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Isometries between quantum convolution algebras

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Abstract

Given locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, we show that if the convolution algebras $L^1(\mathbb{G}_1)$ and $L^1(\mathbb{G}_2)$ are isometrically isomorphic as algebras, then $\mathbb{G}_1$ is isomorphic either to $\mathbb{G}_2$ or the commutant $\mathbb{G}_2'$. Furthermore, given an isometric algebra isomorphism $\theta : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$, the adjoint is a $*$-isomorphism between $L^\infty(\mathbb{G}_1)$ and either $L^\infty(\mathbb{G}_2)$ or its commutant, composed with a twist given by a member of the intrinsic group of $L^\infty(\mathbb{G}_2)$. This extends known results for Kac algebras (although our proofs are somewhat different) which in turn generalised classical results of Wendel and Walter. We show that the same result holds for isometric algebra homomorphisms between quantum measure algebras (either reduced or universal). We make some remarks about the intrinsic groups of the enveloping von Neumann algebras of C*-algebraic quantum groups.

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Keywords: Locally compact quantum group, isometric isomorphism, intrinsic group.

1 Introduction

Locally compact quantum groups generalise Kac algebras, and form an abstract generalisation of Pontryagin duality. For a locally compact quantum group $\mathbb{G}$, we shall write $L^\infty(\mathbb{G})$ for the von Neumann algebraic quantum group, and $C_0(\mathbb{G})$ for the (reduced) C*-algebraic quantum group. As one can move between these algebras, we tend to view them as representing the same object $\mathbb{G}$. Let $L^1(\mathbb{G})$ be the "quantum convolution algebra", which is the predual of $L^\infty(\mathbb{G})$, made into a Banach algebra by using the coproduct. We can alternatively identify $L^1(\mathbb{G})$ as a certain closed ideal in $C_0(\mathbb{G})^*$. Notice that even in the classical case, where G is even an abelian locally compact group, the algebra $L^1(G)$ does not determine G, as if G is finite, then $L^1(G)$ is isomorphic to $C(\hat{G})$, the continuous functions on the dual group $\hat{G}$, and so $L^1(G)$ is isomorphic to $L^1(H)$ if and only if $\hat{G}$ and $\hat{H}$ are of the same cardinality.

However, Wendel's theorem [25] shows that if we take the norm into account, then $L^1(G)$ completely determines G. To be precise, if $\theta : L^1(G_2) \to L^1(G_1)$ is an *isometric* algebra isomorphism, then there is a character χ on G_1, a positive constant $c > 0$, and a continuous group homomorphism $\alpha : G_1 \to G_2$ such that $\theta(f)(s) = c\chi(s)f(\alpha(s))$ almost everywhere for $s \in G_1$. The constant c simply reflects the fact that the Haar measure is only unique up to a constant. This was generalised to Fourier algebras by Walter, [24]: here notice that $A(G)$ and $A(G^{\rm op})$ are also isometrically isomorphic, where $G^{\rm op}$ is the opposite group to G, and indeed Walter's theorem shows (amongst other things) that $A(G_1)$ and $A(G_2)$ are isometrically isomorphic if and only if G_1 is isomorphic to either G_2 or $G_2^{\rm op}$.

The Kac algebra case was shown by De Cannière, Enock and Schwartz in [5] (see also [6]). The proof in the Kac algebra case uses that the antipode is bounded, which is no longer true in the locally compact quantum group case. We instead use a characterisation of the unitary antipode through the Haar weight (see [14, Proposition 5.20] and Section 3.1 below). The intuitive idea is to show that an isometric algebra isomorphism must intertwine the unitary antipode, although our actual argument is slightly indirect.

Our principle result is that when $\theta : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$ is an isometric algebra isomorphism, then there is u, a member of the intrinsic group of $L^\infty(\mathbb{G}_2)$, such that $x \mapsto \theta^*(x)u$ is either a $*$-isomorphism, or an anti-$*$-isomorphism, from $L^\infty(\mathbb{G}_1)$ to $L^\infty(\mathbb{G}_2)$. We briefly study the intrinsic group, and prove that it coincides with the collection of characters of $L^1(\mathbb{G}_2)$, as we expect from Wendel's Theorem. An anti-$*$-isomorphism to $L^\infty(\mathbb{G}_2)$ can be converted to a $*$-isomorphism to the commutant $L^\infty(\mathbb{G}_2)'$ by composing with $x \mapsto Jx^*J$; the possibility of an anti-$*$-isomorphism occurring can of course already be seen in Walter's Theorem. In particular, if $L^1(\mathbb{G}_1)$ and $L^1(\mathbb{G}_2)$ are isometrically isomorphic, then $\mathbb{G}_1$ is isomorphic to either $\mathbb{G}_2$ or $\mathbb{G}_2'$. We can easily remove the possibility of $\mathbb{G}_2'$ occurring by restricting to completely isometric (or even just completely contractive) isomorphisms between $L^1(\mathbb{G}_1)$ and $L^1(\mathbb{G}_2)$, see Section 3.3.

Having established the result for L^1 algebras, we can prove similar results for quantum measure algebras– for example, for isometric algebra isomorphisms between the dual spaces $C_0(\mathbb{G}_2)^*$ and $C_0(\mathbb{G}_1)^*$. Indeed, we work with some generality, and look at C*-bialgebras (A, Δ) which admit a surjection $\pi : A \to C_0(\mathbb{G})$ which intertwines the coproduct, and such that π^* identifies $L^1(\mathbb{G})$ as an ideal in A^*. This includes the reduced and universal C*-algebraic quantum groups associated with $\mathbb{G}$. As in the Kac algebra case, we use order properties of A^{**} to determine $L^1(\mathbb{G})$ inside A^*. Our characterisation of such isometric isomorphisms involves the intrinsic group of A^{**}, but we show that this is always canonically isomorphic to the intrinsic group of $L^\infty(\mathbb{G})$. We finish to showing how, in some sense, the picture becomes clearer by embedding everything into $L^\infty(\hat{\mathbb{G}})$, and here the interaction between multipliers and the antipode becomes important (compare with [3]).

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2 Locally compact quantum groups

We give a quick overview of the theory of locally compact quantum groups. For readable introductions, see [12] or [21]. Our main reference is [14], which is a self-contained account of the C*-algebraic approach to locally compact quantum groups. We shall however mainly work with von Neumann algebras, for which see [13]. However, this paper is not self-contained, and should be read in conjunction with [14]. Indeed, in a number of places, we shall reference [14], where really we need the obvious von Neumann algebraic version of the required result. See also [16] and [23] for the C*-algebraic and von Neumann algebraic approaches, respectively.

A Hopf-von Neumann algebra is a pair (M, Δ) where M is a von Neumann algebra and $\Delta : M \to M \overline{\otimes} M$ is a unital norm $*$-homomorphism which is coassociative: $(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta$. Then Δ induces a Banach algebra product on the predual M_*. We shall write the product in M_* by juxtaposition, so

$$\langle x, \omega\omega' \rangle = \langle \Delta(x), \omega \otimes \omega' \rangle \qquad (x \in M, \omega, \omega' \in M_*).$$

Similarly, the module actions of M on M_* will be denoted by juxtaposition.

Recall the notion of a normal semi-finite faithful weight φ on M (see [20, Chapter VII] for example). We let

$$\mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}, \quad \mathfrak{m}_\varphi = \operatorname{lin}\{x^*y : x, y \in \mathfrak{n}_\varphi\}, \quad \mathfrak{m}_\varphi^+ = \{x \in M^+ : \varphi(x) < \infty\}.$$

Then $\mathfrak{m}_\varphi$ is a hereditary $*$-subalgebra of M, $\mathfrak{n}_\varphi$ is a left ideal, and $\mathfrak{m}_\varphi^+$ is indeed $M^+ \cap \mathfrak{m}_\varphi$. We can perform the GNS construction for φ, which leads to a Hilbert space H, a dense range map $\Lambda : \mathfrak{n}_\varphi \to H$ and a unital normal $*$-representation $\pi : M \to \mathcal{B}(H)$ with $\pi(x)\Lambda(y) = \Lambda(xy)$. In future, we shall tend to suppress π. Then $\Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is full left Hilbert algebra, and this contains a maximal Tomita algebra (see [20, Section 2, Chapter VI]); denote by $\mathcal{T}_\varphi \subseteq \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ the inverse image under Λ of this maximal Tomita algebra. Tomita-Takesaki theory gives us the modular conjugation J and the modular automorphism group (σ_t). Then $\mathcal{T}_\varphi$ is a $*$-algebra, dense in M for the σ-weak topology, all of whose elements are analytic for (σ_t).

A von Neumann algebraic quantum group is a Hopf-von Neumann algebra (M, Δ) together with faithful normal semifinite weights φ, ψ which are left and right invariant, respectively. This means that

$$\varphi\big((\omega \otimes \iota)\Delta(x)\big) = \varphi(x)\langle 1, \omega\rangle, \quad \psi\big((\iota \otimes \omega)\Delta(y)\big) = \psi(y)\langle 1, \omega\rangle \qquad (\omega \in M_*^+, x \in \mathfrak{m}_\varphi^+, y \in \mathfrak{m}_\psi^+).$$

Using these weights, we can construct an antipode S, which will in general be unbounded. Then S has a decomposition $S = R\tau_{-i/2}$, where R is the unitary antipode, and (τ_t) is the scaling group. The unitary antipode R is a normal anti-$*$-automorphism of M, and $\Delta R = \sigma(R \otimes R)\Delta$, where $\sigma : M\overline{\otimes}M \to M\overline{\otimes}M$ is the tensor swap map. As R is normal, it drops to an isometric linear map $R_* : M_* \to M_*$, which is anti-multiplicative. As usual, we make the canonical choice that $\varphi = \psi \circ R$.

Let H be the GNS space of φ, and let $\Lambda : \mathfrak{n}_\varphi \to H$ be the GNS map. There is a unitary W, the fundamental unitary, acting on $H \otimes H$ (the Hilbert space tensor product) such that $\Delta(x) = W^*(1 \otimes x)W$ for $x \in M$. The left-regular representation of M_* is the map $\lambda : \omega \mapsto (\omega \otimes \iota)(W)$. This is a homomorphism, and the σ-weak closure of $\lambda(M_*)$ is a von Neumann algebra $\hat{M}$. We define a coproduct $\hat{\Delta}$ on $\hat{M}$ by $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, where $\hat{W} = \Sigma W^* \Sigma$ (here $\Sigma : H \otimes H \to H \otimes H$ is the swap map). Then we can find invariant weights to turn $(\hat{M}, \hat{\Lambda})$ into a locally compact quantum group– the dual group to M. We have the biduality theorem that $\hat{\hat{M}} = M$ canonically.

As is becoming common, we shall write $\mathbb{G}$ for the abstract "object" to be thought of as a locally compact quantum group. We then write $L^\infty(\mathbb{G})$ for M, $L^1(\mathbb{G})$ for M_*, and $L^2(\mathbb{G})$ for H. In this paper, we shall often have two quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Then we shall denote by S_i the antipode of $\mathbb{G}_i$, for $i = 1, 2$, and similarly for R_i, ψ_i, and so forth.

There is of course a parallel C*-algebraic theory, but we shall introduce this below in Section 4.

2.1 Isomorphisms of quantum groups

Definition 2.1. *A* quantum group isomorphism *between $\mathbb{G}_1$ and $\mathbb{G}_2$ is a normal $*$-isomorphism $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ which intertwines the coproducts.*

Suppose we have a $*$-isomorphism $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ which intertwines the coproducts. Then, arguing as in [14, Proposition 5.45], θ must intertwine the antipode, the unitary antipode, and the scaling group. As the Haar weights are unique up to a constant, we may actually choose the weights to be intertwined by θ. Hence every object associated to $\mathbb{G}_1$ is transfered to $\mathbb{G}_2$ by θ.

Definition 2.2. *A* quantum group commutant isomorphism *between* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *is a normal anti-$*$-isomorphism* $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ *which intertwines the coproducts.*

The *commutant* von Neumann algebraic quantum group to $\mathbb{G}$ is $\mathbb{G}'$, which has $L^\infty(\mathbb{G}') = L^\infty(\mathbb{G})'$, the commutant of $L^\infty(\mathbb{G})$, and $\Delta'(x) = (J \otimes J)\Delta(JxJ)(J \otimes J)$, for $x \in L^\infty(\mathbb{G})'$. All the other objects (such as W', R', φ') associated to $\mathbb{G}'$ can be related to those of $\mathbb{G}$ using the modular conjugation operator J. See [13, Section 4] for further details. Then, if $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ is a commutant isomorphism, then $\theta'(x) = J\theta(x)^*J$ defines a quantum group isomorphism from $\mathbb{G}_1$ to $\mathbb{G}_2'$; this motivates our choice of terminology. Notice that if $\mathbb{G}_2$ is commutative, then $\mathbb{G}_2' = \mathbb{G}_2$; thus we have avoided the terminology "quantum group anti-isomorphism", as this would be misleading in the motivating commutative situation.

3 Isometries of convolution algebras

Throughout this section, fix two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, and let $T_* : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$ be a linear bijective isometry which is an algebra homomorphism (in short, T_* is an isometric algebra isomorphism).

Then $T = (T_*)^* : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ is a bijective linear isometry between von Neumann algebras. Kadison studied such maps in [8] (see also [6, Section 5.4]) where it is shown that $T(1)$ is a unitary in $L^\infty(\mathbb{G}_2)$ and the map $T_1 : x \mapsto T(x)T(1)^*$ is a Jordan $*$-homomorphism. That is,

$$T_1(x)^* = T_1(x^*), \qquad T_1(xy + yx) = T_1(x)T_1(y) + T_1(y)T_1(x) \qquad (x, y \in L^\infty(\mathbb{G}_1)).$$

In our situation, we can say more about the unitary $T(1)$.

Definition 3.1. *Let* $\mathbb{G} = (M, \Delta)$ *be a Hopf-von Neumann algebra. The* intrinsic group *of* $\mathbb{G}$ *is the collection of unitaries* $u \in M$ *with* $\Delta(u) = u \otimes u$.

Recall that a character on a Banach algebra is a non-zero multiplicative functional. The following is more than we need, but is of independent interest; it generalises [6, Theorem 3.6.10] (which again makes extensive use of a bounded antipode for a Kac algebra). Recall that $M(C_0(\mathbb{G}))$ is the multiplier algebra of $C_0(\mathbb{G})$; for further details see Section 4 below.

Theorem 3.2. *Let* $\mathbb{G} = (M, \Delta)$ *be a Hopf-von Neumann algebra. For* $x \in M$*, the following are equivalent:*

1. *x is a character of the Banach algebra M_*;*
2. *$x \neq 0$ and $\Delta(x) = x \otimes x$.*

If $\mathbb{G}$ *is a locally compact quantum group, then a character* $x \in L^\infty(\mathbb{G})$ *is a unitary, and so automatically* x *is a member of the intrinsic group of* $\mathbb{G}$*. Furthermore,* $x \in M(C_0(\mathbb{G}))$ *and* $x \in D(S)$ *with* $S(x) = x^*$*. The maps*

$$L^1(\mathbb{G}) \to L^1(\mathbb{G}); \omega \mapsto \omega x \ , \ x\omega$$

are isometric automorphisms of the algebra $L^1(\mathbb{G})$.

Proof. The equivalence of (1) and (2) is an easy calculation.

Suppose that $\mathbb{G}$ is a locally compact quantum group, and $x \neq 0$ is such that $\Delta(x) = x \otimes x$. Suppose also that $x \geq 0$; we shall prove that $x = 1$. The von Neumann algebra which x generates is abelian, and so isomorphic to $L^\infty(K)$ for some measure space K. Let $\tilde{x}$ be the image of x in $L^\infty(K)$. We note that as $\|x\| = \|\Delta(x)\| = \|x \otimes x\| = \|x\|^2$, necessarily $\|x\| = 1$.

Let $r \in [0,1]$, and using the Borel functional calculus, let $p = \chi_{[r,1]}(x)$. Thus $\tilde{p}$ is the indicator function of the set $\{k \in K : \tilde{x}(k) \geq r\}$. The von Neumann algebra generated by $x \otimes x$ embeds into $L^\infty(K \times K)$ by sending $x \otimes x$ to $\tilde{x} \otimes \tilde{x}$, which is just the function $(k,l) \mapsto \tilde{x}(k)\tilde{x}(l)$. Then $\chi_{[r,1]}(\tilde{x} \otimes \tilde{x})$ is the indicator function of the set $\{(k,l) \in K \times K : \tilde{x}(k)\tilde{x}(l) \geq r\}$. Thus, if $\chi_{[r,1]}(\tilde{x} \otimes \tilde{x})(k,l) = 1$ then $\tilde{x}(k)\tilde{x}(l) \geq r$ so certainly $\tilde{x}(k) \geq r$ (as $\|\tilde{x}\| = 1$) and so $(\tilde{p} \otimes 1)(k,l) = 1$. It follows that

$$\chi_{[r,1]}(\tilde{x} \otimes \tilde{x}) \leq \tilde{p} \otimes 1.$$

By the homomorphism property of the Borel functional calculus,

$$\Delta(p) = \chi_{[r,1]}(\Delta(x)) = \chi_{[r,1]}(x \otimes x) \leq p \otimes 1.$$

However, we can now appeal to [14, Lemma 6.4] to conclude that $p = 0$ or $p = 1$ (as an aside on notation, $\tilde{A}$ as used in [14] is simply $L^\infty(\mathbb{G})$, see [14, Page 874]). So, we have that $\chi_{[r,1]}(x) = 1$ or 0 for every $r \in [0,1]$. It follows that $x = 1$.

Now let $x \in L^\infty(\mathbb{G})$ be non-zero with $\Delta(x) = x \otimes x$. As Δ is a $*$-homomorphism, it follows that $\Delta(x^*x) = x^*x \otimes x^*x$, and so from the previous paragraph, $x^*x = 1$. Similarly, $xx^* = 1$, so x is a unitary, as required.

Then

$$1 \otimes x = (x^* \otimes 1)\Delta(x), \qquad 1 \otimes x^* = \Delta(x^*)(x \otimes 1),$$

and so from (the von Neumann algebraic analogue of) [14, Proposition 5.33] we conclude that $x \in D(S)$ with $S(x) = x^*$. To show that x is a multiplier of $C_0(\mathbb{G})$, we adapt an idea from [26, Section 4], which in turn is inspired by [1, Page 441]. We have that $W \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G})))$, where $\mathcal{K}(L^2(\mathbb{G}))$ is the compact operators on $L^2(\mathbb{G})$, see [14, Section 3.4] or compare [26, Theorem 1.5]. Then

$$x \otimes 1 = (1 \otimes x^*)\Delta(x) = (1 \otimes x^*)W^*(1 \otimes x)W \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G}))),$$

and so $x \in M(C_0(\mathbb{G}))$ as required.

Finally, for $\omega, \omega' \in L^1(\mathbb{G})$, we see that

$$\langle y, (\omega\omega')x \rangle = \langle (x \otimes x)\Delta(y), \omega \otimes \omega' \rangle = \langle y, (\omega x)(\omega' x) \rangle \qquad (y \in L^\infty(\mathbb{G})),$$

so the map $\omega \mapsto \omega x$ is an algebra homomorphism, with inverse $\omega \mapsto \omega x^*$. The case of $\omega \mapsto x\omega$ is analogous. ☐

We remark that similar results to the above theorem have been obtained independently by Neufang and Kalantar, see Kalantar's thesis, [9, Theorem 3.2.11] and [10, Theorem 3.9].

We hence see that if $T_* : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$ is an isometric algebra isomorphism, then so is $T_{1,*} : \omega \mapsto T_*(T(1)^*\omega)$. For the rest of this section, we shall just assume that actually $T(1) = 1$.

Let $p \in L^\infty(\mathbb{G}_2)$ be a central projection, and let T_p be the map $x \mapsto T(x)p$. As in [6, Section 5.4], we define

$$\begin{aligned} \mathcal{P}_h &= \big\{p \text{ a central projection in } L^\infty(\mathbb{G}_2) \text{ with } T_p \text{ an algebra homomorphism}\big\}, \\ \mathcal{P}_a &= \big\{p \text{ a central projection in } L^\infty(\mathbb{G}_2) \text{ with } T_p \text{ an algebra anti-homomorphism }\big\}. \end{aligned}$$

Then [6, Lemma 5.4.5] shows that both $\mathcal{P}_a$ and $\mathcal{P}_h$ have greatest elements, say s_a and s_h. From [19, Theorem 3.3], there is some $p \in \mathcal{P}_a$ with $1 - p \in \mathcal{P}_h$, and so $s_a + s_h \geq 1$.

The following results are also shown in [6], but we give sketch proofs to verify that the results still hold for locally compact quantum groups.

Lemma 3.3. *Let $x \in L^\infty(\mathbb{G})$ be a central projection with $\Delta(x) \geq x \otimes x$ and $R(x) = x$. Then $W(x \otimes x) = (x \otimes x)W$ and $\Delta(x)(x \otimes 1) = \Delta(x)(1 \otimes x) = x \otimes x$.*

Proof. We have that $x \otimes x = (x \otimes x)\Delta(x) = (x \otimes x)W^*(1 \otimes x)W$, and so $(x \otimes x)W^*(1 \otimes x) = (x \otimes x)W^*$. Now we use that $(\hat{J} \otimes J)W(\hat{J} \otimes J) = W^*$, see [13, Corollary 2.2]. Thus

$$x \otimes x = (x \otimes x)(\hat{J} \otimes J)W(\hat{J} \otimes J)(1 \otimes x)(\hat{J} \otimes J)W^*(\hat{J} \otimes J),$$

but $JxJ = x^* = x$ as x is central and self-adjoint, and $\hat{J}x\hat{J} = R(x^*) = R(x) = x$ by assumption. So $x \otimes x = (x \otimes x)W(1 \otimes x)W^*$. Taking adjoints gives $x \otimes x = W(1 \otimes x)W^*(x \otimes x)$. As $W^* \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\hat{\mathbb{G}})$, we see that $W^*(x \otimes x) = (x \otimes 1)W^*(1 \otimes x)$, and so, from above,

$$x \otimes x = W(x \otimes x)W^*(1 \otimes x) = W(x \otimes x)W^*.$$

Thus $W(x \otimes x) = (x \otimes x)W$.

Then, arguing similarly, $\Delta(x)(x \otimes 1) = W^*(1 \otimes x)W(x \otimes 1) = W^*(x \otimes x)W = x \otimes x$. The case of $\Delta(x)(1 \otimes x)$ follows by applying the result to $\mathbb{G}^{\mathrm{op}}$ (see [13, Section 4]). $\square$

Corollary 3.4. *Let $p, q \in L^\infty(\mathbb{G})$ be central projections with $\Delta(p) \geq p \otimes p$ and $\Delta(q) \geq q \otimes q$, with $R(p) = p$ and $R(q) = q$, and with $p + q \geq 1$. Then $p = 1$ or $q = 1$.*

Proof. By the lemma, $\Delta(p)((1-p) \otimes p) = \Delta(p)(1 \otimes p) - p \otimes p = 0$, and $\Delta(q)(q \otimes (1-q)) = 0$. As $1 - q \leq p$ and $1 - p \leq q$, it follows that $\Delta(p)((1-p) \otimes (1-q)) = 0$ and $\Delta(q)((1-p) \otimes (1-q)) = 0$. As $\Delta(p) + \Delta(q) \geq 1$, it follows that $(1-p) \otimes (1-q) = 0$, so $p = 1$ or $q = 1$. $\square$

Proposition 3.5. *Form S_a and S_h as above. Then:*

1. *$(T_{S_h} \otimes T_{S_h})\Delta_1(x) = \Delta_2(T(x))(S_h \otimes S_h)$ for $x \in L^\infty(\mathbb{G}_1)$;*
2. *$(T_{S_a} \otimes T_{S_a})\Delta_1(x) = \Delta_2(T(x))(S_a \otimes S_a)$ for $x \in L^\infty(\mathbb{G}_1)$;*
3. *$\Delta_2(S_h) \geq S_h \otimes S_h$;*
4. *$\Delta_2(S_a) \geq S_a \otimes S_a$.*

Proof. We prove claims for S_a; the proofs for S_h are easier. The preadjoint of T_{S_a} is the map $\omega \mapsto T_*(S_a\omega)$. Firstly, let $\omega, \omega' \in L^1(\mathbb{G}_2)$, and calculate

$$\langle (T_{S_a} \otimes T_{S_a})\Delta_1(x), \omega \otimes \omega' \rangle = \langle x, T_*(S_a\omega)T_*(S_a\omega') \rangle = \langle \Delta_2(T(x)), S_a\omega \otimes S_a\omega' \rangle,$$

which shows (2).

As S_a is central, we see that $S_a \otimes S_a \in L^\infty(\mathbb{G}_2)'\overline{\otimes}L^\infty(\mathbb{G}_2)' \subseteq \Delta_2(L^\infty(\mathbb{G}_2))'$. Let $q \in L^\infty(\mathbb{G}_2)$ be such that $\Delta_2(q)$ is the central support of $S_a \otimes S_a$ (so q is the smallest central projection with $\Delta_2(q)(S_a \otimes S_a) = S_a \otimes S_a$). Then

$$\Phi : \Delta_2(L^\infty(\mathbb{G}_2))(S_a \otimes S_a) \to \Delta_2(L^\infty(\mathbb{G}_2)q); \quad \Delta_2(x)(S_a \otimes S_a) \mapsto \Delta_2(xq) = \Delta_2(x)\Delta_2(q),$$

is readily seen to be an isomorphism. Then, for $x \in L^\infty(\mathbb{G}_1)$,

$$\Delta_2(T_q(x)) = \Delta_2(T(x)q) = \Phi\big(\Delta_2(T(x))(S_a \otimes S_a)\big) = \Phi\big((T_{S_a} \otimes T_{S_a})\Delta_1(x)\big).$$

So $x \mapsto \Delta_2(T_q(x))$ is anti-multiplicative, and so $q \in \mathcal{P}_a$. Thus $q \leq S_a$, and so $\Delta_2(S_a) \geq \Delta(q) \geq S_a \otimes S_a$ as required. $\square$

At this point, we can no longer follow [6]. We would like to show that $TR_1 = R_2T$ (that is, T' as defined in the next proposition, is the identity map) but we have to proceed somewhat indirectly.

Proposition 3.6. *Suppose that the map $T' = T^{-1}R_2TR_1 : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_1)$ is a homomorphism. Then T is either a $*$-homomorphism or an anti-$*$-homomorphism.*

Proof. As the unitary antipode R_2 is an anti-$*$-homomorphism, it is easy to see that $R_2(S_h)$ is a central projection. For $x \in L^\infty(\mathbb{G}_1)$,

$$T_{R_2(S_h)}(x) = T(x)R_2(S_h) = R_2\big(R_2(T(x))S_h\big) = R_2\big(T(T'(R_1(x)))S_h\big).$$

As $y \mapsto T(y)S_h$ is a homomorphism, it follows that $T_{R_2(S_h)}$ is a homomorphism, and so $R_2(S_h) \leq S_h$. As R_2 preserves the order, also $S_h \leq R_2(S_h)$, so $S_h = R_2(S_h)$.

A similar argument establishes that $R(S_a) = S_a$. So, combining the previous proposition and corollary, we conclude that either $S_h = 1$, in which case T is a $*$-homomorphism, or $S_h = 0$, so $S_a = 1$, and T is an anti-$*$-homomorphism. $\square$

We are henceforth motivated to study the map $T' = T^{-1}R_2TR_1$. Notice that this map is normal, and the preadjoint T'_* is an isometric algebra isomorphism from $L^1(\mathbb{G}_2)$ to itself.

3.1 Characterising the unitary antipode

We now study the unitary antipode more closely. For us, an important characterisation of R is the following, given in [14, Proposition 5.20]:

$$R\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) = (\psi \otimes \iota)\big(\Delta(\sigma^\psi_{-i/2}(a^*))(\sigma^\psi_{-i/2}(b) \otimes 1)\big),$$

where $a, b \in \mathcal{T}_\psi$. (We shall shortly explain further exactly what this formula means). We are hence motivated to look at the right Haar weights, and how they interact with T. We shall then split $L^\infty(\mathbb{G}_1)$ into a direct summand, with T acting as a homomorphism in the first component, and as an anti-homomorphism in the second. Then R_1 and R_2 will interact well with T on these components, but less well on the cross-terms. However, this "bad interaction" will cancel out if we consider T'^2, for T' as defined above.

Lemma 3.7. *The map $L^\infty(\mathbb{G}_1)^+ \to [0, \infty]; x \mapsto \psi_2(T(x))$ is a right-invariant, normal semi-finite faithful weight on $L^\infty(\mathbb{G}_1)$, and is hence proportional to ψ_1.*

Proof. As T is a Jordan homomorphism, it restricts to an order isomorphism $L^\infty(\mathbb{G}_1)^+ \to L^\infty(\mathbb{G}_2)^+$. Thus we can define $\psi = \psi_2 \circ T : L^\infty(\mathbb{G}_1)^+ \to [0, \infty]$, and it follows that ψ is a faithful weight, and $\mathfrak{m}^+_\psi = T^{-1}(\mathfrak{m}^+_{\psi_2})$. Thus also $\mathfrak{m}_\psi = T^{-1}(\mathfrak{m}_{\psi_2})$. As T is σ-weakly continuous, it is now routine to establish that ψ is semi-finite, and normal (as T is an order isomorphism on the positive cones).

It remains to check that ψ is right-invariant. For $\omega \in L^1(\mathbb{G}_1)^+$ and $y \in \mathfrak{m}^+_\psi$, a simple calculation shows that $T((\iota \otimes \omega)\Delta_1(y)) = (\iota \otimes T_*^{-1}(\omega))\Delta_2(T(y))$. As $T_*^{-1}(\omega) \geq 0$ and $T(y) \in \mathfrak{m}^+_{\psi_2}$, it follows that

$$\psi\big((\iota \otimes \omega)\Delta(y)\big) = \psi_2\big((\iota \otimes T_*^{-1}(\omega))\Delta_2(T(y))\big) = \langle 1, T_*^{-1}(\omega)\rangle \psi_2(T(y)).$$

As T is unital, this shows that ψ is right-invariant. $\square$

Henceforth, we shall actually assume that that $\psi_1 = \psi_2 \circ T$.

Henceforth, using [19, Theorem 3.3], we fix a central projection $p \in L^\infty(\mathbb{G}_2)$ such that T_p is a homomorphism, and T_{1-p} is an anti-homomorphism. Note that we cannot necessarily assume that $p = S_a$ and $1-p = S_h$. Let $q = T^{-1}(p)$.

Lemma 3.8. *With p, q as above, we have that q is a central projection in $L^\infty(\mathbb{G}_1)$. Then $L^\infty(\mathbb{G}_1)$ decomposes as $qL^\infty(\mathbb{G}_1) \oplus (1-q)L^\infty(\mathbb{G}_1)$, $L^\infty(\mathbb{G}_2)$ decomposes as $pL^\infty(\mathbb{G}_2) \oplus (1-p)L^\infty(\mathbb{G}_2)$, and under these identifications, T decomposes as $T_p \oplus T_{1-p}$.*

Proof. Let $x \in L^\infty(\mathbb{G}_1)$. Then $T(xq)p = T_p(xq) = T_p(x)T_p(q) = T(x)pT(q)p = T(x)p = T_p(x)$, and similarly $T_p(qx) = T_p(x)$, and $T_{1-p}(qx) = T_{1-p}(xq) = 0$. Thus

$$T(xq - qx) = T_p(xq - qx) + T_{1-p}(xq - qx) = T_p(xq) - T_p(qx) = T_p(x) - T_p(x) = 0.$$

So q is central; it is easily seen to be a projection. The remaining claims now follow by simple calculation. □

This lemma means that, for example, given $a \in qL^\infty(\mathbb{G}_1)$ and $x \in L^\infty(\mathbb{G}_1)$,

$$T(ax) = T\big(axq + ax(1-q)\big) = T_p(a)T_p(x) = T_p(a)T(x) = T(a)T_p(x) = T(a)T(x).$$

Thus we understand T quite well; what is unclear is how T interacts with the unitary antipodes R_1 and R_2.

We can then restrict $\psi_1 = \psi_2 \circ T$ to $qL^\infty(\mathbb{G}_1)$ and to $(1-q)L^\infty(\mathbb{G}_2)$, say giving ψ_1^q and ψ_1^{1-q}. As T_p is a $*$-homomorphism, it is clear that T_p gives a bijection from $\mathfrak{n}_{\psi_1^q}$ to $\mathfrak{n}_{\psi_2^p}$. As T_{1-p} is an anti-$*$-homomorphism, we have that $x \in \mathfrak{n}_{\psi_1^{1-q}}$ if and only if $T(x^*) = T(x)^* \in \mathfrak{n}_{\psi_2^{1-p}}$. To ease notation for the modular automorphism groups, for $t \in \mathbb{R}$, we shall let $\sigma_t^{2,p} = \sigma_t^{\psi_2^p}$ and $\sigma_t^{2,1-p} = \sigma_t^{\psi_2^{1-p}}$, and similarly for ψ_1.

Lemma 3.9. *The map T intertwines the modular automorphism groups in the following ways:*

$$T_p \circ \sigma_t^{1,q} = \sigma_t^{2,p} \circ T_p, \qquad T_{1-p} \circ \sigma_t^{1,1-q} = \sigma_{-t}^{2,1-p} \circ T_{1-p} \qquad (t \in \mathbb{R}).$$

Proof. As T_p is a $*$-isomorphism between $L^\infty(\mathbb{G}_1)q$ and $L^\infty(\mathbb{G}_2)p$, it is standard that it intertwines the modular automorphism group, compare [20, Corollary 1.4, Chapter VIII]. As T_{1-p} is an anti-$*$-isomorphism, a variant of the standard argument will show that we get the sign change $t \mapsto -t$. □

As in [12, Section 6] (see also the C*-algebraic version in [14, Section 1.5]) we let

$$\mathfrak{m}_{\psi_1\otimes\iota}^+ = \big\{x \in (L^\infty(\mathbb{G}_1)\overline{\otimes}L^\infty(\mathbb{G}_1))^+ : (\iota\otimes\omega)(x) \in \mathfrak{m}_{\psi_1}^+ \ (\omega \in L^1(\mathbb{G}_1)_+\big\}.$$

Then $\mathfrak{m}_{\psi_1\otimes\iota}^+$ is a hereditary cone in $(L^\infty(\mathbb{G}_1)\overline{\otimes}L^\infty(\mathbb{G}_1))^+$. Let $\mathfrak{m}_{\psi_1\otimes\iota}$ be the $*$-subalgebra generated by $\mathfrak{m}_{\psi_1\otimes\iota}^+$; this agrees with the linear span of $\mathfrak{m}_{\psi_1\otimes\iota}^+$. There is a linear map

$$(\psi_1\otimes\iota) : \mathfrak{m}_{\psi_1\otimes\iota} \to L^\infty(\mathbb{G}_1) \quad \text{with} \quad \langle(\psi_1\otimes\iota)(x), \omega\rangle = \psi_1\big((\iota\otimes\omega)x\big).$$

We then set

$$\mathfrak{n}_{\psi_1\otimes\iota} = \big\{x \in L^\infty(\mathbb{G}_1)\overline{\otimes}L^\infty(\mathbb{G}_1) : x^*x \in \mathfrak{m}_{\psi_1\otimes\iota}^+\big\}.$$

This is a left ideal in $L^\infty(\mathbb{G}_1)\overline{\otimes}L^\infty(\mathbb{G}_1)$, and $\mathfrak{m}_{\psi_1\otimes\iota}$ is the linear span of $\mathfrak{n}_{\psi_1\otimes\iota}^*\mathfrak{n}_{\psi_1\otimes\iota}$.

As ψ_1 is right-invariant, a simple calculation shows that for $a, b \in \mathfrak{n}_{\psi_1}$, we have that $\Delta(b) \in \mathfrak{n}_{\psi_1\otimes\iota}$ and that $a \otimes 1 \in \mathfrak{n}_{\psi_1\otimes\iota}$. Thus $(a^*\otimes 1)\Delta(b) \in \mathfrak{m}_{\psi_1\otimes\iota}$, and similarly $\Delta(a^*)(b\otimes 1) \in \mathfrak{m}_{\psi_1\otimes\iota}$.

In particular, for $a, b \in \mathcal{T}_{\psi_1}$, we can make sense of the formula

$$R_1\big((\psi_1\otimes\iota)((a\otimes 1)\Delta_1(b))\big) = (\psi_1\otimes\iota)\big(\Delta_1(\sigma_{-i/2}^{\psi_1}(a))(\sigma_{-i/2}^{\psi_1}(b)\otimes 1)\big).$$

Lemma 3.10. *Let $a \in \mathcal{T}_{\psi_1}$ and $b \in \mathcal{T}_{\psi_1} q$. Then*

$$T\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_2 \otimes \iota)\big((T(b) \otimes 1)\Delta_2(T(a))\big),$$
$$T\big((\psi_1 \otimes \iota)(\Delta_1(a)(b \otimes 1))\big) = (\psi_2 \otimes \iota)\big(\Delta_2(T(a))(T(b) \otimes 1)\big)$$

Proof. Let $\omega, \omega' \in L^1(\mathbb{G}_2)$. Then, for $x \in L^\infty(\mathbb{G}_1)$,

$$\langle x, T_*(\omega')b\rangle = \langle T(bx), \omega'\rangle = \langle T(b)T(x), \omega'\rangle = \langle x, T_*(\omega' T(b))\rangle,$$

using that $b \in L^\infty(\mathbb{G}_1)q$. Thus

$$\langle T\big((\iota \otimes T_*(\omega))((b \otimes 1)\Delta_1(a))\big), \omega'\rangle = \langle (b \otimes 1)\Delta_1(a), T_*(\omega') \otimes T_*(\omega)\rangle$$
$$= \langle \Delta_1(a), T_*(\omega' T(b)) \otimes T_*(\omega)\rangle = \langle T(a), (\omega' T(b))\omega\rangle = \langle (T(b) \otimes 1)\Delta_2(T(a)), \omega' \otimes \omega\rangle.$$

Hence finally

$$\langle T\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big), \omega\rangle = \psi_1\big((\iota \otimes T_*(\omega))((b \otimes 1)\Delta_1(a))\big)$$
$$= \psi_2\big((\iota \otimes \omega)(T(b) \otimes 1)\Delta_2(T(a))\big) = \langle (\psi_2 \otimes \iota)\big((T(b) \otimes 1)\Delta_2(T(a))\big), \omega\rangle,$$

as required.

Now, as ψ_2 is a weight, we have that $\psi_2(x^*) = \overline{\psi_2(x)}$ for $x \in \mathfrak{m}_{\psi_2}$. We can also verify that $(\iota \otimes \omega)(x^*) = (\iota \otimes \omega^*)(x)^*$ for $x \in \mathfrak{m}_{\psi_2 \otimes \iota}$ and $\omega \in L^1(\mathbb{G}_2)^+$. It follows that $(\psi_2 \otimes \iota)(x^*) = (\psi_2 \otimes \iota)(x)^*$. As $\mathcal{T}_{\psi_1}$ is a $*$-algebra, and T respects the involution, applying this calculation to a^* and b^* and then taking the adjoint yields the second claimed equality. □

Lemma 3.11. *Let $a \in \mathcal{T}_{\psi_1}$ and $b \in \mathcal{T}_{\psi_1}(1-q)$. Then*

$$T\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_2 \otimes \iota)\big(\Delta_2(T(a))(T(b) \otimes 1)\big),$$
$$T\big((\psi_1 \otimes \iota)(\Delta_1(a)(b \otimes 1))\big) = (\psi_2 \otimes \iota)\big((T(b) \otimes 1)\Delta_2(T(a))\big).$$

Proof. As in the previous proof, but now using that $b \in L^\infty(\mathbb{G}_1)(1-q)$, we check that for $\omega, \omega' \in L^1(\mathbb{G}_2)$, we have that $T_*(\omega')b = T_*\big(T(b)\omega'\big)$, which leads to

$$T\big((\iota \otimes T_*(\omega))((b \otimes 1)\Delta_1(a))\big) = (\iota \otimes \omega)\big(\Delta_2(T(a))(T(b) \otimes 1)\big),$$

which gives the first result. The second equality now follows by taking adjoints. □

Proposition 3.12. *As before, let $T' = T^{-1}R_2TR_1$. If $a, b \in \mathcal{T}_{\psi_1}q$ or $a, b \in \mathcal{T}_{\psi_1}(1-q)$, we have that*

$$T'\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a)).$$

Proof. Suppose that $a, b \in \mathcal{T}_{\psi_1}(1-q)$, the other case being analogous. We have that

$$TR_1\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = T\big((\psi_1 \otimes \iota)\big(\Delta_1(\sigma^{1,1-q}_{-i/2}(b))(\sigma^{1,1-q}_{-i/2}(a) \otimes 1)\big)\big)$$
$$= (\psi_2 \otimes \iota)\big((T_{1-p}\sigma^{1,1-q}_{-i/2}(a) \otimes 1)\Delta_2(T_{1-p}\sigma^{1,1-q}_{-i/2}(b))\big)$$
$$= (\psi_2 \otimes \iota)\big((\sigma^{2,1-p}_{i/2}T_{1-p}(a) \otimes 1)\Delta_2(\sigma^{2,1-p}_{i/2}T_{1-p}(b))\big)$$

using first Lemma 3.11 (applied to $\sigma^{1,1-q}_{-i/2}(a) \in \mathcal{T}_{\psi_1}(1-q)$) and then Lemma 3.9.

Thus, taking adjoints gives that

$$TR_1\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_2 \otimes \iota)\big(\Delta_2(\sigma^{2,1-p}_{-i/2}T_{1-p}(b^*))(\sigma^{2,1-p}_{-i/2}T_{1-p}(a^*) \otimes 1)\big)^*$$
$$= R_2\big((\psi_2 \otimes \iota)\big((T_{1-p}(b^*) \otimes 1)\Delta_2(T_{1-p}(a^*))\big)^*\big)$$
$$= R_2T\big((\psi_1 \otimes \iota)\big(\Delta_1(a^*)(b^* \otimes 1)\big)^*\big) = R_2T\big((\psi_1 \otimes \iota)\big((b \otimes 1)\Delta_1(a)\big)\big),$$

as required. □

Proposition 3.13. *As before, let $T' = T^{-1}R_2TR_1$. If $a \in \mathcal{T}_{\psi_1}(1-q)$ and $b \in \mathcal{T}_{\psi_1}q$, or vice versa, we have that*

$$T'\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_1 \otimes \iota)(\Delta_1(a)(\sigma_{-i}^{\psi_1}(b) \otimes 1)),$$
$$T'\big((\psi_1 \otimes \iota)(\Delta_1(a)(b \otimes 1))\big) = (\psi_1 \otimes \iota)((\sigma_i^{\psi_1}(b) \otimes 1)\Delta_1(a)).$$

Proof. Suppose that $a \in \mathcal{T}_{\psi_1}(1-q)$ and $b \in \mathcal{T}_{\psi_1}q$, so we can follow the previous proof through to get that

$$TR_1\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_2 \otimes \iota)\big((\sigma_{i/2}^{2,1-p}T_{1-p}(a) \otimes 1)\Delta_2(\sigma_{-i/2}^{2,p}T_p(b))\big),$$

where here we remember that $b \in \mathcal{T}_{\psi_1}q$. Thus

$$\begin{aligned}TR_1\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) &= (\psi_2 \otimes \iota)\big(\Delta_2(\sigma_{i/2}^{2,p}T_p(b^*))(\sigma_{-i/2}^{2,1-p}T_{1-p}(a^*) \otimes \iota)\big)^* \\ &= R_2\big((\psi_2 \otimes \iota)\big((\sigma_i^{2,p}(T_p(b^*)) \otimes 1)\Delta_2(T_{1-p}(a^*))\big)^*\big) \\ &= R_2T\big((\psi_1 \otimes \iota)\big((\sigma_i^{1,q}(b^*) \otimes 1)\Delta_1(a^*)\big)^*\big) = R_2T\big((\psi_1 \otimes \iota)\big(\Delta_1(a)(\sigma_{-i}^{\psi_1}(b) \otimes 1)\big)\big),\end{aligned}$$

as required, using Lemma 3.10. The case when $a \in \mathcal{T}_{\psi_1}q$ and $b \in \mathcal{T}_{\psi_1}(1-q)$ follows similarly. Again, taking adjoints (and remembering that $\sigma_i^{\psi_1}(b)^* = \sigma_{-i}^{\psi_1}(b^*)$ gives the second claimed equality). ☐

Corollary 3.14. *We have that $T'^2 = \iota$.*

Proof. By density, it is enough to verify these identities on elements of the form $(\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))$ for $a, b \in \mathcal{T}_{\psi_1}$. By linearity, we may suppose that $a, b \in \mathcal{T}_{\psi_1}q$ or $a, b \in \mathcal{T}_{\psi_1}(1-q)$, in which case the result follows from Proposition 3.12, or that $a \in \mathcal{T}_{\psi_1}q, b \in \mathcal{T}_{\psi_1}(1-q)$ or vice versa, in which case the result follows from Proposition 3.13. ☐

Finally, we wish to show that T' commutes with the scaling group (τ_t). For this, recall from [14, Proposition 6.8] that $\Delta_1\sigma_t^{\psi_1} = (\sigma_t^{\psi_1} \otimes \tau_{-t})\Delta_1$.

Proposition 3.15. *We have that $\tau_tT' = T'\tau_t$ for each $t \in \mathbb{R}$.*

Proof. Let $a \in \mathcal{T}_{\psi_1}(1-q)$ and $b \in \mathcal{T}_{\psi_1}q$. Then, from Proposition 3.13,

$$\tau_tT'\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = \tau_t(\psi_1 \otimes \iota)(\Delta_1(a)(\sigma_{-i}^{\psi_1}(b) \otimes 1)).$$

Now, for $\omega, \omega' \in L^1(\mathbb{G}_1)$,

$$\begin{aligned}\langle(\iota \otimes \omega \circ \tau_t)(\Delta_1(a)(\sigma_{-i}^{\psi_1}(b) \otimes 1)), \omega'\rangle &= \langle(\iota \otimes \tau_t)\Delta_1(a), \sigma_{-i}^{\psi_1}(b)\omega' \otimes \omega\rangle \\ &= \langle(\sigma_t^{\psi_1} \otimes \iota)\Delta_1(\sigma_{-t}^{\psi_1}(a)), \sigma_{-i}^{\psi_1}(b)\omega' \otimes \omega\rangle = \langle(\sigma_t^{\psi_1} \otimes \iota)\big(\Delta_1(\sigma_{-t}^{\psi_1}(a))(\sigma_{-t-i}^{\psi_1}(b) \otimes 1)\big), \omega' \otimes \omega\rangle \\ &= \langle\sigma_t^{\psi_1}\big((\iota \otimes \omega)\big(\Delta_1(\sigma_{-t}^{\psi_1}(a))(\sigma_{-t-i}^{\psi_1}(b) \otimes 1)\big)\big), \omega'\rangle\end{aligned}$$

Thus also

$$\begin{aligned}\langle\tau_t(\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a)), \omega\rangle &= \psi_1\big((\iota \otimes \omega \circ \tau_t)(\Delta_1(a)(\sigma_{-i}^{\psi_1}(b) \otimes 1))\big) \\ &= \psi_1\big(\sigma_t^{\psi_1}\big((\iota \otimes \omega)\big(\Delta_1(\sigma_{-t}^{\psi_1}(a))(\sigma_{-t-i}^{\psi_1}(b) \otimes 1)\big)\big)\big) \\ &= \langle(\psi_1 \otimes \iota)\big(\Delta_1(\sigma_{-t}^{\psi_1}(a))(\sigma_{-t-i}^{\psi_1}(b) \otimes 1)\big), \omega\rangle,\end{aligned}$$

and so we conclude that

$$\tau_t T'\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = (\psi_1 \otimes \iota)\big(\Delta_1(\sigma^{\psi_1}_{-t}(a))(\sigma^{\psi_1}_{-t-i}(b) \otimes 1)\big).$$

Similarly, we find that

$$\begin{aligned}&\langle(\iota \otimes \omega \circ \tau_t)((b \otimes 1)\Delta_1(a)), \omega'\rangle = \langle(\iota \otimes \tau_t)\Delta_1(a), \omega' b \otimes \omega\rangle \\ &\quad = \langle(\sigma^{\psi_1}_t \otimes \iota)\Delta_1(\sigma^{\psi_1}_{-t}(a)), \omega' b \otimes \omega\rangle = \langle(\sigma^{\psi_1}_t \otimes \iota)\big((\sigma^{\psi_1}_{-t}(b) \otimes 1)\Delta_1(\sigma^{\psi_1}_{-t}(a))\big), \omega' \otimes \omega\rangle.\end{aligned}$$

So arguing similarly,

$$\begin{aligned}&T'\tau_t\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big) = T'\big((\psi_1 \otimes \iota)((\sigma^{\psi_1}_{-t}(b) \otimes 1)\Delta_1(\sigma^{\psi_1}_{-t}(a)))\big) \\ &\quad = (\psi_1 \otimes \iota)\big(\Delta_1(\sigma^{\psi_1}_{-t}(a))(\sigma^{\psi_1}_{-t-i}(b) \otimes 1)\big) = \tau_t T'\big((\psi_1 \otimes \iota)((b \otimes 1)\Delta_1(a))\big).\end{aligned}$$

The same argument works if $a \in \mathcal{T}_{\psi_1} q$ and $b \in \mathcal{T}_{\psi_1}(1-q)$. Similarly, by using Proposition 3.12, a similar calculation works for $a, b \in \mathcal{T}_{\psi_1} q$ or $a, b \in \mathcal{T}_{\psi_1}(1-q)$. By linearity and density, the result follows. □

3.2 The main result

We are now in a position to state and prove our main result.

Theorem 3.16. *Let $T_* : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$ be an isometric algebra isomorphism. Then $u = T(1) \in L^\infty(\mathbb{G}_2)$ is a member of the intrinsic group, and there is a quantum group isomorphism, or quantum group commutant isomorphism, $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ such that*

$$T_*(\omega) = \theta_*(u\omega) \qquad (\omega \in L^\infty(\mathbb{G}_2)).$$

In particular, $\mathbb{G}_1$ is isomorphic to either $\mathbb{G}_2$ or $\mathbb{G}_2'$.

Proof. Suppose that the result holds (with $u = 1$) when $T = (T_*)^*$ is unital. Then we apply this to T_1 to find that

$$T_*(T(1)^*\omega) = T_{1,*}(\omega) = \theta_*(\omega) \qquad (\omega \in L^1(\mathbb{G}_2)),$$

from which the general case follows.

So, we may suppose that T is unital. We wish to prove that T is either a $*$-homomorphism, or an anti-$*$-homomorphism. Form $T' = T^{-1}R_2 T R_1$. By Corollary 3.14, $T'^2 = \iota$, so $R_1 T' R_1 = R_1 T^{-1} R_2 T = T'^{-1} = T'$; thus T' commutes with R_1.

Now, $T'(1) = 1$ and T'_* is an isometric algebra isomorphism. By Proposition 3.6, as $T'^{-1}R_1 T' R_1 = \iota$, it follows that T' is either a $*$-homomorphism, or an anti-$*$-homomorphism. If T' is a $*$-homomorphism, then Proposition 3.6 now shows that T itself is either a $*$-homomorphism or an anti-$*$-homomorphism, as required.

If we reverse the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$, and work with T^{-1}, then the same arguments show that $(T^{-1})' = TR_1T^{-1}R_2$ is either a $*$-homomorphism or an anti-$*$-homomorphism. If it is a $*$-homomorphism, then T^{-1} (and so T) is either a $*$-homomorphism or an anti-$*$-homomorphism, as required.

So, the remaining case is when both T' and $(T^{-1})'$ are anti-$*$-homomorphisms (and, to avoid special cases, by this we mean that T' and $(T^{-1})'$ are not also $*$-homomorphisms). Then we can consider the map $\Phi : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_1') = L^\infty(\mathbb{G}_1)'; x \mapsto JT'(x)^*J$, which will be a $*$-isomorphism which intertwines the coproducts. Thus Φ will also intertwine the antipode, the unitary antipode, and in particular the scaling group. The scaling group of $L^\infty(\mathbb{G}_1')$ is $\tau_t'(x) = J\tau_{-t}(JxJ)J$, see [13, Section 4]. So, for $x \in L^\infty(\mathbb{G}_1)$,

$$JT'(\tau_t(x))^*J = \Phi(\tau_t(x)) = \tau_t'(\Phi(x)) = J\tau_{-t}(T'(x)^*)J.$$

Thus $T'\tau_t = \tau_{-t}T'$. However, Proposition 3.15 shows that $T'\tau_t = \tau_t T'$; as T' bijects, it follows that $\tau_t = \iota$ for all t.

So the scaling group of $\mathbb{G}_1$ is trivial; arguing with $(T^{-1})'$ in place of T' shows that the same is true of $\mathbb{G}_2$. This does not quite show that $\mathbb{G}_1$ and $\mathbb{G}_2$ are Kac algebras (see [22, Page 7]) but it does give us enough that we can now easily follow the proof in the Kac algebra case, see [6, Section 5.5]. Indeed, let

$$X = (T_p \otimes \iota)(W_1) + (T_{1-p}R_1 \otimes \iota)(W_1^*) \in L^\infty(\mathbb{G}_2)\overline{\otimes}L^\infty(\hat{\mathbb{G}}_1),$$

where W_1 is the fundamental unitary for $\mathbb{G}_1$. This makes sense, as both T_p and $T_{1-p}R_1$ are $*$-homomorphisms.

Then X is unitary. This follows, as for $x, y \in L^\infty(\mathbb{G}_1)$, $T_p(x)T_{1-p}(y) = T(x)T(y)p(1-p) = 0$. Thus

$$\begin{aligned} X^*X &= \big((T_p \otimes \iota)(W_1^*) + (T_{1-p}R_1 \otimes \iota)(W_1)\big)\big((T_p \otimes \iota)(W_1) + (T_{1-p}R_1 \otimes \iota)(W_1^*)\big) \\ &= (T_p \otimes \iota)(1) + (T_{1-p}R_1 \otimes \iota)(1) = 1, \end{aligned}$$

and similarly $XX^* = 1$.

As the scaling group is trivial, the familiar formula for the antipode, [14, Proposition 8.3] or [13, Page 79], becomes

$$R_1\big((\iota \otimes \omega)(W_1)\big) = (\iota \otimes \omega)(W_1^*) \qquad (\omega \in L^1(\hat{\mathbb{G}}_1)).$$

Thus, for $\omega \in L^1(\mathbb{G}_2)$ and $\omega' \in L^1(\hat{\mathbb{G}}_1)$, as $R_1^2 = \iota$,

$$\begin{aligned} \langle(\omega \otimes \iota)(X), \omega'\rangle &= \langle T_p((\iota \otimes \omega')W_1), \omega\rangle + \langle T_{1-p}R_1((\iota \otimes \omega')(W_1^*)), \omega\rangle \\ &= \langle T_p((\iota \otimes \omega')W_1), \omega\rangle + \langle T_{1-p}((\iota \otimes \omega')(W_1)), \omega\rangle \\ &= \langle W_1, T_*(\omega) \otimes \omega'\rangle = \langle \lambda_1(T_*(\omega)), \omega'\rangle. \end{aligned}$$

So the map $L^1(\mathbb{G}_2) \to C_0(\hat{\mathbb{G}}_1); \omega \mapsto \lambda_1(T_*(\omega))$ is a homomorphism, and now a simple calculation shows that $(\Delta_2 \otimes \iota)(X) = X_{13}X_{23}$.

Again, as $S_2 = R_2$, we can turn $L^1(\mathbb{G}_2)$ into a Banach $*$-algebra for the involution

$$\langle x, \omega^\sharp\rangle = \overline{\langle R_2(x)^*, \omega\rangle} \qquad (x \in L^\infty(\mathbb{G}_2), \omega \in L^1(\mathbb{G}_2)),$$

compare with Section 4 below. As X is unitary, [11, Proposition 5.2] shows that $\lambda_1 T_*$ is a $*$-homomorphism. Hence T_* is a $*$-homomorphism. So, for $x \in L^\infty(\mathbb{G}_1)$ and $\omega \in L^1(\mathbb{G}_2)$,

$$\begin{aligned} \langle x, T_*(\omega)^\sharp\rangle &= \overline{\langle R_1(x)^*, T_*(\omega)\rangle} = \overline{\langle T(R_1(x))^*, \omega\rangle} \\ &= \langle T(x), \omega^\sharp\rangle = \overline{\langle R_2(T(x))^*, \omega\rangle}, \end{aligned}$$

showing that $R_2T = TR_1$. Thus $T' = \iota$, so T' is a $*$-homomorphisms, a contradiction, completing the proof. ∎

We remark that a corollary of the proof is that, actually, $T' = \iota$ all along!

3.3 Completely isometric homomorphisms

It is increasingly common in non-abelian harmonic analysis to study objects in the category of operator spaces and completely bounded maps; see for example the survey [17]. It is well-known that the transpose mapping is the canonical example of an isometric, but not completely isometric, linear mapping. So we might suspect that a complete isometry cannot give rise to a anti-$*$-homomorphism, and this is indeed the case– this is well-known, but we include a sketch proof for completeness.

Theorem 3.17. *Let A and B be unital C*-algebras, and let $T : A \to B$ be completely isometric bijection. Then $T(1)$ is a unitary, and the map $A \to B; a \mapsto T(a)T(1)^*$ is a $*$-homomorphism.*

Proof. By Kadison, $T(1)$ is unitary, and $S : a \mapsto T(a)T(1)^*$ is a unital, completely isometric bijection. We can now follow [2, Section 1.3] to conclude that S is a $*$-homomorphism, as required. Indeed, S is unital and completely contractive, and so is completely positive. Then the Stinespring construction allows us to prove the Kadison-Schwarz inequality: $S(a)^*S(a) \leq S(a^*a)$. Applying this to S^{-1} as well, and using polarisation, yields the result. ☐

Theorem 3.18. *Let $T_* : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$ be a completely isometric algebra isomorphism. Then $u = T(1) \in L^\infty(\mathbb{G}_2)$ is a member of the intrinsic group, and there is a quantum group isomorphism $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ such that*

$$T_*(\omega) = \theta_*(u\omega) \qquad (\omega \in L^\infty(\mathbb{G}_2)).$$

In particular, $\mathbb{G}_1$ is isomorphic to $\mathbb{G}_2$.

Proof. The previous result shows that $\theta(x) = T(x)u^*$ defines a $*$-homomorphism, and so the result is immediate. ☐

We remark that if $T_* : L^1(\mathbb{G}_2) \to L^1(\mathbb{G}_1)$ is isometric, and completely contractive, then T_* is induced by a quantum group isomorphism as above. Indeed, we only need to rule out the possibility that $\theta : x \mapsto T(x)T(1)^*$ is an anti-$*$-isomorphism. As θ is still a complete contraction, the Kadison-Schwarz inequality would yield that $\theta(aa^*) = \theta(a)^*\theta(a) \leq \theta(a^*a)$, so applying θ^{-1} (which is an order-isomorphism) gives $aa^* \leq a^*a$, a contradiction (unless $L^\infty(\mathbb{G}_1)$ is commutative, in which case θ is a homomorphism, as required).

4 Isometries between measure algebras

In this section, we extend our results to isometric algebra isomorphisms between quantum measure algebras. We thus start with a survey of the C*-algebraic theory of locally compact quantum groups.

A *morphism* between two C*-algebras A and B is a non-degenerate $*$-homomorphism $\phi : A \to M(B)$ from A to the multiplier algebra of B. That ϕ is non-degenerate is equivalent to ϕ extending to a unital, strictly continuous $*$-homomorphism $M(A) \to M(B)$. Thus morphisms can be composed; for further details see [16, Appendix A].

Given $\mathbb{G}$ and its fundamental unitary W, the space $\{(\iota \otimes \omega)(W) : \omega \in \mathcal{B}(H)_*\}$ is an algebra σ-weakly dense in $L^\infty(\mathbb{G})$. However, the norm closure turns out to be a C*-algebra, which we shall denote by $C_0(\mathbb{G})$. Then Δ restricts to give a morphism $C_0(\mathbb{G}) \to M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$, and R, τ_t

and so forth all restrict to $C_0(\mathbb{G})$. It is possible to define locally compact quantum groups purely at the C*-algebra level, although the necessary weight theory is more complicated; see [14].

As for $L^1(\mathbb{G})$, we use Δ to turn $C_0(\mathbb{G})^* = M(\mathbb{G})$ into a Banach algebra. In the commutative case, this is the measure algebra of a group, which justifies the notation. As $C_0(\mathbb{G})$ is σ-weakly dense in $L^\infty(\mathbb{G})$, the embedding of $L^1(\mathbb{G})$ into $M(\mathbb{G})$ is an isometry; clearly it is an algebra homomorphism, and actually $L^1(\mathbb{G})$ becomes an ideal in $M(\mathbb{G})$, see [14, Page 914].

Actually, we work in a little generality, and introduce the following (non-standard) terminology.

Definition 4.1. *A* quantum group above $C_0(\mathbb{G})$ *is a triple* (A, Δ_A, π) *where* A *is a* C*-*algebra,* $\Delta_A : A \to M(A \otimes A)$ *is a morphism, coassociative in the sense that* $(\iota \otimes \Delta_A)\Delta_A = (\Delta_A \otimes \iota)\Delta_A$, *and* $\pi : A \to C_0(\mathbb{G})$ *is a surjective* $*$*-homomorphism with* $\Delta\pi = (\pi \otimes \pi)\Delta_A$. *Then* $\pi^* : M(\mathbb{G}) \to A^*$ *is an algebra homomorphism, and we make the further requirement that* $\pi^*(L^1(\mathbb{G}))$ *is an essential ideal in* A^*. *Here* essential *means that if* $\pi^*(\omega)\mu = 0$ *for all* $\omega \in L^1(\mathbb{G})$, *then* $\mu = 0$, *and similarly with the orders reversed.*

For example, $C_0(\mathbb{G})$ itself is a quantum group above $C_0(\mathbb{G})$. In the cocommutative case, $C_0(\mathbb{G}) = C_r^*(G)$ the reduced group C*-algebra of a locally compact group G, and so $M(\mathbb{G}) = B_r(G)$, the reduced Fourier-Stieltjes algebra. We could alternatively study the full group C*-algebra $C^*(G)$, whose dual is $B(G)$ the Fourier-Stieltjes algebra. Then $C^*(G)$ is a quantum group above $C_r^*(G)$. It turns out that this example can be generalised to the quantum setting.

We follow [11]. Let $\mathbb{G}$ be a locally compact quantum group, and let $L^1_\sharp(\hat{\mathbb{G}})$ be the collection of $\omega \in L^1(\hat{\mathbb{G}})$ such that there is $w^\sharp \in L^1(\hat{\mathbb{G}})$ with

$$\langle x, \omega^\sharp \rangle = \overline{\langle S(x)^*, \omega \rangle} \qquad (x \in D(\hat{S})).$$

Then $L^1_\sharp(\hat{\mathbb{G}})$ is a $*$-algebra for the involution $\sharp$. Let $C_u(\mathbb{G})$ be the universal enveloping C^*-algebra of $L^1_\sharp(\hat{\mathbb{G}})$, and let $\hat{\lambda}_u : L^1_\sharp(\hat{\mathbb{G}}) \to C_u(\mathbb{G})$ be the canonical homomorphism. Then $C_u(\mathbb{G})$ becomes a "quantum group" which is very similar to $C_0(\mathbb{G})$, the essential difference being that the left and right invariant weights are no longer faithful. For us, the important features are:

- There is a non-degenerate $*$-homomorphism $\Delta_u : C_u(\mathbb{G}) \to M(C_u(\mathbb{G}) \otimes C_u(\mathbb{G}))$ which is coassociative;
- There is a surjective $*$-homomorphism $\pi : C_u(\mathbb{G}) \to C_0(\mathbb{G})$ with $\Delta\pi = (\pi \otimes \pi)\Delta_u$.

We note here that there are many differences in notation between [11] and that for Kac algebras used in [6]. We shall continue to follow [11]. It is shown in [4, Section 8] that $L^1(\mathbb{G})$ is an essential ideal in $C_u(\mathbb{G})^*$, and so $C_u(\mathbb{G})$ is a quantum group above $C_0(\mathbb{G})$.

In [15] examples of discrete groups G are given so that there is a compact quantum group (A, Δ_A) which "sits between" $C_r^*(G)$ and $C^*(G)$, in the sense that we have proper quotient maps $C^*(G) \to A \to C_r^*(G)$ which intertwine the coproducts. Then the inclusion maps $B_r(G) \to A^* \to B(G)$ are isometric algebra homomorphisms. As the Fourier algebra $A(G)$ is an essential ideal in $B(G)$, it follows that A is a quantum group above $C_r^*(G)$. Indeed, this argument would work for any quantum group sitting between $C_0(\mathbb{G})$ and $C_u(\mathbb{G})$ (but to our knowledge, [15] gives the first example of this phenomena).

4.1 Quantum group isomorphisms revisited

Let $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$ be a quantum group isomorphism. Assuming we have normalised the Haar weights, θ will induce an isomorphism between the Hilbert spaces which intertwines the fundamental unitaries. Thus θ will restrict to give a $*$-isomorphism $C_0(\mathbb{G}_1) \to C_0(\mathbb{G}_2)$.

Similarly, θ induces a quantum group isomorphism $\hat{\theta} : L^\infty(\hat{\mathbb{G}}_2) \to L^\infty(\hat{\mathbb{G}}_1)$ which satisfies $\hat{\theta}\lambda_2 = \lambda_1\theta_*$. Then $\hat{\theta}_*$ will restrict to give a $*$-isomorphism between $L^1_\sharp(\mathbb{G}_1)$ and $L^1_\sharp(\mathbb{G}_2)$. This will induce a $*$-isomorphism $\theta_u : C_u(\mathbb{G}_1) \to C_u(\mathbb{G}_2)$ which intertwines the coproducts, and satisfies $\pi_2\theta_u = \theta\pi_1$.

For a quantum group commutant isomorphism θ we simply compose θ with the map $x \mapsto Jx^*J$ to get a quantum group isomorphism from $\mathbb{G}_1$ to $\mathbb{G}_2'$. In [13, Section 4] it is shown that $(\mathbb{G}_2')\hat{} = \hat{\mathbb{G}}_2^{\mathrm{op}}$, where $L^\infty(\hat{\mathbb{G}}_2^{\mathrm{op}}) = L^\infty(\hat{\mathbb{G}}_2)$ with the opposite coproduct $\hat{\Delta}^{\mathrm{op}} = \sigma\hat{\Delta}$. Hence $L^1_\sharp(\hat{\mathbb{G}}_2^{\mathrm{op}})$ agrees with $L^1_\sharp(\hat{\mathbb{G}}_2)$, but with the reversed product, and similarly $C_u(\mathbb{G}_2')$ is canonically equal to the opposite C*-algebra to $C_u(\mathbb{G}_2)$, but has the same coproduct. Thus, for example, θ lifts to an anti-$*$-isomorphism $\theta_u : C_u(\mathbb{G}_1) \to C_u(\mathbb{G}_2)$ which intertwines the coproduct (somewhat as we might hope).

4.2 Normal extensions

Let B be a C*-algebra non-degenerately represented on a Hilbert space H. Let $M = B''$ be the von Neumann algebra generated by B. We can identify the multiplier algebra of B with

$$M(B) = \{x \in M : xa, ax \in B \ (a \in B)\}.$$

Let A be a C*-algebra, and consider the enveloping C^*-algebra A^{**}. Let $\phi : A \to M(B)$ be a morphism. By the universal property of A^{**}, there is a unique normal $*$-homomorphism $\tilde{\phi} : A^{**} \to B''$ extending ϕ. As ϕ is non-degenerate, $\tilde{\phi}$ is unital. In the special case when $B'' = B^{**}$ (say with $B \subseteq \mathcal{B}(H)$ the universal representation) the extension $\tilde{\phi}$ is nothing but the second adjoint ϕ^{**}.

Now let (A, Δ_A, π) be a quantum group above $C_0(\mathbb{G})$, and let $A \subseteq \mathcal{B}(H)$ be the universal representation, so that both $A \otimes A$ (the spacial C*-tensor product) and $A^{**}\overline{\otimes}A^{**}$ are subalgebras of $\mathcal{B}(H \otimes H)$. We can hence form the extension $\tilde{\Delta}_A : A^{**} \to A^{**}\overline{\otimes}A^{**}$. Notice that then $(A^{**}, \tilde{\Delta}_A)$ becomes a Hopf-von Neumann algebra.

Similarly, we form $\tilde{\pi} : A^{**} \to L^\infty(\mathbb{G})$. The preadjoint of this map is simply the embedding $\tilde{\pi}_* : L^1(\mathbb{G}) \to A^*$, which is the composition of the isometry $L^1(\mathbb{G}) \to C_0(\mathbb{G})^*$ with the isometry $\pi^* : C_0(\mathbb{G})^* \to A^*$. Let $\operatorname{supp}\tilde{\pi}$ be the support projection of $\tilde{\pi}$, so $\operatorname{supp}\tilde{\pi} \in A^{**}$ is the unique central projection with, for $x \in A^{**}$, $x \operatorname{supp}\tilde{\pi} = 0$ if and only if $\tilde{\pi}(x) = 0$. Then

$$\tilde{\pi}_*(L^1(\mathbb{G}))^\perp = \{x \in A^{**} : \langle x, \tilde{\pi}_*(\omega)\rangle = 0 \ (\omega \in L^1(\mathbb{G}))\} = \ker\tilde{\pi} = (1 - \operatorname{supp}\tilde{\pi})A^{**}.$$

It follows that

$$\begin{aligned}\tilde{\pi}_*(L^1(\mathbb{G})) &= \{\mu \in A^* : \langle x, \mu\rangle = 0 \ (x \in (1 - \operatorname{supp}\tilde{\pi})A^{**})\} \\ &= \{\mu \in A^* : (1 - \operatorname{supp}\tilde{\pi})\mu = 0\} = (\operatorname{supp}\tilde{\pi})A^*.\end{aligned}$$

Temporarily, let Δ_0 be the coproduct on $C_0(\mathbb{G})$, and let Δ_∞ be the coproduct on $L^\infty(\mathbb{G})$. Identifying $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ with a subalgebra of $L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, we see that Δ_∞ extends Δ_0. It is easy to verify that as $(\pi \otimes \pi)\Delta_A = \Delta_0\pi$, also $(\tilde{\pi} \otimes \tilde{\pi})\tilde{\Delta}_A = \Delta_\infty\tilde{\pi}$. We shall use this, and similar relations, without comment in the next section.

We remark that we could work with a more general notion of a quantum group above $C_0(\mathbb{G})$. Indeed, suppose that (A, Δ_A) is a C*-bialgebra, and that $\pi : A \to L^\infty(\mathbb{G})$ is a $*$-homomorphism with σ-weak dense range. Then $\pi \otimes \pi$ is a $*$-homomorphism $A \otimes A \to L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G}) \subseteq L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, and so, by taking a normal extension, we have a $*$-homomorphism $\pi : M(A \otimes A) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$. We can thus make sense of the requirement that $(\pi \otimes \pi)\Delta_A = \Delta\pi$. Then π^* restricted to $L^1(\mathbb{G})$ gives a homomorphism $L^1(\mathbb{G}) \to A^*$ (which is an isometry, as π has σ-weakly dense range, and using Kaplansky Density). We again insist that $\pi^*(L^1(\mathbb{G}))$ is an essential ideal in A^*. A careful examination of the following proofs show that they would all work in this more general setting; but in the absence of any examples, we do not make this a formal definition.

4.3 Isometries of duals of quantum groups

For $i = 1, 2$ let $(A_i, \Delta_{A_i}, \pi_i)$ be a quantum group above $C_0(\mathbb{G}_i)$. Let $T_* : A_2^* \to A_1^*$ be an isometric algebra isomorphism, and set $T = (T_*)^* : A_1^{**} \to A_2^{**}$. The following is now proved in an entirely analogous way to the arguments in Section 3.

Lemma 4.2. *$T(1)$ is a unitary element of A_2^{**} which is a member of the intrinsic group. The map $T_{1*} : A_1^{**} \to A_2^{**}; \omega \mapsto T_*(T(1)^*\omega)$ is an isometric algebra isomorphism.*

Again, we find that $T_1 = (T_{1*})^*$ is a Jordan homomorphism. We now show (in a similar, but more general, fashion to the arguments in [6, Section 5.6]) a link between $\tilde{\pi}$ and the order properties of A^{**}, for (A, Δ_A, π) a quantum group above $C_0(\mathbb{G})$.

Proposition 4.3. *Let $\mathbb{G}$ be a locally compact quantum group, and let (A, Δ_A) a quantum group above $C_0(\mathbb{G})$. Let*

$$\mathcal{Q} = \{Q \in A^{**} : Q \text{ is a projection}, Q \neq 1, \tilde{\Delta}_A(Q) \leq Q \otimes Q\}.$$

Then $\mathcal{Q}$ has a maximal element, which is $1 - \operatorname{supp} \tilde{\pi}$.

Proof. Let $e = 1 - \operatorname{supp} \tilde{\pi}$, so as in Section 4.2 above, $eA^{**} = \ker \tilde{\pi}$ and $\tilde{\pi}_*(L^1(\mathbb{G})) = (1 - e)A^*$. Let $\mu, \mu' \in A^*$, so there are $\omega, \omega' \in L^1(\mathbb{G})$ with $\tilde{\pi}_*(\omega) = (1 - e)\mu$ and $\tilde{\pi}^*(\omega') = (1 - e)\mu'$. Then

$$\begin{aligned}
\langle \tilde{\Delta}_A(e)(e \otimes e), \mu \otimes \mu' \rangle &= \langle e, (e\mu)(e\mu') \rangle = \langle e, (\mu - \tilde{\pi}_*(\omega))(\mu' - \tilde{\pi}_*(\omega')) \rangle \\
&= \langle e, \mu\mu' \rangle + \langle e, \tilde{\pi}_*(\omega\omega') - \mu\tilde{\pi}_*(\omega') - \tilde{\pi}_*(\omega)\mu' \rangle \\
&= \langle e, \mu\mu' \rangle = \langle \tilde{\Delta}_A(e), \mu \otimes \mu' \rangle,
\end{aligned}$$

as $\tilde{\pi}_*(L^1(\mathbb{G}))$ is an ideal in A^*. It follows that $\tilde{\Delta}_A(e)(e \otimes e) = \tilde{\Delta}_A(e)$, and so $\tilde{\Delta}_A(e) \leq e \otimes e$. Thus $e \in \mathcal{Q}$.

Now let $Q \in \mathcal{Q}$, so that $\Delta\tilde{\pi}(Q) = (\Delta\pi)\tilde{}(Q) = ((\pi \otimes \pi)\Delta_u)\tilde{}(Q) = (\tilde{\pi} \otimes \tilde{\pi})\tilde{\Delta}_u(Q) \leq \tilde{\pi}(Q) \otimes \tilde{\pi}(Q) \leq 1 \otimes \tilde{\pi}(Q)$. By [14, Lemma 6.4], this can only occur when $\tilde{\pi}(Q) = 0$ or 1.

If $\tilde{\pi}(Q) = 1$, then $Q \geq \operatorname{supp} \tilde{\pi}$, and so $Q + e \geq 1$. Thus $\tilde{\Delta}_u(Q) + \tilde{\Delta}_u(e) \geq 1 \otimes 1$, but as $Q, e \in \mathcal{Q}$, it follows that

$$1 \otimes 1 \leq Q \otimes Q + e \otimes e.$$

Thus also

$$(1 - Q) \otimes (1 - e) \leq \big((1 - Q) \otimes (1 - e)\big)\big(Q \otimes Q + e \otimes e\big)\big((1 - Q) \otimes (1 - e)\big) = 0,$$

and so $Q = 1$ or $e = 1$, a contradiction. Thus $\tilde{\pi}(Q) = 0$, showing that $Q \leq e$ as required. $\square$

Proposition 4.4. *With T_*, T, T_1 as above, we have that:*

1. *$T_1(1 - \operatorname{supp} \tilde{\pi}_1) = 1 - \operatorname{supp} \tilde{\pi}_2$.*
2. *$T(\ker \tilde{\pi}_1) = \ker \tilde{\pi}_2$.*

3. $T_*(\tilde\pi_{2,*}(L^1(\mathbb{G}_2))) = \tilde\pi_{1,*}(L^1(\mathbb{G}_1))$.

Proof. For $i=1,2$, form $\mathcal{Q}_i$ for A_i as in Proposition 4.3. We claim that T_1 gives a bijection $\mathcal{Q}_1$ to $\mathcal{Q}_2$. Let $Q\in\mathcal{Q}_1$, so as T_1 is Jordan homomorphism, $T_1(Q)$ is a projection which is not equal to 1 (as $T_1(1)=1$ and T_1 bijects). For $\mu,\mu'\in A_2^*$ positive, we see that

$$\begin{aligned}\langle \tilde\Delta_{A_2}(T_1(Q)), \mu\otimes\mu'\rangle &= \langle \Delta_{A_1}(Q), T_{1*}(\mu)\otimes T_{1*}(\mu')\rangle \\ &\leq \langle Q\otimes Q, T_{1*}(\mu)\otimes T_{1*}(\mu')\rangle = \langle T_1(Q)\otimes T_1(Q), \mu\otimes\mu'\rangle.\end{aligned}$$

It follows that $\tilde\Delta_{A_2}(T_1(Q))\leq T_1(Q)\otimes T_1(Q)$, and so $T_1(\mathcal{Q}_1)\subseteq\mathcal{Q}_2$. Applying the same argument to T_1^{-1} yields that $T_1(\mathcal{Q}_1)\supseteq\mathcal{Q}_2$, giving the claim. As T_1 preserves the order, and $1-\operatorname{supp}\tilde\pi_i$ is the maximal element of $\mathcal{Q}_i$, it follows that $T_1(1-\operatorname{supp}\tilde\pi_1) = 1-\operatorname{supp}\tilde\pi_2$ showing (1).

For $i=1,2$, we know that $x\in\ker\tilde\pi_i$ if and only if $x\operatorname{supp}\tilde\pi_i=0$. For $x\in A_1^{**}$, as T_1 is a Jordan homomorphism, we see that

$$\begin{aligned}2T_1(x)\operatorname{supp}\tilde\pi_2 &= (\operatorname{supp}\tilde\pi_2)T_1(x) + T_1(x)\operatorname{supp}\tilde\pi_2 = T_1(\operatorname{supp}\tilde\pi_1)T_1(x) + T_1(x)T_1(\operatorname{supp}\tilde\pi_1) \\ &= T_1((\operatorname{supp}\tilde\pi_2)x + x\operatorname{supp}\tilde\pi_2) = 2T_1(x\operatorname{supp}\tilde\pi_2),\end{aligned}$$

using (1). Thus $T_1(\ker\tilde\pi_1)=\ker\tilde\pi_2$. As $\ker\tilde\pi_1$ is an ideal, and $T(1)$ is unitary, it follows that $\ker\tilde\pi_1 T(1) = \ker\tilde\pi_1$, and so $T(\ker\tilde\pi_1) = T_1(\ker\tilde\pi_1 T(1)) = \ker\tilde\pi_2$ showing (2).

As in Section 4.2 above, we have that $\tilde\pi_{i,*}(L^1(\mathbb{G}_i)) = {}^\perp(\ker\tilde\pi_i)$, for $i=1,2$. Hence (3) follows immediately from (2). ☐

For $i=1,2$ we have that $L^1(\mathbb{G}_i)\subseteq A_i^*$ isometrically, and so the restriction of T yields an isometric algebra homomorphism $T_r : L^1(\mathbb{G}_2)\to L^1(\mathbb{G}_1)$. We have already characterised such maps, and we next bootstrap this to determine the structure of T_* on all of A_2^*. For the moment, we restrict attention to the cases when $A_i = C_0(\mathbb{G}_i)$ for $i=1,2$, or $A_i = C_u(\mathbb{G}_i)$, for $i=1,2$. In the next section we use quantum group duality to say something about the general case.

Given a quantum group (commutant) isomorphism $\theta : L^\infty(\mathbb{G}_1)\to L^\infty(\mathbb{G}_2)$, we recall from Section 4.1 that θ restricts to a (anti-) $*$-isomorphism $\theta : C_0(\mathbb{G}_1)\to C_0(\mathbb{G}_2)$, and lifts to a (anti-) $*$-isomorphism $\theta_u : C_u(\mathbb{G}_1)\to C_u(\mathbb{G}_2)$. In the following, we call such a map "associated".

Theorem 4.5. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be locally compact quantum groups. Suppose that either $A_1 = C_0(\mathbb{G}_1), A_2 = C_0(\mathbb{G}_2)$, or $A_1 = C_u(\mathbb{G}_1)$, $A_2 = C_u(\mathbb{G}_2)$. Let $T_* : A_2^*\to A_1^*$ be a bijective isometric algebra homomorphism, and set $T=(T_*)^*$. Then $v=T(1)$ and $u=T_r^*(1)$ are in the intrinsic groups of A_2^{**} and $L^\infty(\mathbb{G}_2)$, respectively. There is either:*

1. *A quantum group isomorphism $\theta : L^\infty(\mathbb{G}_1)\to L^\infty(\mathbb{G}_2)$ and associated $*$-isomorphism $\theta_0 : A_1\to A_2$ which intertwines the coproducts; or*
2. *A quantum group commutant isomorphism $\theta : L^\infty(\mathbb{G}_1)\to L^\infty(\mathbb{G}_2)$ and associated anti-$*$-isomorphism $\theta_0 : A_1\to A_2$ which intertwines the coproducts.*

In either case, for $\omega\in M_u(\mathbb{G}_2)$ and $\omega'\in L^1(\mathbb{G}_2)$,

$$T_*(\omega) = \theta_0^*(v\omega), \qquad T_r(\omega') = \theta_*(u\omega').$$

Proof. By previous work, $u=T_r^*(1)$ is a member of the intrinsic group of $L^\infty(\mathbb{G}_2)$, and the map $\theta : L^\infty(\mathbb{G}_1)\to L^\infty(\mathbb{G}_2); x\mapsto T_r^*(x)u^*$ is either a normal $*$-isomorphism, or a normal anti-$*$-isomorphism, which in either case intertwines the coproduct.

Suppose we are in the first case, where θ is a $*$-isomorphism. Then we have an associated $*$-isomorphism $\theta_0 : A_1 \to A_2$ which intertwines the coproducts, and which satisfies $\pi_2\theta_0 = \theta\pi_1$. Taking adjoints gives that $\theta_0^*\tilde{\pi}_{2,*} = \tilde{\pi}_{1,*}\theta_*$.

For $\omega \in L^1(\mathbb{G}_2)$, we have that $\theta_*(\omega) = T_r(u^*\omega)$. Also, T_r is constructed so that $T_*\tilde{\pi}_{2,*} = \tilde{\pi}_{1,*}T_r$. Thus

$$T_*\big(\tilde{\pi}_{2,*}(u^*\omega)\big) = \tilde{\pi}_{1,*}\big(T_r(u^*\omega)\big) = \tilde{\pi}_{1,*}\big(\theta_*(\omega)\big) = \theta_0^*\big(\tilde{\pi}_{2,*}(\omega)\big) \qquad (\omega \in L^1(\mathbb{G}_2)).$$

Recall that $v = T(1) \in A_2^{**}$ is also unitary. Then $u = T_r^*(1) = T_r^*\tilde{\pi}_1(1) = \tilde{\pi}_2 T(1) = \tilde{\pi}_2(v)$. A simple calculation shows that $v\tilde{\pi}_{2,*}(\omega) = \tilde{\pi}_{2,*}(u\omega)$ for $\omega \in L^1(\mathbb{G}_2)$.

Let $\mu \in A_2^*$ and $\omega \in L^1(\mathbb{G}_2)$, so we can find $\omega' \in L^1(\mathbb{G}_2)$ with $\tilde{\pi}_{2,*}(\omega') = \tilde{\pi}_{2,*}(\omega)\mu$. Then

$$\begin{aligned} T_*\big(\tilde{\pi}_{2,*}(\omega)\big)T_*(\mu) &= T_*\big(\tilde{\pi}_{2,*}(\omega')\big) = \theta_0^*\big(\tilde{\pi}_{2,*}(u\omega')\big) = \theta_0^*\big(v\tilde{\pi}_{2,*}(\omega')\big) = \theta_0^*\big(v\big(\tilde{\pi}_{2,*}(\omega)\mu\big)\big) \\ &= \theta_0^*\big((v\tilde{\pi}_{2,*}(\omega))(v\mu)\big) = \theta_0^*(v\tilde{\pi}_{2,*}(\omega))\theta_0^*(v\mu) = T_*\big(\tilde{\pi}_{2,*}(\omega)\big)\theta_0^*(v\mu). \end{aligned}$$

Recall that, from the hypothesis, $\tilde{\pi}_{1,*}(L^1(\mathbb{G}_2))$ is an essential ideal in A_1^*. As T_* bijects $\tilde{\pi}_{2,*}(L^1(\mathbb{G}_2))$ to $\tilde{\pi}_{1,*}(L^1(\mathbb{G}_2))$, we see that

$$T_*(\mu) = \theta_0^*(v\mu) \qquad (\mu \in A_2^*),$$

as claimed.

The other case, when θ is a quantum group commutant isomorphism, is entirely analogous. □

The previous theorem needs a characterisation of the intrinsic group of A^{**}, for A a quantum group above $C_0(\mathbb{G})$. The following results show that it is enough to know the intrinsic group of $L^\infty(\mathbb{G})$.

Lemma 4.6. *Let A be a Banach algebra, and let $I \subseteq A$ be a closed ideal. Let Φ_I be the character space I, and let X be the collection of characters on A which do not restrict to the zero functional on I. Then restriction of linear functionals gives a bijection from X to Φ_I.*

Proof. Let $f, g \in X$ induce the same (non-zero) character on I. Pick $a_0 \in I$ with $f(a_0) = g(a_0) = 1$. Then, for $a \in A$, we see that $f(a) = f(a)f(a_0) = f(aa_0) = g(aa_0) = g(a)g(a_0) = g(a)$, using that $aa_0 \in I$. Thus $f = g$, so the restriction map is injective.

Now let $u \in \Phi_I$, and pick $a_0 \in I$ with $u(a_0) = 1$. Define $f \in A^*$ by $f(a) = u(aa_0)$ for each $a \in A$. Then, for $a, b \in A$,

$$\begin{aligned} f(ab) &= u(aba_0) = u(a_0)u(aba_0) = u(a_0aba_0) = u(a_0a)u(ba_0) = u(a_0a)u(a_0)f(b) \\ &= u(a_0aa_0)f(b) = u(a_0)u(aa_0)f(b) = f(a)f(b). \end{aligned}$$

So f is a character on A. For $a \in I$, also $f(a) = u(aa_0) = u(a)u(a_0) = u(a)$, and so $f \in X$ and f restricts to u. Thus the restriction map is a bijection. □

The following should be compared with [24, Theorem 1] where Walter shows this in the co-commutative case.

Theorem 4.7. *Let (A, Δ_A, π) be a quantum group above $C_0(\mathbb{G})$. For a character u on A^*, the following are equivalent:*

1. *u is a member of the intrinsic group of A^{**};*
2. *u is invertible in A^{**};*

3. $\tilde{\pi}(u) \neq 0$*, that is,* u *does not induce the zero functional on* $\tilde{\pi}_*(L^1(\mathbb{G}))$*.*

Moreover, $\tilde{\pi} : A^{**} \to L^\infty(\mathbb{G})$ *restricts to a bijection between the intrinsic groups of* A^{**} *and* $L^\infty(\mathbb{G})$*.*

Proof. Let Y be the intrinsic group of $L^\infty(\mathbb{G})$, which by Theorem 3.2 is the character space of $L^1(\mathbb{G})$. Let X_1 be the intrinsic group of A^{**}, let X_2 be the collection of invertible characters, and let X_3 be the collection of characters not sent to zero by $\tilde{\pi}$. If $u \in X_2$ then $1 = \tilde{\pi}(1) = \tilde{\pi}(uu^{-1}) = \tilde{\pi}(u)\tilde{\pi}(u^{-1})$, showing that $\tilde{\pi}(u) \neq 0$ and hence $u \in X_3$. Thus $X_1 \subseteq X_2 \subseteq X_3$. By the lemma, $\tilde{\pi}$ restricts to a bijection between X_3 and Y.

Let $u \in X_3$, so by Theorem 3.2, $\tilde{\Delta}_A(u) = u \otimes u$. As $\tilde{\Delta}_A$ is a $*$-homomorphism, also u^*u is a character. As $\tilde{\pi}(u) \in L^\infty(\mathbb{G})$ is a (non-zero) character, it is unitary, and so $1 = \tilde{\pi}(u)^*\tilde{\pi}(u) = \tilde{\pi}(u^*u)$. Thus $u^*u \in X_3$, and as $\tilde{\pi}$ injects on X_3, and $1 \in X_3$, we conclude that $u^*u = 1$. Similarly, $uu^* = 1$. Thus u is a member of the intrinsic group of A^{**}, that is, $u \in X_1$. We hence have the required equalities $X_1 = X_2 = X_3$. □

In special cases, we can say more.

Proposition 4.8. *The intrinsic group of* $C_u(\mathbb{G})^{**}$*, respectively* $C_0(\mathbb{G})^{**}$*, is a subgroup of the unitary group of* $M(C_u(\mathbb{G}))$*, respectively* $M(C_0(\mathbb{G}))$*.*

Proof. Let $x \in C_u(\mathbb{G})^{**}$ be a member of the intrinsic group, and set $y = \tilde{\pi}(x) \in L^\infty(\mathbb{G})$. By Theorem 3.2, we have that y is unitary, and $y \in M(C_0(\mathbb{G}))$. Thus, in the language of [11, Proposition 6.6], y is a unitary corepresentation of $C_0(\mathbb{G})$ on $\mathbb{C}$, and so there is $x_0 \in M(C_u(\mathbb{G}))$ with $\pi(x_0) = y$ and $\Delta_u(x_0) = x_0 \otimes x_0$. By uniqueness (from the previous theorem) we must have that $x_0 = x$, treating $M(C_u(\mathbb{G}))$ as a subalgebra of $C_u(\mathbb{G})^{**}$.

Now let $x \in C_0(\mathbb{G})^{**}$ be a member of the intrinsic group. Again, $\tilde{\pi}(x) = y \in M(C_0(\mathbb{G}))$, so let x_0 be the image of y under the embedding $M(C_0(\mathbb{G})) \to C_0(\mathbb{G})^{**}$. Thus x_0 is a member of the intrinsic group of $C_0(\mathbb{G})^{**}$ and $\tilde{\pi}(x_0) = \tilde{\pi}(x)$, so again by uniqueness, we conclude that $x_0 = x$. □

4.4 The picture under duality

We now show that by using the duality theory of locally compact quantum groups, we can handle the more general situation; this also gives results more reminiscent of those for Kac algebras, see [6, Section 5.6].

Let (A, Δ_A, π) be a quantum group above $C_0(\mathbb{G})$. As $L^1(\mathbb{G})$ is an essential ideal in A^*, each member of A^* induces a (completely bounded) multiplier (or centraliser) of $L^1(\mathbb{G})$. Let us introduce the notation that given $\mu \in A^*$, we have maps $L_\mu, R_\mu : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ with

$$\tilde{\pi}_* L_\mu(\omega) = \mu\tilde{\pi}_*(\omega), \quad \tilde{\pi}_* R_\mu(\omega) = \tilde{\pi}_*(\omega)\mu \qquad (\omega \in L^1(\mathbb{G})).$$

Let us denote by $M_{cb}(L^1(\mathbb{G}))$ the algebra of completely bounded multipliers of $L^1(\mathbb{G})$. In [4, Theorem 8.9], a homomorphism $\Lambda : M_{cb}(L^1(\mathbb{G})) \to C^b(\hat{\mathbb{G}})$ was constructed (and a more general construction, with one-sided multipliers, is given in [7]). We hence find a map, which we shall continue to denote by Λ, from A^* to $C^b(\hat{\mathbb{G}})$, which is uniquely determined by the properties that

$$\Lambda(\mu)\lambda(\omega) = \lambda(L_\mu(\omega)), \quad \lambda(\omega)\Lambda(\mu) = \lambda(R_\mu(\omega)) \qquad (\mu \in A^*, \omega \in L^1(\mathbb{G})).$$

An important link between multipliers and the antipode is established in [3]. In particular, given $\mu \in A^*$, define an associated left multiplier $L^\dagger_\mu$ by

$$L^\dagger_\mu(\omega) = L_\mu(\omega^*)^* \quad \text{so} \quad \tilde{\pi}_* L^\dagger_\mu(\omega) = \big(\mu\tilde{\pi}_*(\omega^*)\big)^* = \mu^*\tilde{\pi}_*(\omega),$$

that is, $L_\mu^\dagger = L_{\mu^*}$. (Recall that ω^* is the normal functional $x \mapsto \overline{\langle x^*, \omega\rangle}$ so this calculation follows immediately from the fact that $\tilde{\Delta}_A$ and $\tilde{\pi}$ are $*$-homomorphisms). Then [3, Theorem 5.9] shows that $\Lambda(\mu^*) \in D(\hat{S})^*$ and $\Lambda(\mu) = \hat{S}(\Lambda(\mu^*)^*)$.

For $\lambda : L^1(\mathbb{G}) \to C_0(\mathbb{G})$ (which Λ extends) we can see this directly. Recall (see [14, Proposition 8.3]) that $\hat{S}((\iota \otimes \omega)(\hat{W})) = (\iota \otimes \omega)(\hat{W}^*)$. As $\hat{W} = \sigma W^* \sigma$, we see that $\lambda(\omega) = (\omega \otimes \iota)(W) = \hat{S}((\omega \otimes \iota)(W^*)) = \hat{S}(\lambda(\omega^*)^*)$.

Lemma 4.9. *Let $u \in L^\infty(\mathbb{G})$ be a member of the intrinsic group. For $x \in L^\infty(\hat{\mathbb{G}})$, let $\hat{\gamma}_u(x) = uxu^*$. Then $\hat{\gamma}_u$ is a $*$-automorphism of $L^\infty(\hat{\mathbb{G}})$, which restricts to a $*$-automorphism of $C_0(\hat{\mathbb{G}})$. Furthermore, $\hat{\gamma}_u \lambda(\omega) = \lambda(u\omega)$ for $\omega \in L^1(\mathbb{G})$.*

Proof. We have that $\Delta(u) = u \otimes u$, so $W^*(1 \otimes u)W = u \otimes u$. Using that W and u are unitary, it follows that $(1 \otimes u)W(1 \otimes u^*) = W(u \otimes 1)$. Then, for $\omega \in L^1(\mathbb{G})$,

$$\hat{\gamma}_u \lambda(\omega) = u(\omega \otimes \iota)(W)u^* = (\omega \otimes \iota)\big((1 \otimes u)W(1 \otimes u^*)\big) = (\omega \otimes \iota)\big(W(u \otimes 1)\big) = \lambda(u\omega),$$

as claimed. By density, it follows that $\hat{\gamma}_u$ is a self-map of $C_0(\hat{\mathbb{G}})$, which clearly has the inverse $\hat{\gamma}_{u^*}$. As $\hat{\gamma}_u$ is normal, it follows that $\hat{\gamma}_u$ is also an automorphism of $L^\infty(\hat{\mathbb{G}})$. $\square$

For the construction of $\hat{\theta}$ in the following, we again refer the reader to Section 4.1.

Theorem 4.10. *For $i = 1, 2$, let $\mathbb{G}_i$ be a locally compact quantum groups, and let $(A_i, \Delta_{A_i}, \pi_i)$ be a quantum group above $C_0(\mathbb{G}_i)$. Let $T_* : A_2^* \to A_1^*$ be a bijective isometric algebra homomorphism, and set $T = (T_*)^*$. Then $v = T(1)$ and $u = \tilde{\pi}_2(v)$ are members of the intrinsic groups of A_2^{**} and $L^\infty(\mathbb{G}_2)$, respectively. Then either:*

1. *There is a quantum group isomorphism $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$, leading to a quantum group isomorphism $\hat{\theta} : L^\infty(\hat{\mathbb{G}}_2) \to L^\infty(\hat{\mathbb{G}}_1)$, with*
$$\Lambda_1 T_* = \hat{\theta}\hat{\gamma}_u \Lambda_2.$$
2. *There is a quantum group commutant isomorphism $\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2)$, leading to a quantum group isomorphism $\hat{\theta} : L^\infty(\hat{\mathbb{G}}_2^{\mathrm{op}}) \to L^\infty(\hat{\mathbb{G}}_1)$, with*
$$\Lambda_1 T_* = \hat{\theta}\hat{R}_2 \hat{S}_2^{-1} \hat{\gamma}_u \Lambda_2.$$

In particular, $\mathbb{G}_1$ is isomorphic to either $\mathbb{G}_2$ or $\mathbb{G}_2'$.

Proof. In this more general situation, the proof of Theorem 4.5, and Theorem 4.7, still give the facts about v and u, and yields θ such that

$$T_*\big(\tilde{\pi}_{2,*}(u^*\omega)\big) = \tilde{\pi}_{1,*}\big(\theta_*(\omega)\big) \qquad (\omega \in L^1(\mathbb{G}_2)).$$

Suppose first that θ is a quantum group isomorphism. Let $\hat{\theta} : L^\infty(\hat{\mathbb{G}}_2) \to L^\infty(\hat{\mathbb{G}}_1)$ be the quantum group isomorphism induced by θ, which satisfies $\lambda_1 \theta_* = \hat{\theta}\lambda_2$.

Let $\omega \in L^1(\mathbb{G}_2)$ and $\mu \in A_2^*$. There is $\omega' \in L^1(\mathbb{G}_2)$ with $\mu\tilde{\pi}_{2,*}(\omega) = \tilde{\pi}_{2,*}(\omega')$. Then

$$\begin{aligned}\Lambda_1(T_*(\mu))\Lambda_1(T_*(\tilde{\pi}_{2,*}(\omega))) &= \Lambda_1(T_*(\tilde{\pi}_{2,*}(\omega'))) = \lambda_1(\theta_*(u\omega')) = \hat{\theta}(\lambda_2(u\omega')) = \hat{\theta}(\hat{\gamma}_u(\lambda_2(\omega'))) \\ &= \hat{\theta}\hat{\gamma}_u\big(\Lambda_2(\mu)\lambda_2(\omega)\big) = \hat{\theta}\hat{\gamma}_u\Lambda_2(\mu)\Lambda_1(T_*(\tilde{\pi}_{2,*}(\omega))),\end{aligned}$$

using that, similarly, $\Lambda_1(T_*(\tilde{\pi}_{2,*}(\omega))) = \hat{\theta}(\hat{\gamma}_u(\lambda_2(\omega)))$. As the set

$$\{\hat{\gamma}_u(\lambda_2(\omega)) : \omega \in L^1(\mathbb{G}_2)\}$$

is norm dense in $C_0(\hat{\mathbb{G}}_2)$, working in $M(C_0(\hat{\mathbb{G}}_2))$, we conclude that

$$\Lambda_1 T_*(\mu) = \hat{\theta}\hat{\gamma}_u\Lambda_2(\mu) \qquad (\mu \in A_2^*),$$

as required.

In the case when θ is a quantum group commutant isomorphism, define $\Phi : L^\infty(\mathbb{G}_2) \to L^\infty(\mathbb{G}_2'); x \mapsto Jx^*J$, and set $\theta' = \Phi\theta : L^\infty(\mathbb{G}_1) \to L^\infty(\mathbb{G}_2')$, which is a quantum group isomorphism. As in Section 4.1 we find a quantum group isomorphism $\hat{\theta}' : L^\infty((\mathbb{G}_2')\hat{}\,) \to L^\infty(\hat{\mathbb{G}}_1)$. As $(\mathbb{G}_2')\hat{}\, = (\hat{\mathbb{G}}_2)^{\mathrm{op}}$, this gives a normal $*$-isomorphism $\hat{\theta} : L^\infty(\hat{\mathbb{G}}_2) \to L^\infty(\hat{\mathbb{G}}_1)$ with $\hat{\Delta}_1\hat{\theta} = \sigma(\hat{\theta}\otimes\hat{\theta})\hat{\Delta}_2$. Then $\hat{\theta}\lambda_2' = \lambda_1\theta_*' = \lambda_1\theta_*\Phi_*$.

We now calculate $\lambda_2'\Phi_*^{-1}$. Let $\xi,\eta,\alpha,\beta \in L^2(\mathbb{G})$, so

$$\begin{aligned}
\big(\lambda_2'\Phi_*^{-1}(\omega_{\xi,\eta})\alpha\big|\beta\big) &= \big(\lambda_2'(\omega_{J\eta,J\xi})\alpha\big|\beta\big) = \big(W_2'(J\eta\otimes\alpha)\big|J\xi\otimes\beta\big) \\
&= \big((J\otimes J)W_2(\eta\otimes J\alpha)\big|J\xi\otimes\beta\big) = \big(W_2^*(\xi\otimes J\beta)\big|\eta\otimes J\alpha\big) \\
&= \big((\omega_{\xi,\eta}\otimes\iota)(W_2^*)J\beta\big|J\alpha\big) = \big(J((\omega_{\xi,\eta}\otimes\iota)(W_2^*))^*J\alpha\big|\beta\big),
\end{aligned}$$

using that $W_2' = (J\otimes J)W_2(J\otimes J)$. With reference to the discussion before Lemma 4.9, we see that

$$\lambda_2'\Phi_*^{-1}(\omega) = \hat{R}_2\big((\omega\otimes\iota)(W_2^*)\big) = \hat{R}_2\big(\lambda_2(\omega^*)^*\big) = \hat{R}_2\hat{S}_2^{-1}\lambda_2(\omega) \qquad (\omega \in L^1(\mathbb{G}_2)).$$

In particular,

$$\lambda_1\theta_* = \hat{\theta}\hat{R}_2\hat{S}_2^{-1}\lambda_2.$$

Finally, we follow the previous argument through. So let $\omega,\omega' \in L^1(\mathbb{G}_2)$ and $\mu \in A_2^*$ with $\mu\tilde{\pi}_{2,*}(\omega) = \tilde{\pi}_{2,*}(\omega')$. Then

$$\begin{aligned}
\Lambda_1(T_*(\mu))\Lambda_1(T_*(\tilde{\pi}_{2,*}(\omega))) &= \lambda_1(\theta_*(u\omega')) = \hat{\theta}\hat{R}_2\hat{S}_2^{-1}\hat{\gamma}_u\lambda_2(\omega') = \hat{\theta}\hat{R}_2\hat{S}_2^{-1}\hat{\gamma}_u\big(\Lambda_2(\mu)\lambda_2(\omega)\big) \\
&= \big(\hat{\theta}\hat{R}_2\hat{S}_2^{-1}\hat{\gamma}_u\Lambda_2(\mu)\big)\Lambda_1(T_*(\tilde{\pi}_{2,*}(\omega))),
\end{aligned}$$

using that $\hat{R}_2\hat{S}_2^{-1}$ is a homomorphism on $D(S_2^{-1})$. This completes the proof. $\square$

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