proof. First check that for $\tau, \sigma \in C_0(\mathbb{G}) \otimes K$, we have that

$$((\theta \otimes J_K)\tau | (\theta \otimes J_K)\sigma) = J(\tau | \sigma)J.$$

Then, for $a, b \in C_0(\mathbb{G})$,

$$(\alpha(a) | \alpha(b)) = J(\alpha'(JaJ) | \alpha'(JbJ))J = a^*J\alpha'^*\alpha'Jb,$$

where here $\alpha'^*\alpha' \in C^b(\mathbb{G}')$, and so $J\alpha'^*\alpha'J \in C^b(\mathbb{G})$. It follows that α is well-defined and bounded. We can similarly show that $\alpha^* = \theta\alpha'^*(\theta^{-1} \otimes J_K)$, so in particular, α is adjointable.

Let $a \in C_0(\mathbb{G}), \xi \in L^2(\mathbb{G})$ and $\eta \in K$. With reference to Proposition 3.1, we have that $e_\xi(\theta \otimes J_K)(a \otimes \eta) = JaJ\xi \otimes J_k(\eta) = (J \otimes J_k)e_{J\xi}(a \otimes \eta)$. It follows that

$$\tilde{\alpha}(a\xi) = e_\xi(\theta \otimes J_K)\alpha'\theta^{-1}(a) = (J \otimes J_K)e_{J\xi}\alpha'(JaJ) = (J \otimes J_K)\tilde{\alpha}'(Ja\xi),$$

and so $\tilde{\alpha} = (J \otimes J_K)\tilde{\alpha}'J$. \square

Lemma 5.5. *Let $\alpha', \beta' \in \mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$ and $\alpha, \beta \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ be associated as in the previous lemma. Then the pair (α', β') is invariant with respect to $JbJ \in C^b(\mathbb{G}')$ if and only if the pair (α, β) is invariant with respect to $b \in C^b(\mathbb{G})$.*

Proof. We have that

$$\begin{aligned}
(\Delta' * \alpha)^\sim &= (W')_{12}^*(1 \otimes \tilde{\alpha}')W' \\
&= (J \otimes J \otimes J_K)W_{12}^*(J \otimes J \otimes J_K)(J \otimes J \otimes J_K)(1 \otimes \tilde{\alpha})(J \otimes J)(J \otimes J)W(J \otimes J) \\
&= (J \otimes J \otimes J_K)W_{12}^*(1 \otimes \tilde{\alpha})W(J \otimes J) = (J \otimes J \otimes J_K)(\Delta * \alpha)^\sim(J \otimes J).
\end{aligned}$$

Hence (α', β') being invariant with respect to JbJ is equivalent to

$$\begin{aligned}
JbJ \otimes 1 &= (1 \otimes \tilde{\beta}'^*)(J \otimes J \otimes J_K)(\Delta * \alpha)^\sim(J \otimes J) \\
&= (J \otimes J)(1 \otimes \tilde{\beta}'^*)(\Delta * \alpha)^\sim(J \otimes J).
\end{aligned}$$

By applying $J \otimes J$ to both sides, this is equivalent to (α, β) being invariant with respect to b , as claimed. \square

For $x \in L^\infty(\mathbb{G})$, we have that $R(x) = \tilde{\beta}'^*(x \otimes 1)\tilde{\alpha}'$. By using Lemma 5.4, we see that

$$R(x) = J\tilde{\beta}'^*(JxJ \otimes 1)\tilde{\alpha}J = \hat{\kappa}(\tilde{\alpha}^*(Jx^*J \otimes 1)\tilde{\beta}) = \hat{\kappa}(\tilde{\alpha}^*(\hat{\kappa}(x) \otimes 1)\tilde{\beta}) \quad (x \in L^\infty(\hat{\mathbb{G}})).$$

So to make links with Lemma 5.2, we are led to look at the pair (β, α) .

Proposition 5.6. *Let (α, β) be an invariant pair in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$, and let (α', β') be the associated invariant pair in $\mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$. Let R_*^{op} be the left multiplier of $L^1(\hat{\mathbb{G}}^{\text{op}})$ induced by (α', β') , and let R_* (a right multiplier of $L^1(\hat{\mathbb{G}})$) be represented by $a \in C^b(\mathbb{G})$. Then (β, α) is invariant with respect to $\kappa(a)$.*

Proof. Form R_*^{op} using (α', β') , so that R_* is a completely bounded right multiplier of $L^1(\mathbb{G})$. By Proposition 5.3, R_* is represented, say by $a \in C^b(\mathbb{G})$. Let $L_* = \hat{\kappa}_*R_*\hat{\kappa}_*$, so by Lemma 5.2, L_* is a left multiplier represented by $\kappa(a)$. For $x \in L^\infty(\hat{\mathbb{G}})$, we have that $\hat{\kappa}L\hat{\kappa}(x) = R(x) = \hat{\kappa}(\tilde{\alpha}^*(\hat{\kappa}(x) \otimes 1)\tilde{\beta})$, using the above calculation. Hence $L(x) = \tilde{\alpha}^*(x \otimes 1)\tilde{\beta}$. By Proposition 4.1, it follows that (β, α) is invariant with respect to $\kappa(a)$. \square

We now show what happens with the induced left multipliers of $L^1(\hat{\mathbb{G}})$, without reference to $L^1(\hat{\mathbb{G}}^{\text{op}})$. We first need a lemma: remember that $\hat{\lambda}^{\text{op}}$ is the homomorphism $L^1(\hat{\mathbb{G}}^{\text{op}}) \rightarrow C_0(\mathbb{G}')$.

Lemma 5.7. *For $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, we have that $\hat{\lambda}^{\text{op}}(\hat{\omega}) = J\hat{J}\hat{\lambda}(\hat{\omega}^*)^*\hat{J}J$.*

Proof. From [17, Section 4], we have that $W^{\text{op}} = (\hat{J} \otimes \hat{J})W(\hat{J} \otimes \hat{J})$, and so by duality, $\hat{W}^{\text{op}} = (J \otimes J)\hat{W}(J \otimes J)$. For $\hat{\omega} = \hat{\omega}_{\xi_0, \eta_0} \in L^1(\hat{\mathbb{G}})$, we have that

$$\langle x, \hat{\omega}_{J\xi_0, J\eta_0} \rangle = (J\xi_0 | xJ\eta_0) = (\eta_0 | Jx^*J\xi_0) = \langle \hat{\kappa}(x), \hat{\omega}^* \rangle \quad (x \in L^\infty(\hat{\mathbb{G}})).$$

Thus, for $\xi, \eta \in L^2(\mathbb{G})$, we have

$$\begin{aligned}
(\xi | \hat{\lambda}^{\text{op}}(\hat{\omega})\eta) &= (\xi | (\hat{\omega} \otimes \iota)(\hat{W}^{\text{op}})\eta) = (\xi_0 \otimes \xi | (J \otimes J)\hat{W}(J \otimes J)(\eta_0 \otimes \eta)) \\
&= (\hat{W}(J\eta_0 \otimes J\eta) | J\xi_0 \otimes J\xi) = \overline{(J\xi_0 \otimes J\xi | \hat{W}(J\eta_0 \otimes J\eta))} \\
&= \overline{(J\xi | (\hat{\omega}_{J\xi_0, J\eta_0} \otimes \iota)(\hat{W})J\eta)} = \overline{(J\xi | \hat{\lambda}(\hat{\kappa}_*(\hat{\omega}^*))J\eta)} \\
&= (\kappa\hat{\lambda}(\hat{\omega}^*)J\eta | J\xi) = (\hat{J}\hat{\lambda}(\hat{\omega}^*)^*\hat{J}J\eta | J\xi) = (\hat{J}J\xi | \hat{\lambda}(\hat{\omega}^*)^*\hat{J}J\eta).
\end{aligned}$$

Thus $\hat{\lambda}^{\text{op}}(\hat{\omega}) = J\hat{J}\hat{\lambda}(\hat{\omega}^*)^*\hat{J}J$. \square

Given a left multiplier L_* of $L^1(\hat{\mathbb{G}})$ define

$$L_*^\dagger(\hat{\omega}) = L_*(\hat{\omega}^*)^* \quad (\hat{\omega} \in L^1(\hat{\mathbb{G}})).$$

For $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, recall that $\hat{\omega}^* \in L^1(\hat{\mathbb{G}})$ satisfies $\langle x, \hat{\omega}^* \rangle = \overline{\langle x^*, \hat{\omega} \rangle}$ for $x \in L^\infty(\hat{\mathbb{G}})$. As the coproduct Δ is a $*$ -homomorphism, it is easy to see that $L^1(\hat{\mathbb{G}}) \rightarrow L^1(\hat{\mathbb{G}}); \hat{\omega} \mapsto \hat{\omega}^*$ is a conjugate-linear algebra homomorphism. It follows that L_*^\dagger is a left multiplier; completely bounded if L_* is (compare with the proof of Lemma 5.2). Similarly, we define R_*^\dagger for a right multiplier.

Proposition 5.8. *For $L_* \in M_{cb}^l(L^1(\hat{\mathbb{G}}))$, let L_* be given by an invariant pair (α, β) . Then the invariant pair (β, α) induces the left multiplier L_*^\dagger .*

Proof. Let (α, β) be invariant with respect to $b \in C^b(\mathbb{G})$, and let (β, α) be invariant with respect to $\kappa(a)$. Thus (β', α') is invariant with respect to $J\kappa(a)J = J\hat{J}a^*\hat{J}J$. Let T_*^{op} be the associated left multiplier of $L^1(\hat{\mathbb{G}}^{\text{op}})$, and let T_* be the associated right multiplier of $L^1(\hat{\mathbb{G}})$. Then, as in Proposition 5.6, we have that

$$T(x) = (\tilde{\alpha}')^*(x \otimes 1)\tilde{\beta}' = J\tilde{\alpha}^*(JxJ \otimes 1)\tilde{\beta}J = \hat{\kappa}L\hat{\kappa}(x) \quad (x \in L^\infty(\hat{\mathbb{G}})).$$

It follows that

$$\hat{\lambda}^{\text{op}}(\hat{\kappa}_*L_*\hat{\kappa}_*(\hat{\omega})) = \hat{\lambda}^{\text{op}}(T_*^{\text{op}}(\hat{\omega})) = J\hat{J}a^*\hat{J}J\hat{\lambda}^{\text{op}}(\hat{\omega}) \quad (\hat{\omega} \in L^1(\hat{\mathbb{G}})).$$

Now, for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, by Lemma 5.7, we have that $\hat{\lambda}^{\text{op}}(\hat{\kappa}_*(\hat{\omega})) = J\hat{J}\hat{\lambda}(\hat{\kappa}(\hat{\omega}^*))^*\hat{J}J = J\hat{J}\kappa(\hat{\lambda}(\hat{\omega}^*))^*\hat{J}J = J\hat{\lambda}(\hat{\omega}^*)J$. For $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, let $\hat{\sigma} = \hat{\kappa}_*(\hat{\omega}^*)$, so also $\hat{\omega} = \hat{\kappa}_*(\hat{\sigma}^*)$. Then

$$\hat{\lambda}^{\text{op}}(\hat{\kappa}_*L_*\hat{\kappa}_*(\hat{\omega})) = J\hat{\lambda}(L_*\hat{\kappa}_*(\hat{\omega})^*)J = J\hat{\lambda}(L_*^\dagger\hat{\kappa}_*(\hat{\omega}^*))J = J\hat{\lambda}(L_*^\dagger(\hat{\sigma}))J,$$

and also

$$J\hat{J}a^*\hat{J}J\hat{\lambda}^{\text{op}}(\hat{\omega}) = J\kappa(a)J\hat{\lambda}^{\text{op}}(\hat{\kappa}_*(\hat{\omega}^*)) = J\kappa(a)JJ\hat{\lambda}(\hat{\omega})J.$$

As these two are equal, we see that

$$\hat{\lambda}(L_*^\dagger(\hat{\sigma})) = \kappa(a)\hat{\lambda}(\hat{\sigma}) \quad (\hat{\sigma} \in L^1(\hat{\mathbb{G}})).$$

Thus L_*^\dagger is represented by $\kappa(a)$, which (β, α) is invariant with respect to, as required. \square

5.2 Taking a coordinate approach

We have shown that an invariant pair (α, β) , say represented by $b \in C^b(\mathbb{G})$, gives rise to another invariant pair (β, α) , say represented by $\kappa(a) \in C^b(\mathbb{G})$. In this section, we show that the relationship between a and b is given by the (in general, unbounded) antipode S .

Let us recall from [18, Section 5.5] that $MC_I(C_0(\mathbb{G}))$ is the collection of $(x_i)_{i \in I} \subseteq M(C_0(\mathbb{G}))$ such that $\sum_i x_i^*x_i$ is strictly convergent in $M(C_0(\mathbb{G}))$. Similarly, define $MR_I(C_0(\mathbb{G}))$ to be the collection of those families $(x_i^*)_{i \in I}$ with $(x_i) \in MC_I(C_0(\mathbb{G}))$.

Let K be a Hilbert space, and let $\alpha \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. Let (e_i) be an orthonormal basis for K , and let $\alpha_i = (\iota \otimes e_i)\alpha \in \mathcal{L}(C_0(\mathbb{G})) \cong C^b(\mathbb{G})$ for each i . A simple calculation shows that $(\iota \otimes e_i)^*(\iota \otimes e_i) = 1 \otimes \theta_{e_i, e_i} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$, and so $\sum_i (\iota \otimes e_i)^*(\iota \otimes e_i)$ converges strictly to the identity. Thus $\sum_i \alpha_i^*\alpha_i$ converges strictly to $\alpha^*\alpha$, and so $(\alpha_i) \in MC_I(C_0(\mathbb{G}))$. Furthermore, we have that

$$\alpha(a) = \sum_i \alpha_i a \otimes e_i \in C_0(\mathbb{G}) \otimes K \quad (a \in C_0(\mathbb{G})),$$

with the sum converging in norm.

Similarly, from Proposition 3.2, we have that $(\Delta * \alpha)_i = \Delta(\alpha_i)$ for all i . Hence, a pair (α, β) is invariant with respect to $b \in C^b(\mathbb{G})$ precisely when

$$\sum_i (1 \otimes \beta_i^*) \Delta(\alpha_i) = b \otimes 1 \in C^b(\mathbb{G}) \otimes 1.$$

Theorem 5.9. *For $a, b \in C^b(\mathbb{G})$, the following are equivalent:*

1. *there is $R_* \in M_{cb}^r(L^1(\hat{\mathbb{G}}))$ represented by a , with R_*^{op} being represented by JbJ ;*
2. *there is a pair (α, β) of maps in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ which is invariant with respect to b , and with (β, α) being invariant with respect to $\kappa(a)$;*
3. *there is $L_* \in M_{cb}^l(L^1(\hat{\mathbb{G}}))$ represented by $b \in C^b(\mathbb{G})$, with L_*^\dagger being represented by $\kappa(a)$.*

Furthermore, if these hold, then $a \in D(S)^* = D(S^{-1}) = D(\tau_{-i/2})^*$ and $b = \tau_{-i/2}(a^*) = \hat{J}S^{-1}(a)\hat{J}$.

Proof. By Proposition 5.6, (i) and (ii) are equivalent, and by Proposition 5.8, (ii) and (iii) are equivalent.

We shall assume (ii). As (β, α) is invariant with respect to $\kappa(a)$, applying the adjoint shows that

$$\sum_i \Delta(\beta_i^*)(1 \otimes \alpha_i) = \hat{J}a\hat{J} \otimes 1 \in C^b(\mathbb{G}) \otimes 1.$$

By [18, Corollary 5.34] (and, as we are working with $C^b(\mathbb{G})$ and not $C_0(\mathbb{G})$ here, we need also to look at [18, Remark 5.44]) it follows that $\kappa(a^*) \in D(S)$ with $S\kappa(a^*) = b$. Thus $\tau_{-i/2}(a^*) = b$, as claimed. \square

For each $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, we have that $\hat{\lambda}(\hat{\omega}^*)^* \in D(S) = D(\tau_{-i/2})$ and $S(\hat{\lambda}(\hat{\omega}^*)^*) = \hat{\lambda}(\hat{\omega})$. Furthermore, $\{\hat{\lambda}(\hat{\omega}^*)^* : \hat{\omega} \in L^1(\hat{\mathbb{G}})\}$ forms a core for S (either as an operator on $C_0(\mathbb{G})$ or on $L^\infty(\mathbb{G})$). These results follow easily from [18, Proposition 8.3] and [17, Proposition 2.4]. Combined with the work of Kustermans in [16] on strict extensions of one-parameter groups on C^* -algebras, these observations would give another way to show the above theorem. The proof of Lemma 5.7 can be adapted to show that $\hat{\lambda}^{\text{op}}(\hat{\omega}) = J\hat{J}S^{-1}(\hat{\lambda}(\hat{\omega}))\hat{J}J$ for $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, and this could then be used to argue purely at the level of multipliers, instead of with invariant pairs.

Notice that the ‘‘coordinate’’ approach is very close in spirit to how Vaes and Van Daele gave a definition of a *Hopf C^* -algebra* in [28]. It would be interesting to explore this further, together with the implicit link with Haagerup tensor products (which Spronk used extensively in his study of the completely bounded multipliers of $A(G)$ in [23]). Indeed, if one looks at the proof of [12, Corollary 4.4], then there are two steps. Firstly, the adjoint of a right multiplier is extended from $L^\infty(\mathbb{G})$ to a map on $B(L^2(\mathbb{G}))$ with certain commutation properties (see [12, Proposition 4.3]) and then an argument using the extended (or weak*) Haagerup tensor product is used, [12, Proposition 3.2] (compare with [2, Theorem 4.2], where it is more explicit as to how the Haagerup tensor product appears). Indeed, with this perspective, what we have done is to finesse where we can take the elements in the extended Haagerup tensor product expansion (that is, from $C^b(\mathbb{G})$ and not $L^\infty(\mathbb{G})$). We note that [23, Corollary 5.6] shows that in the motivating example of $A(G)$, we can even work with $\text{wap}(G)$ and not $C^b(G)$: it’s unclear what the ‘‘quantum’’ analogue of this would be.

We curiously get the following strengthening of [18, Corollary 5.34] (and [18, Remark 5.44]) where it is a hypothesis that there exists $b \in C^b(\mathbb{G})$ with $b \otimes 1 = \sum_i (1 \otimes p_i)\Delta(q_i)$, and the conclusion is that $b = \overline{S}(a)$. To be careful, we now do not identify S with its strict closure.

Corollary 5.10. *Let $a \in C^b(\mathbb{G})$ be such that for some $(p_i) \in MR_I(C_0(\mathbb{G}))$ and $(q_i) \in MC_I(C_0(\mathbb{G}))$, we have that*

$$a \otimes 1 = \sum_i \Delta(p_i)(1 \otimes q_i).$$

Let \bar{S} be the strict closure of S on $C^b(\mathbb{G})$. Then $a \in D(\bar{S})$ and

$$\bar{S}(a) \otimes 1 = \sum_i (1 \otimes p_i) \Delta(q_i).$$

Proof. Let (q_i) and (p_i^*) induce, respectively, α and β in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \ell^2(I))$, so that by applying the adjoint, we see that (β, α) is invariant with respect to a^* . Then (α, β) is invariant say with respect to $b \in C^b(\mathbb{G})$. Thus

$$b \otimes 1 = \sum_i (1 \otimes \beta_i^*) \Delta(\alpha_i) = \sum_i (1 \otimes p_i) \Delta(q_i).$$

By [18, Remark 5.44], or from Theorem 5.9, it follows that $a \in D(\bar{S})$ and $\bar{S}(a) = b$, as required. \square

A slight subtlety here is the following. Suppose that actually $a \in C_0(\mathbb{G})$, so that the above theorem tells us that $a \in D(\bar{S})$. However, this is seemingly not enough to ensure that $a \in D(S)$ (where S is considered as a densely defined operator on $C_0(\mathbb{G})$). Indeed, using that $S = \kappa\tau_{-i/2}$, by [16, Proposition 2.15], we have that $a \in D(S)$ if and only if $S(a) = b \in C_0(\mathbb{G})$ (as κ leaves $C_0(\mathbb{G})$ invariant). It is not clear to us whether this is likely to be true or not.

We could have used this ‘‘coordinate’’ approach to $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ throughout. However, this would have been much harder to motivate from Gilbert’s theorem. Furthermore, in Section 3 above, we used that $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ was a ‘‘slice’’ of $\mathcal{L}(C_0(\mathbb{G}) \otimes K)$. This seemed like a technical tool, but in the next two sections, we shall see how this viewpoint actually appears quite natural and profitable.

6 Links with universal quantum groups

For a locally compact group G , we always have that $B(G)$, the Fourier-Stieltjes algebra of G , embeds into $M_{cb}A(G)$. Furthermore, we can construct the maps α, β in the Gilbert representation by using unitary representations of G .

An analogous result holds for quantum groups. Firstly, we consider the analogue of $B(G)$. Given a locally compact quantum group \mathbb{G} , we can consider the Banach $*$ -algebra $L^1_{\sharp}(\mathbb{G})$, and then take its universal enveloping C^* -algebra, say $C_0^u(\hat{\mathbb{G}})$. In [15], it is shown that $C_0^u(\hat{\mathbb{G}})$ admits a coproduct, left and right invariant weights, and so forth, all of these objects interacting very well with the natural quotient map $\hat{\pi} : C_0^u(\hat{\mathbb{G}}) \rightarrow C_0(\hat{\mathbb{G}})$. Indeed, we call $C_0^u(\hat{\mathbb{G}})$ the *universal quantum group* of $\hat{\mathbb{G}}$, the essential difference with the *reduced quantum group* $C_0(\hat{\mathbb{G}})$ being that the invariant weights are no longer faithful. This is a generalisation of the difference between $C^*(G)$ and $C_r^*(G)$ for a non-amenable locally compact group G . Then $C_0^u(\hat{\mathbb{G}})^*$ becomes a Banach algebra, and $\hat{\pi}^* : M(\hat{\mathbb{G}}) = C_0(\hat{\mathbb{G}})^* \rightarrow C_0^u(\hat{\mathbb{G}})^*$ a homomorphism.

We showed in [6], adapting the argument given in [18, page 914], that $C_0^u(\hat{\mathbb{G}})^*$ embeds into $M_{cb}L^1(\hat{\mathbb{G}})$. To be precise, let $\iota : L^1(\hat{\mathbb{G}}) \rightarrow C_0^u(\hat{\mathbb{G}})^*$ be the natural inclusion, given by composing the map $L^1(\hat{\mathbb{G}}) \rightarrow C_0(\hat{\mathbb{G}})^*$ with $\hat{\pi}^*$. Then [6, Proposition 8.3] shows that $\iota(L^1(\hat{\mathbb{G}}))$ is an ideal in $C_0^u(\hat{\mathbb{G}})^*$ and that the induced map $C_0^u(\hat{\mathbb{G}})^* \rightarrow M_{cb}(L^1(\hat{\mathbb{G}}))$ is an injection.

If $L^1(\hat{\mathbb{G}})$ has a bounded approximate identity (that is, $\hat{\mathbb{G}}$ is *coamenable*) then $M_{cb}(L^1(\hat{\mathbb{G}})) = C_0^u(\hat{\mathbb{G}})^* = M(\hat{\mathbb{G}})$. We remark that we don't know if the converse is true or not. In particular, in the commutative case, for a locally compact group, $L^1(G)$ always has a bounded approximate identity, and so $M_{cb}(L^1(G)) = M(G)$ (which is the classical Wendel's Theorem). The following result thus shows how measures in $M(G)$ arise from invariant pairs in $\mathcal{L}(C_r^*(G), C_r^*(G) \otimes K)$ for a suitable Hilbert space K .

Theorem 6.1. *There exists a Hilbert space K with an involution J_K , and a unitary $\mathcal{U} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$ with the following property. For each $\mu \in C_0^u(\hat{\mathbb{G}})^*$, say giving a multiplier $(L_*, R_*) \in M_{cb}(L^1(\hat{\mathbb{G}}))$, there exist $\xi_0, \eta_0 \in K$ with $\|\xi_0\| \|\eta_0\| = \|\mu\|$, and such that:*

1. *with $\alpha = \mathcal{U}^*(\iota \otimes \xi_0)^*$ and $\beta = \mathcal{U}^*(\iota \otimes \eta_0)^*$, we have that (α, β) is an invariant pair which gives L_* ;*
2. *with $\gamma = \mathcal{U}^*(\iota \otimes J_K \eta_0)^*$ and $\delta = \mathcal{U}^*(\iota \otimes J_K \xi_0)^*$, we have that (γ, δ) is invariant, and gives $\hat{\kappa}_* R_* \hat{\kappa}_*$ (and thus, using Section 5, gives R_*).*

Proof. Let $\theta : C_0^u(\hat{\mathbb{G}}) \rightarrow \mathcal{B}(K)$ be the universal representation. That is, for each state $\mu \in C_0^u(\hat{\mathbb{G}})^*$, let $(H_\mu, \theta_\mu, \xi_\mu)$ be the cyclic GNS construction for μ , and let $K = \bigoplus_\mu H_\mu$ with θ the direct sum representation.

We next find our unitary \mathcal{U} . Let $\lambda_u : L_\#^1(\mathbb{G}) \rightarrow C_0^u(\hat{\mathbb{G}})$ be the natural map. As in the proof of Theorem 4.2, using [15], as the map $L_\#^1(\mathbb{G}) \rightarrow M(\mathcal{B}_0(K)); \omega \mapsto \theta(\lambda_u(\omega))$ is a non-degenerate $*$ -representation, there is a unitary corepresentation $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(K))$ with

$$\theta(\lambda_u(\omega)) = (\omega \otimes \iota)(U) \quad (\omega \in L_\#^1(\mathbb{G})) \quad (\Delta \otimes \iota)(U) = U_{13}U_{23}.$$

Then U induces $\mathcal{U} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$.

Actually, the unitary U is actually given by a ‘‘universal’’ unitary $\hat{U} \in M(C_0(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{G}}))$, by which we mean satisfies $U = (\iota \otimes \theta)(\hat{U})$, see the proof of [15, Corollary 4.3]. Kustermans works on the dual side in [15], but as explained on [15, Page 311], we can use biduality to recover results for $C_0^u(\hat{\mathbb{G}})$. In particular, \hat{U} induces the coproduct in the sense that

$$(\hat{\pi} \otimes \iota)(\sigma(\hat{\Delta}_u(y))) = \hat{U}(\hat{\pi}(y) \otimes 1)\hat{U}^* \quad (y \in C_0^u(\hat{\mathbb{G}})).$$

Define (α, β) as in (i), where we choose ξ_0 and η_0 so that $\omega_{\eta_0, \xi_0} \circ \theta = \mu$. We then have that $\tilde{\alpha}(\xi) = U^*(\xi \otimes \xi_0)$ and $\tilde{\beta}(\xi) = U^*(\xi \otimes \eta_0)$, for $\xi \in L^2(\mathbb{G})$. Then

$$\begin{aligned} (1 \otimes \tilde{\beta}^*)W_{12}^*(1 \otimes \tilde{\alpha})W &= (\iota \otimes \iota \otimes \omega_{\eta_0, \xi_0})(U_{23}W_{12}^*U_{23}^*W_{12}) = (\iota \otimes \iota \otimes \omega_{\eta_0, \xi_0})(U_{23}(\Delta \otimes \iota)(U)^*) \\ &= (\iota \otimes \iota \otimes \omega_{\eta_0, \xi_0})(U_{23}U_{23}^*U_{13}^*) = (\iota \otimes \iota \otimes \omega_{\eta_0, \xi_0})(U_{13}^*) \in C^b(\mathbb{G}) \otimes 1, \end{aligned}$$

as $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(K))$, so the right slice of U is in $M(C_0(\mathbb{G})) = C^b(\mathbb{G})$. Thus (α, β) is an invariant pair, inducing $L'_* \in \mathcal{CB}(L^1(\hat{\mathbb{G}}))$, say.

We wish to show that L'_* is given by left multiplication by μ . Let $\hat{\omega} = \hat{\omega}_{\eta_1, \xi_1} \in L^1(\hat{\mathbb{G}})$, so that $\mu(\hat{\omega}) \in \iota(L^1(\hat{\mathbb{G}}))$. Let $\omega \in L_\#^1(\mathbb{G})$, and set $x = \lambda(\omega) \in C_0(\hat{\mathbb{G}})$. Then $\hat{\pi}(\lambda_u(\omega)) = x$, so

$$\begin{aligned} \langle x, \iota^{-1}(\mu(\hat{\omega})) \rangle &= \langle \mu(\hat{\omega}), \lambda_u(\omega) \rangle = \langle \mu \otimes \iota(\hat{\omega}), \hat{\Delta}_u(\lambda_u(\omega)) \rangle = \langle \iota(\hat{\omega}) \otimes \mu, \sigma \hat{\Delta}_u(\lambda_u(\omega)) \rangle \\ &= \langle \hat{\omega} \otimes \mu, (\hat{\pi} \otimes \iota)(\sigma \hat{\Delta}_u(\lambda_u(\omega))) \rangle = \langle \hat{\omega} \otimes \mu, \hat{U}(\hat{\pi}(\lambda_u(\omega)) \otimes 1)\hat{U}^* \rangle \\ &= \langle \hat{\omega} \otimes \omega_{\eta_0, \xi_0}, U(x \otimes 1)U^* \rangle = (U^*(\eta_1 \otimes \eta_0)|(x \otimes 1)U^*(\xi_1 \otimes \xi_0)) \\ &= (\tilde{\beta}(\eta_1)|(x \otimes 1)\tilde{\alpha}(\xi_1)) = \langle L(x), \hat{\omega} \rangle = \langle x, L_*(\hat{\omega}) \rangle, \end{aligned}$$

as we hoped. By density, this holds for all $x \in L^\infty(\hat{\mathbb{G}})$, so that $L'_* = L_*$ as required to show (i).

We next define J_K . By [15, Proposition 7.2], there is an anti- $*$ -automorphism $\hat{\kappa}_u : C_0^u(\hat{\mathbb{G}}) \rightarrow C_0^u(\hat{\mathbb{G}})$ which ‘‘lifts’’ $\hat{\kappa}$, in the sense that $\hat{\pi}\hat{\kappa}_u = \hat{\kappa}\hat{\pi}$. For each state $\mu \in C_0^u(\hat{\mathbb{G}})^*$, let $\mu' = \hat{\kappa}_u^*(\mu)$, which is still a state, as $\hat{\kappa}_u^*$ is an anti- $*$ -automorphism. On each H_μ , (densely) define J_K by

$$J_K(\theta_\mu(a)\xi_\mu) = \theta_{\mu'}(\hat{\kappa}_u(a^*))\xi_{\mu'} \quad (a \in C_0^u(\hat{\mathbb{G}})).$$

Then, for $a \in C_0^u(\hat{\mathbb{G}})$, we have

$$\|J_K(\theta_\mu(a)\xi_\mu)\|^2 = \langle \mu', \hat{\kappa}_u(a)\hat{\kappa}_u(a^*) \rangle = \langle \hat{\kappa}_u^*(\mu), \hat{\kappa}_u(a^*a) \rangle = \langle \mu, a^*a \rangle = \|\theta_\mu(a)\xi_\mu\|^2.$$

Thus J_K extends by linearity and continuity to all of K . Clearly J_K is an involution. Then, for $a, b \in C_0^u(\hat{\mathbb{G}})$, we have

$$\begin{aligned} J_K\theta(a^*)J_K\theta_\mu(b)\xi_\mu &= J_K\theta_{\mu'}(a^*\hat{\kappa}_u(b^*))\xi_{\mu'} = \theta_\mu(\hat{\kappa}_u(\hat{\kappa}_u(b)a))\xi_\mu \\ &= \theta_\mu(\hat{\kappa}_u(a)b)\xi_\mu = \theta(\hat{\kappa}_u(a))\theta_\mu(b)\xi_\mu. \end{aligned}$$

It follows that $\theta\hat{\kappa}_u(a) = J_K\theta(a^*)J_K$ for each $a \in C_0^u(\hat{\mathbb{G}})$.

Now define (γ, δ) as in (ii), so by the argument just given, (γ, δ) is an invariant pair which induces the left multiplier given by multiplication by $\omega_{J_K\xi_0, J_K\eta_0} \circ \theta \in C_0^u(\hat{\mathbb{G}})^*$. Now, for $x \in C_0^u(\hat{\mathbb{G}})$,

$$\langle \omega_{J_K\xi_0, J_K\eta_0} \circ \theta, x \rangle = (J_K\xi_0|\theta(x)J_K\eta_0) = (\eta_0|J_K\theta(x)^*J_K\xi_0) = (\eta_0|\theta(\hat{\kappa}_u(x))\xi_0) = \langle \mu, \hat{\kappa}_u(x) \rangle.$$

Thus (γ, δ) gives the left multiplier induced by $\hat{\kappa}_u^*(\mu)$. For $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, we have that $\iota(\hat{\omega})\mu = \iota(R_*(\hat{\omega}))$, and so

$$\iota(\hat{\kappa}_*R_*\hat{\kappa}_*(\hat{\omega})) = \hat{\kappa}_u^*\iota(R_*\hat{\kappa}_*(\hat{\omega})) = \hat{\kappa}_u^*(\iota(\hat{\kappa}_*(\hat{\omega}))\mu) = \hat{\kappa}_u^*(\mu)\iota(\hat{\omega}).$$

Thus (γ, δ) gives $\hat{\kappa}_*R_*\hat{\kappa}_*(\hat{\omega})$, showing (ii). \square

Consider further (γ, δ) as in (ii) above. By [15, Proposition 7.2] we have that $(\kappa \otimes \hat{\kappa}_u)(\hat{U}) = \hat{U}$. As $U = (\iota \otimes \theta)(\hat{U})$ and $\theta\hat{\kappa}_u(\cdot) = J_K\theta(\cdot)^*J_K$, we see that

$$U = (\kappa \otimes \theta\hat{\kappa}_u)(\hat{U}) = (J \otimes J_K)U^*(J \otimes J_K).$$

Now, we have that $\tilde{\gamma}(\xi) = U^*(\xi \otimes J_K\eta_0)$ for $\xi \in L^2(\mathbb{G})$. It follows that

$$(J \otimes J_K)\tilde{\gamma}(\xi) = U(J\xi \otimes \eta_0) \quad (\xi \in L^2(\mathbb{G})),$$

and a similar formula holds for $\tilde{\delta}$. Thus $\tilde{\gamma}$ and $\tilde{\delta}$ are given by right slices of U ; however, it is not clear what, if any, meaning we can give to taking a right slice of \mathcal{U} .

7 For two-sided multipliers

In this final section, we look at two-sided multipliers. Firstly, as we saw in Section 2.1, a two-sided multiplier $(L_*, R_*) \in M_{cb}(L^1(\hat{\mathbb{G}}))$ gives rise to represented multipliers, represented by the same $a \in C^b(\mathbb{G})$.

Let $(L_*, R_*) \in M_{cb}(L^1(\hat{\mathbb{G}}))$, and recall the definitions of L_*^\dagger and R_*^\dagger from Section 5.1. For $\hat{\omega}, \hat{\sigma} \in L^1(\hat{\mathbb{G}})$ we have that

$$\hat{\omega}L_*^\dagger(\hat{\sigma}) = (\hat{\omega}^*L_*(\hat{\sigma}^*))^* = (R_*(\hat{\omega}^*)\hat{\sigma}^*)^* = R_*^\dagger(\hat{\omega})\hat{\sigma}.$$

Thus the map $(L_*, R_*) \rightarrow (L_*^\dagger, R_*^\dagger)$ is a conjugate-linear, period two algebra homomorphism from $M_{cb}(L^1(\mathbb{G}))$ to $M_{cb}(L^1(\mathbb{G}))$. This map extends the map $L^1(\hat{\mathbb{G}}) \rightarrow L^1(\hat{\mathbb{G}}); \hat{\omega} \mapsto \hat{\omega}^*$. The following is easy to deduce from Theorem 5.9.

Proposition 7.1. *The homomorphism $\hat{\Lambda} : M_{cb}(L^1(\hat{\mathbb{G}})) \rightarrow C^b(\mathbb{G})$ maps into $D(S^{-1}) = D(S)^*$. Furthermore, for $(L_*, R_*) \in M_{cb}(L^1(\hat{\mathbb{G}}))$, we have that $\hat{\Lambda}(L_*^\dagger, R_*^\dagger) = S(\hat{\Lambda}(L_*, R_*)^*)$.*

Informally, this means that we can “see” the (unbounded) antipode at the level of two-sided multipliers. From the remarks after Theorem 5.9 that the image of $\hat{\lambda}$, and hence certainly the image of $\hat{\Lambda}$, is a strict core for S (as an operator on $C^b(\mathbb{G})$). We remark that in the classical case, when $\mathbb{G} = G$ a locally compact group, then S is bounded, but $M_{cb}A(G)$ need not be norm dense in $C^b(G)$ (but it is of course always strictly dense).

To finish, we make links with Section 6, and show how our consideration of $\mathcal{L}(A, A \otimes K)$ as a “slice” of $\mathcal{L}(A \otimes K)$ is more than a technical tool.

Theorem 7.2. *Let (α, β) be an invariant pair in $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. There exists a contraction $\mathcal{T} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$ and $\xi_0, \eta_0 \in K$ with $\|\xi_0\| = \|\alpha\|$ and $\|\eta_0\| = \|\beta\|$ such that $\alpha = \mathcal{T}(\iota \otimes \xi_0)^*$ and $\beta = \mathcal{T}(\iota \otimes \eta_0)^*$.*

Proof. We shall suppose, by rescaling, that $\|\alpha\| = \|\beta\| \leq 1$. We first show that $\beta^*\alpha = \epsilon 1$ for some $\epsilon \in \mathbb{C}$ with $|\epsilon| \leq 1$. Indeed, let $L_* \in \mathcal{CB}(L^1(\hat{\mathbb{G}}))$ be the left multiplier induced by (α, β) . Then $\beta^*\alpha = \tilde{\beta}^*\tilde{\alpha} = \tilde{\beta}^*(1 \otimes 1)\tilde{\alpha} = L(1)$. Now, for $\hat{\omega}, \hat{\sigma} \in L^1(\hat{\mathbb{G}})$, we have that

$$\langle \Delta(L(1)), \hat{\omega} \otimes \hat{\sigma} \rangle = \langle 1, L_*(\hat{\omega}\hat{\sigma}) \rangle = \langle 1, L_*(\hat{\omega})\hat{\sigma} \rangle = \langle \Delta(1), L_*(\hat{\omega}) \otimes \hat{\sigma} \rangle = \langle L(1) \otimes 1, \hat{\omega} \otimes \hat{\sigma} \rangle.$$

Thus $\Delta(L(1)) = L(1) \otimes 1$. It follows from (the von Neumann version of) [18, Result 5.13] (see also [1, Lemma 4.6]) that $L(1) \in \mathbb{C}1$, as required. As $\|\beta^*\alpha\| \leq 1$, it follows that $|\epsilon| \leq 1$.

Suppose for now that $|\epsilon| < 1$. Let ξ_0 and ξ_1 be orthogonal unit vectors in K . Choose δ with $|\epsilon|^2 + |\delta|^2 = 1$; by our assumption, $\delta \neq 0$. Set $\eta_0 = \bar{\epsilon}\xi_0 + \delta\xi_1$, and define

$$\mathcal{T} = \alpha(\iota \otimes \xi_0) + \delta^{-1}(\beta - \bar{\epsilon}\alpha)(\iota \otimes \xi_1).$$

Then $\mathcal{T}(\iota \otimes \xi_0)^* = \alpha$ and $\mathcal{T}(\iota \otimes \eta_0)^* = \bar{\epsilon}\alpha + \delta\delta^{-1}(\beta - \bar{\epsilon}\alpha) = \beta$, as required. It remains to show that \mathcal{T} is a contraction. It suffices to show that $\|\mathcal{T}(\tau)\| \leq \|\tau\|$ for all $\tau \in A \otimes K$ of the form $T = a \otimes \xi_0 + b \otimes \xi_1$, for some $a, b \in C_0(\mathbb{G})$. Indeed, as the span of ξ_0 and ξ_1 agrees with the span of ξ_0 and η_0 , we may suppose that $\tau = a \otimes \xi_0 + b \otimes \eta_0$. Then $\mathcal{T}(\tau) = \alpha(a) + \beta(b)$, so

$$\begin{aligned} \|\mathcal{T}(\tau)\|^2 &= (\alpha^*\alpha(a)|a) + (\beta^*\alpha(a)|b) + (b|\beta^*\alpha(a)) + (\beta^*\beta(b)|b) \\ &\leq \|a\|^2 + \bar{\epsilon}(a|b) + \epsilon(b|a) + \|b\|^2 \\ &= (a|a) + (\xi_0|\eta_0)(a|b) + (\eta_0|\xi_0)(b|a) + (b|b) \\ &= (a \otimes \xi_0 + b \otimes \eta_0|a \otimes \xi_0 + b \otimes \eta_0) = \|\tau\|^2. \end{aligned}$$

Thus \mathcal{T} is a contraction.

If $|\epsilon| = 1$, then α must be an isometry, for if $\|\alpha(a)\| < \|a\|$ for some $a \in C_0(\mathbb{G})$, then $\|a\| > \|\beta^*\alpha(a)\| = |\epsilon|\|a\|$, a contradiction. Similarly, β is an isometry. It follows that $(\alpha - \epsilon\beta)^*(\alpha - \epsilon\beta) = 0$, showing that $\alpha = \epsilon\beta$. Hence in this case, we can simply set $\eta_0 = \bar{\epsilon}\xi_0$ and $\mathcal{T} = \alpha(\iota \otimes \xi_0)$. \square

If $\alpha = \mathcal{T}(\iota \otimes \xi_0)^*$ and $\beta = \mathcal{T}(\iota \otimes \eta_0)^*$, then the proof of Proposition 3.3 shows that

$$(1 \otimes \beta)^*(\Delta * \alpha) = (\iota \otimes \iota \otimes \omega_{\eta_0, \xi_0})T_{23}^*W_{12}^*T_{23}W_{12}.$$

Hence invariance can be expressed directly at the level of T ; this of course is taking us very far from our analogies with $M_{cb}A(G)$ and Gilbert’s result. Let us finish by looking at two-sided multipliers.

Theorem 7.3. *Let (L_*, R_*) be a completely bounded two-sided multiplier of $L^1(\hat{\mathbb{G}})$. There exists a Hilbert space K with an involution J_K , $\mathcal{T} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$, and $\xi_0, \eta_0 \in K$ such that:*

1. *with $\alpha = \mathcal{T}(\iota \otimes \xi_0)^*$ and $\beta = \mathcal{T}(\iota \otimes \eta_0)^*$, we have that (α, β) is invariant, and induces L_* ;*
2. *with $\gamma = \mathcal{T}(\iota \otimes J_K \eta_0)^*$ and $\delta = \mathcal{T}(\iota \otimes J_K \xi_0)^*$, we have that (γ, δ) is invariant, and induces $\hat{\kappa}_* R_* \hat{\kappa}_*$ (and thus, using Section 5, induces R_*).*

Proof. By rescaling, suppose that $\|(L, R)\|_{cb} = 1$, so that $\|L\|_{cb} \leq 1$ and $\|R\|_{cb} \leq 1$. Apply the previous theorem to an invariant pair which induces L_* to form $T_1 \in \mathcal{L}(C_0(\mathbb{G}) \otimes K_1)$, say, with $\xi_0^{(1)}, \eta_0^{(1)} \in K_1$. Similarly, find $T_2 \in \mathcal{L}(C_0(\mathbb{G}) \otimes K_2)$ and $\xi_0^{(2)}, \eta_0^{(2)} \in K_2$ for $\hat{\kappa}_* R_* \hat{\kappa}_*$. Indeed, looking at the proof of Theorem 7.2, we have that $\xi_0^{(1)}$ and $\xi_1^{(1)}$ are orthogonal unit vectors, and that $\eta_0^{(1)} = \bar{\epsilon}_1 \xi_0^{(1)} + \gamma_1 \xi_1^{(1)}$, where $L(1) = \epsilon_1 1$. We have a similar construction for $\hat{\kappa}_* R_* \hat{\kappa}_*$; in particular, $\epsilon_2 1 = \hat{\kappa} R \hat{\kappa}(1) = R(1)$. Now, that (L_*, R_*) is a two-sided multiplier means that $\hat{\omega} L_*(\hat{\sigma}) = R_*(\hat{\omega}) \hat{\sigma}$ for $\hat{\omega}, \hat{\sigma} \in L^1(\hat{\mathbb{G}})$. Equivalently, $(\iota \otimes L)\hat{\Delta} = (R \otimes \iota)\hat{\Delta}$, and so

$$\epsilon_1 1 \otimes 1 = 1 \otimes L(1) = (\iota \otimes L)\hat{\Delta}(1) = (R \otimes \iota)\hat{\Delta}(1) = R(1) \otimes 1 = \epsilon_2 1 \otimes 1,$$

showing that $\epsilon_1 = \epsilon_2$. Remember that we have a free choice for γ_1 and γ_2 , subject to the condition that $|\gamma_1|^2 = 1 - |\epsilon_1|^2 = 1 - |\epsilon_2|^2 = |\gamma_2|^2$. We shall assume that $\gamma_1 = \bar{\gamma}_2$.

Let $\{\xi_0^{(1)}, \xi_1^{(1)}\} \cup \{e_i\}$ be an orthonormal basis for K_1 , and let $\{\xi_0^{(2)}, \xi_1^{(2)}\} \cup \{f_i\}$ be an orthonormal basis for K_2 . By embedding K_1 or K_2 in a larger Hilbert space, if necessary, we may suppose that $\{e_i\}$ and $\{f_i\}$ are indexed by the same set. Let $K = K_1 \oplus K_2$, and let J_K be the unique involution on K which satisfies

$$J_K(\xi_0^{(1)}) = \eta_0^{(2)}, \quad J_K(\xi_1^{(1)}) = \gamma_1 \xi_0^{(2)} - \epsilon_1 \xi_1^{(2)}, \quad J_K(e_i) = f_i.$$

For this to make sense, we need that for all $a, b, c, d \in \mathbb{C}$, we have

$$\begin{aligned} \bar{a}c + \bar{b}d &= (a\xi_0^{(1)} + b\xi_1^{(1)} | c\xi_0^{(1)} + d\xi_1^{(1)}) = (J_K(c\xi_0^{(1)} + d\xi_1^{(1)}) | J_K(a\xi_0^{(1)} + b\xi_1^{(1)})) \\ &= (\bar{c}\bar{\epsilon}_1 \xi_0^{(2)} + \bar{c}\gamma_2 \xi_1^{(2)} + \bar{d}\gamma_1 \xi_0^{(2)} - \bar{d}e^{it} \epsilon_1 \xi_1^{(2)} | \bar{a}\bar{\epsilon}_1 \xi_0^{(2)} + \bar{a}\gamma_2 \xi_1^{(2)} + \bar{b}\gamma_1 \xi_0^{(2)} - \bar{b}e^{it} \epsilon_1 \xi_1^{(2)}) \\ &= (c\epsilon_1 + d\bar{\gamma}_1)(\bar{a}\bar{\epsilon}_1 + \bar{b}\gamma_1) + (c\bar{\gamma}_2 - de^{-it}\bar{\epsilon}_1)(\bar{a}\gamma_2 - \bar{b}e^{it}\epsilon_1) \\ &= \bar{a}c(|\epsilon_1|^2 + |\gamma_2|^2) + \bar{b}d(|\gamma_1|^2 + |\epsilon_1|^2) + c\bar{b}(\gamma_1 - \bar{\gamma}_2)\epsilon_1 + \bar{a}d(\bar{\gamma}_1 - \gamma_2)\bar{\epsilon}_1. \end{aligned}$$

This holds, as $\gamma_1 = \bar{\gamma}_2$, $\epsilon_1 = \epsilon_2$, and $|\epsilon_1|^2 + |\gamma_1|^2 = 1$. Notice that

$$\begin{aligned} J_K(\eta_0^{(1)}) &= \epsilon_1 J_K(\xi_0^{(1)}) + \bar{\gamma}_1 J_K(\xi_1^{(1)}) = \epsilon_1 \eta_0^{(2)} + \bar{\gamma}_1 (\gamma_1 \xi_0^{(2)} - \epsilon_1 \xi_1^{(2)}) \\ &= \epsilon_1 \bar{\epsilon}_1 \xi_0^{(2)} + \epsilon_1 \bar{\gamma}_1 \xi_0^{(2)} + \bar{\gamma}_1 (\gamma_1 \xi_0^{(2)} - \epsilon_1 \xi_1^{(2)}) = \xi_0^{(2)}. \end{aligned}$$

We have that $C_0(\mathbb{G}) \otimes K = C_0(\mathbb{G}) \otimes K_1 \oplus C_0(\mathbb{G}) \otimes K_2$ for the obvious isomorphism. Let

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_1 & 0 \\ 0 & \mathcal{T}_2 \end{pmatrix} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K).$$

Then, with $\alpha = \mathcal{T}(\iota \otimes \xi_0^{(1)})^* = \mathcal{T}_1(\iota \otimes \xi_0^{(1)})^*$ and $\beta = \mathcal{T}(\iota \otimes \eta_0^{(1)})^* = \mathcal{T}_1(\iota \otimes \xi_0^{(1)})^*$, we have that (α, β) induces L_* . Also, with

$$\gamma = \mathcal{T}(\iota \otimes J_K \eta_0^{(1)})^* = \mathcal{T}(\iota \otimes \xi_0^{(2)})^*, \quad \delta = \mathcal{T}(\iota \otimes J_K \xi_0^{(1)})^* = \mathcal{T}(\iota \otimes \eta_0^{(2)})^*,$$

we have that (γ, δ) induces $\hat{\kappa}_* R_* \hat{\kappa}_*$, as we hoped. \square

While the formulas in the above theorem are nicely symmetric, the proof feels a little like a “trick” (although it is far from being completely artificial, as we do use that L_* and R_* interact as a two-sided multiplier). It is still our belief that there should be a more elegant approach to two-sided multipliers.

In particular, let us finish with a question. Let (α, β) be an invariant pair, leading to a left multiplier L . Can we “see”, at the level of the maps α and β , when there is a right multiplier R making the pair (L, R) a two-sided multiplier?

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