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A GENERALIZED FREQUENCY RESPONSE FOR
NONLINEAR SYSTEMS

by

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Abstract

The principal aim of this paper is to present the idea of a 'generalized frequency response' of a nonlinear input-output map S . It is defined as $\mathcal{J}S\mathcal{J}^{-1}$ where \mathcal{J} is the usual isomorphism from $L^2(0, \infty)$ to ℓ^2 . Realization results are presented pertaining to linear, bilinear and nonlinear systems.

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1. Introduction

The theory of linear systems has been developed from two points of view.- the time-domain state-space theory and the frequency-domain transfer-function approach. Each method has its advantages and disadvantages and, as is well-known, the classical theory of control is developed mainly in the (complex) frequency domain, while the theory of optimal control was initially developed in state space. The two methods can, however, be regarded as equivalent since the Fourier transform provides an isomorphism between the two representations of a system. Thus it is not surprising that state-space methods have frequency-domain counterparts and vice-versa ; consider, for example, the recent H^∞ methods for frequency-domain optimisation.

When we come to nonlinear systems, however, we are faced with what is essentially a state-space theory since it does not appear to be sensible even to consider the frequency response of such a system. Nevertheless, there have been attempts in the literature to define some kind of 'frequency domain' theory for nonlinear systems (see [1] , [6]). The method consists of finding a Volterra series expansion of the input-output map of the system and associating a sequence of transfer functions $H_k(s_1, \dots, s_k)$, $k \geq 1$, each one being the multi-dimensional Laplace transform of the corresponding Volterra series kernel. Setting $s_i = j\omega_i$ gives the 'frequency response' of the k^{th} kernel, namely $H_k(j\omega_1, \dots, j\omega_k)$. However this approach has two immediate difficulties; firstly, we obtain frequency spaces of increasing dimension and secondly, it is not at all clear how the functions $H_k(j\omega_1, \dots, j\omega_k)$ relate to the responses of the system to standard inputs (e.g. complex exponentials).

In this paper we shall take a more pragmatic approach and define a 'generalized frequency response' for a nonlinear system in terms of the way it responds to any input in terms of the components of the input with respect to some 'standard functions'. Thus we shall suppose that the input and output functions belong to $L^2[0, \infty]$ and take as standard functions some basis of this space (as a Hilbert space). Using the induced isomorphism of $L^2[0, \infty]$ with ℓ^2 we obtain a 'generalized

frequency response' which is just an analytic map $s: \ell^2 \rightarrow \ell^2$.

Throughout the paper we shall discuss the method relative to linear, bilinear and general nonlinear analytic systems. In particular, in the linear case we shall show that the method reduces essentially to the classical frequency response of a linear system in the sense that the map $s: \ell^2 \rightarrow \ell^2$ is linear and has a block diagonal matrix representation, each block being a 2×2 matrix which can be decomposed into a rotation and dilation corresponding to the phase shift and amplitude response of the system at the frequency of the corresponding basis function.

In section 2 we shall specify some notation and in section 3 we shall present a new input-output representation for a nonlinear analytic system. In section 4 we shall define the generalized frequency response of a nonlinear system and in section 5 the realization of frequency response maps $s: \ell^2 \rightarrow \ell^2$ will be discussed.

2. Notation and terminology:

In this paper we shall use the following notation. An n-multi-index is an n-tuple $i=(i_1, \dots, i_n)$ of non-negative integers; its length (or order) is given by $|i|=i_1+\dots+i_n$. The sum of two multi-indices i and l is defined as $i+l=(i_1+l_1, \dots, i_n+l_n)$. We say that $i \leq l$ if $i_k \leq l_k$ for $k=1, \dots, n$. When $i \leq l$ we define $l-i$ as $(l_1-i_1, \dots, l_n-i_n)$. We also define $i! = i_1! \dots i_n!$ and $x^i = x_1^{i_1} \dots x_n^{i_n}$ for $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. $l(r)$ will denote the n-multi-index with l in the r th place and zero elsewhere, and $\delta_i^l = \delta_{i_1}^{l_1} \dots \delta_{i_n}^{l_n}$ where $\delta_{i_k}^{l_k} = 1$ if $i_k = l_k$ and $\delta_{i_k}^{l_k} = 0$ if $i_k \neq l_k$.

For an analytic function $h: \mathbb{R}^n \rightarrow \mathbb{R}$, Taylor's formula becomes

$$h(x) = \sum_{|i| \geq 0} \frac{x^i}{i!} h^{(|i|)}(0) \quad (o)$$

where

$$h^{(|i|)}(x) = \partial^{(|i|)} h(x) / \partial x_1^{i_1} \dots \partial x_n^{i_n}$$

Let ℓ^2 denote the standard Banach space of square summable sequences and let ℓ_e^2 [2] denote the Banach space of sequences $(\alpha_n)_{n \geq 0}$ such that the sequence $(\alpha_n/n!)_{n \geq 0}$ belongs to ℓ^2 . Define a norm on ℓ_e^2 by

$$\|(\alpha_n)_{n \geq 0}\| = \left(\sum_{n \geq 0} \frac{\alpha_n^2}{(n!)^2} \right)^{1/2}, \quad (\alpha_n)_{n \geq 0} \in \ell_e^2$$

Now consider the algebraic tensor product of n copies of ℓ_e^2 , $\mathcal{L}_n = \otimes_n \ell_e^2$, and let $\|\cdot\|$ be any cross norm on \mathcal{L}_n . For a simple tensor ϕ we have $\phi = (\phi_{i_1 \dots i_n}) = (\alpha_{i_1}^1 \dots \alpha_{i_n}^n) = \alpha^1 \otimes \dots \otimes \alpha^n$ where $\alpha^k = (\alpha_{i_k}^k)_{i_k \geq 0} \in \ell_e^2$, $k=1, \dots, n$.

Then, $\|\phi\| = \prod_{k=1}^n \|\alpha^k\|_e$.

The standard $L^2[0, \infty]$ and ℓ^2 spaces will also be used.

3. The Nonlinear Input-Output map:

In this section we shall consider an analytic system of the form

$$\begin{cases} \dot{x} = f(x, u) & , x(0) = x_0 & (3.1) \\ y = g(x) & & (3.2) \end{cases}$$

where $x_0 \in \mathbb{R}^n$, $u \in \mathbb{R}$; $f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ are analytic functions. (Thus, we are considering the single-input, single-output case for simplicity). We shall assume that the solution of (3.1) exist for all time and for each $x_0 \in \mathbb{R}^n$, although systems with finite escape times can be treated similarly (except that the solutions $x(t, x_0)$ of (3.1) are not analytic in t for all t).

We shall require the input-output map $S: u \rightarrow y$ for the system (3.1), (3.2) which can be obtained by using the method of Carleman Linearization [3]. (See also [5]).

To obtain this map we introduce the functions

$$\phi_{i_1 \dots i_n}(x) = x_1^{i_1} \dots x_n^{i_n} \triangleq x^i$$

Since $f(x, \cdot)$ is analytic we have

$$f(x, u) = \sum_{j \geq 0} \frac{u^j}{j!} f^{(0, j)}(x, 0)$$

where $f^{(0, j)}$ denotes the j^{th} derivation of $f(x, u)$ with respect to u . By the analyticity of f with respect to x , we have

$$f(x, u) = \sum_{j \geq 0} \frac{u^j}{j!} \sum_{|i| \geq 0} \frac{1}{i!} f^{(|i|, j)}(0, 0) x^i$$

where $i = (i_1, \dots, i_n)$.

Hence we have

$$\dot{\phi}_{i_1 \dots i_n} = \sum_{r=1}^n i_r x^{i-1(r)} \dot{x}_r$$

and so

$$\begin{aligned} \dot{\phi}_{i_1 \dots i_n} &= \sum_{r=1}^n i_r x^{i-1(r)} \dot{f}_r(x, u) \\ &= \sum_{r=1}^n i_r x^{i-1(r)} \sum_{j \geq 0} \frac{u^j}{j!} \sum_{|\ell| \geq 0} \frac{1}{\ell!} f_r^{(|\ell|, j)}(0, 0) x^\ell \\ &= \sum_{j \geq 0} u^j \sum_{r=1}^n i_r \sum_{|\ell| \geq 0} \frac{f_r^{(|\ell|, j)}(0, 0)}{\ell! j!} x^{i-1(r)+\ell} \\ &= \sum_{j \geq 0} u^j \sum_{r=1}^n i_r \sum_{|\ell| \geq 0} \sum_{|p| \geq 0} \frac{f_r^{(|\ell|, j)}(0, 0)}{\ell! j!} \delta_p^{i-1(r)+\ell} x^p \end{aligned}$$

We therefore have

$$\dot{\phi}_{i_1 \dots i_n} = \sum_{j \geq 0} u^j \sum_{|p| \geq 0} a_i^p(j) x^p$$

where

$$a_i^p(j) = \begin{cases} \sum_{r=1}^n \frac{i_r}{(p-i+1(r))! j!} f_r^{(|p-i+1(r)|, j)} & (0,0) \text{ if } p-i+1(r) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally, we have

$$\dot{\phi} = A_0 \phi + \sum_{j \geq 1} u^j A_j \phi \tag{3.3}$$

where ϕ is the tensor with components $\phi_{i_1 \dots i_n}$ and A_j is the tensor operator defined by

$$(A_j \phi)_i = \sum_{|p| \geq 0} a_i^p(j) x^p = \sum_{|p| \geq 0} a_i^p(j) \phi_p$$

We can solve equation (3.3) by standard Picard iteration ; thus define

$$\begin{aligned} \psi_0(t) &= e^{A_0 t} \phi_0 \\ \psi_k(t) &= \sum_{j \geq 1} \int_0^t e^{A_0(t-\tau)} u^j(\tau) A_j \psi_{k-1}(\tau) d\tau, k \geq 1 \end{aligned} \tag{3.4}$$

Using methods developed in [3], it can be shown that the solution of (3.3) is given by

$$\phi(t) = \sum_{k \geq 0} \psi_k(t)$$

where the series on the right hand side converges in ℓ_e^2 for bounded controls.

Explicitly we have

$$\psi_k(t) = \sum_{i_1 \geq 1} \dots \sum_{i_k \geq 1} \int_0^t \int_0^{\sigma_k} \dots \int_0^{\sigma_2} \frac{i_1 \dots i_k}{v_k} (t, \sigma_1, \dots, \sigma_k) u^{i_k}(\sigma_k) \dots \dots u^{i_1}(\sigma_1) d\sigma_1 \dots d\sigma_k \tag{3.5}$$

where

$$\frac{i_1 \dots i_k}{v_k} (t, \sigma_1, \dots, \sigma_k) = e^{A_0(t-\sigma_k)} A_{i_k} e^{A_0(\sigma_k-\sigma_{k-1})} A_{i_{k-1}} \dots \dots A_{i_1} e^{A_0 \sigma_1} \phi_0$$

From (3.2) we have

$$\begin{aligned} y &= g(x) \\ &= \sum_{|p| \geq 0} \frac{g|p|(0)}{p!} x^p \\ &= G \phi \end{aligned}$$

for some tensor operator G . Hence the input-output map of the system (3.1),

(3.2) is given by

$$y(t) = v_0(t) + \sum_{k \geq 1} \sum_{i_1 \geq 1} \dots \sum_{i_k \geq 1} \int_0^t \int_0^{\sigma_k} \dots \int_0^{\sigma_2} v_k^{i_1 \dots i_k}(t, \sigma_1, \dots, \sigma_k) u_k^{i_1 \dots i_k}(\sigma_k) \dots u_1^{i_1}(\sigma_1) d\sigma_1 \dots d\sigma_k \quad (3.6)$$

when

$$v_0(t) = G e^{A_0 t} \phi_0$$

and

$$v_k^{i_1 \dots i_k}(t, \sigma_1, \dots, \sigma_k) = G \bar{v}_k^{i_1 \dots i_k}(t, \sigma_1, \dots, \sigma_k)$$

4. Spectral theory of Nonlinear Input-Output Maps:

Consider a System S given in terms of an input-output map

$$S : \mathbb{R}^n \times \mathcal{L}^2 [0, \infty] \rightarrow \mathcal{L}^2 [0, \infty] \quad (4.1)$$

$$\text{defined by } y(t) = S(x_0, u(\cdot))(t) \quad (4.2)$$

We assume that the input u and the output y belong to $\mathcal{L}^2 [0, \infty]$. $x_0 \in \mathbb{R}^n$ is the initial state in some given state-space realization. Again for simplicity, we have assumed scalar inputs and outputs, the multivariable case will be considered in a future paper.

We have seen an example of such an input-output map in section 3, generated by a nonlinear analytic differential equation. Two simple examples are given by the linear-convolution systems and by the Volterra series of a bilinear system.

Example 1: Linear Systems

Consider the Linear system

$$\begin{cases} \dot{x} = Ax + Bu, & x(0) = x_0 \\ y = Cx \end{cases} \quad (4.3)$$

then the input-output map is given by

$$y(t) = Ce^{At} x_0 + \int_0^t C e^{A(t-\tau)} Bu(\tau) d\tau \quad (4.4)$$

In this case

$$S(x_0, u(\cdot))(t) = g_0(t) + (g * u)(t) \quad (4.5)$$

where

$$g_0(t) = Ce^{At} x_0, \quad g(t) = C e^{At} B$$

and

* denotes the convolution operator.

Example 2: Bilinear Systems

Consider the bilinear system

$$\begin{cases} \dot{x} = Ax + uNx + Bu, & x(0) = x_0 \\ y = Cx \end{cases} \quad (4.6)$$

where A, N, B, C are constant matrices of suitable dimensions. Then the input-output map is given by [4]:

$$\begin{aligned} y(t) = & C e^{At} x_0 + \sum_{j \geq 1} \int_0^t \int_0^{\sigma_j} \dots \int_0^{\sigma_2} v_j(t, \sigma_1, \dots, \sigma_j) \cdot \\ & \cdot u(\sigma_1) \dots u(\sigma_j) d\sigma_1 \dots d\sigma_j + \\ & + \sum_{j \geq 1} \int_0^t \int_0^{\sigma_j} \dots \int_0^{\sigma_2} w_j(t, \sigma_1, \dots, \sigma_j) \cdot \\ & \cdot u(\sigma_1) \dots u(\sigma_j) d\sigma_1 \dots d\sigma_j \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} v_1(t, \sigma_1) &= C e^{A(t-\sigma_1)} B \\ v_j(t, \sigma_1, \dots, \sigma_j) &= C e^{A(t-\sigma_j)} N e^{A(\sigma_j-\sigma_{j-1})} N \dots \\ & \dots N e^{A(\sigma_2-\sigma_1)} B, \quad j > 1 \end{aligned}$$

and

$$\begin{aligned} w_1(t, \sigma_1) &= C e^{A(t-\sigma_1)} x_0 \\ w_j(t, \sigma_1, \dots, \sigma_j) &= C e^{A(t-\sigma_j)} N e^{A(\sigma_j-\sigma_{j-1})} N \dots \\ & \dots N e^{A(\sigma_2-\sigma_1)} N e^{A\sigma_1} x_0, \quad j > 1 \end{aligned}$$

so $y(t) = S(x_0, u(\cdot))(t)$.

Returning to (4-1), for each fixed initial state x_0 , we have a map

$$S_{x_0} \triangleq S(x_0, \cdot): L^2[0, \infty] \rightarrow L^2[0, \infty] \quad (4.8)$$

Let $\{e_j\}_{j \geq 0}$ be a basis of $L^2[0, \infty]$ and let \mathcal{J} denote the usual isomorphism

$$\mathcal{J}: L^2[0, \infty] \rightarrow \ell^2$$

given by

$$\mathcal{J}(f) = \{f_j\}_{j \geq 0}$$

where

$$f \in L^2[0, \infty], \quad f = \sum_{j \geq 0} f_j e_j$$

The S_{x_0} induces a map

$$s_{x_0}: \ell^2 \rightarrow \ell^2$$

such that the diagram

$$\begin{array}{ccc}
 L^2 [0, \infty] & \xrightarrow{S_{x_0}} & L^2 [0, \infty] \\
 \downarrow & & \downarrow \\
 \ell^2 & \xrightarrow{S_{x_0}} & \ell^2
 \end{array}
 \tag{4.9}$$

commutes.

generalized

We shall call s_{x_0} the generalized frequency response of S_{x_0} (with respect to the basis $\{e_k\}_{k \geq 0}$). Explicitly s_{x_0} is given by

$$s_{x_0} (\{u_k\}_{k \geq 0}) = \{ \langle S_{x_0} (\sum_{j \geq 0} u_j e_j), e_k \rangle \}_{k \geq 0}
 \tag{4.10}$$

where $\langle \cdot, \cdot \rangle$ denotes a scalar product in $L^2(0, \infty)$.

As an example consider the linear system in example 1 above. The map S_{x_0} , and hence the map s_{x_0} , is affine and is linear if $x_0 = 0$. s_{x_0} has a matrix representation given by

$$s_{x_0} (\{u_k\}_{k \geq 0}) = w_\ell + \sum_{j \geq 0} G_{\ell j} \cdot u_j$$

where

$$w_\ell = \langle C e^{At} x_0, e_\ell \rangle$$

In order to see the relationship to the familiar frequency response consider the input and output functions over $[0, T]$ for fixed $T > 0$. Let $x_0 = 0$ and introduce the basis

$$B = \left\{ \frac{1}{\sqrt{T}} \right\} \cup \left\{ \frac{\sqrt{2}}{T} \cos 2 n \pi t / T \right\}_{n \geq 1} \cup \left\{ \frac{\sqrt{2}}{T} \sin 2 n \pi t / T \right\}_{n \geq 1}$$

of $L^2 [0, T]$. Then, as is well-known from linear systems theory we can write $g*u$ in the form

$$(g*u)(t) = \int_0^t g(\tau) u(t-\tau) d\tau$$

and so if $u(t) = e^{i w_n t}$, where $w_n = 2\pi n/T$ we have

$$\begin{aligned}
 (g*u)(t) &= e^{i w_n t} \int_0^\infty g(\tau) e^{-i w_n \tau} d\tau - e^{i w_n t} \int_t^\infty g(\tau) e^{-i w_n \tau} d\tau \\
 &\triangleq e^{i w_n t} G(i w_n) + E_n(t)
 \end{aligned}
 \tag{4.11}$$

for $t \in [0, T]$. Here, $E_n(t)$ is called the transient term. Of course, this is just the familiar expression for the frequency response $G(i w_n)$ (i.e., the Fourier

transform of the impulse response). If we order the basis B in the following way:

$$1/\sqrt{T}, \sqrt{\frac{2}{T}} \cos \pi(2\pi t/T), \sqrt{\frac{2}{T}} \sin (2\pi t/T), \\ \sqrt{\frac{2}{T}} \cos (4\pi t/T), \sqrt{\frac{2}{T}} \sin (4\pi t/T). \dots$$

then the commutative diagram (4.9) induces by the map \mathcal{J} a one-to-one correspondence of this basis with the basis

$$(1,0,0,\dots), (0,1,0,\dots), (0,0,1,0,\dots), \dots \text{ of } \ell^2,$$

From (4.10), we therefore see that the matrix representation of the linear operator $s_{x_0} : \ell^2 \rightarrow \ell^2$ for the linear system above, with respect to the basis given

is

$$\begin{pmatrix} \gamma & & 0 \\ & \Gamma_1 & \\ 0 & & \Gamma_2 \end{pmatrix} + \Delta \tag{4.12}$$

(assuming the system is stable). Here

$$\gamma = \int_0^T g(t) dt$$

is just the gain of the system, Γ_k ($1 \leq k < \infty$) is a 2×2 matrix given by

$$\Gamma_k = \begin{pmatrix} \operatorname{Re} G(i\omega_k) & -\operatorname{Im} G(i\omega_k) \\ \operatorname{Im} G(i\omega_k) & \operatorname{Re} G(i\omega_k) \end{pmatrix}$$

and Δ is the matrix representation of the isomorphic image of the transient term in ℓ^2 .

$$\text{If we write } \Gamma_k = (\det \Gamma_k)^{1/2} \begin{pmatrix} \cos \theta_k & -\sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{pmatrix}$$

then Γ_k represents a rotation and a dilation corresponding to the phase shift and the amplitude response of the system.

To illustrate the expression (4.10) for the bilinear input-output map (4.7) let $\{e_i\}$ be a basis of $L^2[0, \infty]$ and define

$$w_0(t) = Ce^{At} x_0$$

and for $j \geq 1$

$$V_{jk_1, \dots, k_j}(t) = \int_0^t \int_0^{\sigma_1} \dots \int_0^{\sigma_{j-1}} v_j(t, \sigma_1, \dots, \sigma_j) e_{k_1}(\sigma_1) \dots e_{k_j}(\sigma_j) \cdot d\sigma_1 \dots d\sigma_j \quad (4.13)$$

$$W_{jk_1, \dots, k_j}(t) = \int_0^t \int_0^{\sigma_1} \dots \int_0^{\sigma_{j-1}} w_j(t, \sigma_1, \dots, \sigma_j) e_{k_1}(\sigma_1) \dots e_{k_j}(\sigma_j) \cdot d\sigma_1 \dots d\sigma_j$$

then, if $u = \sum_{k \geq 0} u_k e_k$, $u \in L^2[0, \infty]$

the input-output map (4.7) becomes

$$y(t) = w_0(t) + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} V_{jk_1 \dots k_j}(t) + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} W_{jk_1 \dots k_j}(t) \quad (4.14)$$

and so

$$\begin{aligned} \langle y(t), e_\ell \rangle &= \langle w_0(t), e_\ell \rangle + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} \cdot \langle V_{jk_1 \dots k_j}(t), e_\ell \rangle + \\ &+ \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} \langle W_{jk_1 \dots k_j}(t), e_\ell \rangle \end{aligned}$$

Hence

$$y_\ell = w_{0\ell} + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} V_{jk_1 \dots k_j, \ell} + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} W_{jk_1 \dots k_j, \ell}$$

where $y(t) = \sum_{\ell \geq 0} y_\ell e_\ell(t)$, $w_0(t) = \sum_{\ell \geq 0} w_{0\ell} e_\ell(t)$

$$V_{jk_1 \dots k_j}(t) = \sum_{\ell \geq 0} V_{jk_1 \dots k_j, \ell} e_\ell(t), \quad W_{jk_1 \dots k_j}(t) = \sum_{\ell \geq 0} W_{jk_1 \dots k_j, \ell} e_\ell(t)$$

It follows that the diagram (4.9) induces the map $s_{x_0}: \ell^2 \rightarrow \ell^2$ given by

$$\begin{aligned} s_{x_0}((u_0, u_1, \dots))_\ell &= w_{0\ell} + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} \cdot V_{jk_1 \dots k_j, \ell} + \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} W_{jk_1 \dots k_j, \ell} \quad (4.15) \end{aligned}$$

We shall require the generalization of the Taylor's formula of an analytic function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ to an analytic function $f: \ell^2 \rightarrow \mathbb{R}$ defined on the infinite-dimensional space ℓ^2 . From the general theory of higher-order derivatives of functions defined on an infinite-dimensional space [7] and from the familiar finite-dimensional formula, we can write an analytic function $f: \ell^2 \rightarrow \mathbb{R}$ in the form

$$f(u_0, u_1, u_2, \dots) = \sum_{|i| \geq 0} \frac{f^{(|i|)}(0)}{i!} u^i \quad (4.16)$$

where

$$u = (u_0, u_1, u_2, \dots) \in \ell^2$$

$$i = (i_0, i_1, i_2, \dots) \text{ with only a finite number of non-zero terms.}$$

$$u^i = u_0^{i_0} u_1^{i_1} u_2^{i_2} \dots$$

$$i! = i_0! i_1! i_2! \dots$$

$$|i| = \sum_{l \geq 0} i_l$$

$$0 = (0, 0, 0, \dots)$$

and

$$f^{(|i|)}(0) = \frac{\partial^i f(0)}{\partial u_0^{i_0} \partial u_1^{i_1} \partial u_2^{i_2} \dots}$$

Define the map $s_{x_0, j}: \ell^2 \rightarrow \mathbb{R}$ in the usual way by

$$s_{x_0, j}((u_0, u_1, \dots)) = (s_{x_0}((u_0, u_1, \dots)))_j$$

then, comparing (4.14) with (4.16) we obtain

$$w_{0l} = s_{x_0 l}((0, 0, \dots)) \quad (4.17)$$

$$V_{j k_1 \dots k_{j l}} + W_{j k_1 \dots k_{j l}} = \frac{\partial^j}{\partial u_0^{p_1} \partial u_1^{p_1} \dots} s_{x_0 l}((0, 0, \dots))$$

where p_r is the number of terms in the sequence k_1, \dots, k_j equal to r (note that

$$\sum_{r \geq 0} p_r = j).$$

It follows that in the 'frequency domain' representation a bilinear system is just an analytic map from ℓ^2 to ℓ^2 (or if we vary the initial value from $\mathbb{R} \times \ell^2$ to ℓ^2).

Consider, finally, the general non-linear analytic system (3.1), (3.2) and its

input-output map (3.6). As before let $u = \sum_{j \geq 0} u_j e_j$.

since (3.6) involves $u^i(\sigma)$, we consider $u^i(\sigma) = \sum_{j \geq 0} U_{ij} e_j(\sigma)$ where

$$U_{ij} = \langle u^i, e_j \rangle$$

we have

$$\begin{aligned} U_{ij} &= \langle u^{i-1} \sum_{k \geq 0} u_k e_k, e_j \rangle = \sum_{k \geq 0} u_k \langle u^{i-1} e_k, e_j \rangle \\ &= \sum_{k \geq 0} u_k \langle u^{i-1}, \bar{e}_k e_j \rangle \end{aligned}$$

let $\beta_{kp}^j = \langle \bar{e}_k e_j, e_p \rangle$ (4.18)

then
$$\begin{aligned} U_{ij} &= \sum_{k \geq 0} u_k \langle u^{i-1}, \sum_{p \geq 0} \beta_{kp}^j e_p \rangle \\ &= \sum_{k \geq 0} \sum_{p \geq 0} u_k \beta_{kp}^j \langle u^{i-1}, e_p \rangle \end{aligned}$$

that is

$$\begin{cases} U_{ij} = \sum_{k \geq 0} \sum_{p \geq 0} u_k \beta_{kp}^j U_{i-1,p} \\ U_{0j} = \langle 1, e_j \rangle \end{cases} \quad (4.19)$$

therefore

$$\begin{aligned} U_{ij} &= \sum_{k_1 \geq 0} \sum_{p_1 \geq 0} \dots \sum_{k_i \geq 0} \sum_{p_i \geq 0} u_{k_1} \beta_{k_1 p_1}^j u_{k_2} \beta_{k_2 p_2}^{p_1} \dots u_{k_i} \beta_{k_i p_i}^{p_{i-1}} U_{0 p_i} \\ &= \sum_{k_1 \geq 0} \dots \sum_{k_i \geq 0} V_{ij}^{k_1 \dots k_i} u_{k_1} \dots u_{k_i} \end{aligned}$$

where

$$V_{ij}^{k_1 \dots k_i} = \sum_{p_1 \geq 0} \dots \sum_{p_i \geq 0} \beta_{k_1 p_1}^j \beta_{k_2 p_2}^{p_1} \dots \beta_{k_i p_i}^{p_{i-1}} u_{0 p_i}$$

Thus (3.6) becomes

$$y(t) = v_0(t) + \sum_{k \geq 1} \sum_{i_1 \geq 1} \dots \sum_{i_k \geq 1} \sum_{j_1 \geq 0} \dots \sum_{j_k \geq 0} U_{i_1 j_1} \dots U_{i_k j_k} V_{k, i_1 \dots i_k}^{*j_1 \dots j_k}(t)$$

where

$$V_{k, i_1 \dots i_k}^{*j_1 \dots j_k}(t) = \int_0^t \int_0^{\sigma_k} \dots \int_0^{\sigma_2} V_k^{i_1 \dots i_k}(t, \sigma_1, \dots, \sigma_k) e_{j_k}(\sigma_k) \dots e_{j_1}(\sigma_1) d\sigma_1 \dots d\sigma_k \quad (4.20)$$

ie.,

$$\begin{aligned} y(t) &= v_0(t) + \sum_{k \geq 1} \sum_{i_1 \geq 1} \dots \sum_{i_k \geq 1} \sum_{j_1 \geq 0} \dots \sum_{j_k \geq 0} \sum_{l_{1, i_1} \geq 0} \dots \sum_{l_{i_1, i_1} \geq 0} \dots \\ &\dots \sum_{l_{1, i_k} \geq 0} \dots \sum_{l_{i_k, i_k} \geq 0} V_{i_1 j_1}^{l_{1, i_1} \dots l_{i_1, i_1}} \dots V_{i_k j_k}^{l_{1, i_k} \dots l_{i_k, i_k}} \end{aligned}$$

$$\cdot u_{l_1, i_1} \dots u_{l_{i_1}, i_1} \dots u_{l_1, i_k} \dots u_{l_{i_k}, i_k} v_{k i_1 \dots i_k}^{*j_1 \dots j_k} (t)$$

Hence

$$y_j = w_{oj} + \sum_{k \geq 1} \sum_{i_1 \geq 0} \dots \sum_{i_k \geq 0} \sum_{l_1, i_1 \geq 0} \dots \sum_{l_{i_1}, i_1 \geq 0} \dots \sum_{l_1, i_1 \geq 0} \dots \sum_{l_{i_k}, i_k \geq 0} W_{k, i_1 \dots i_k}^{j_1 \dots j_k} \cdot u_{l_1, i_1} \dots u_{l_{i_1}, i_1} \dots u_{l_1, i_k} \dots u_{l_{i_k}, i_k} \triangleq S_{X_0}((u_0, u_1, \dots))_j \quad (4.21)$$

where

$$y_j = \langle y(t), e_j \rangle$$

$$w_{oj} = \langle v_0(t), e_j \rangle$$

and

$$W_{k i_1 \dots i_k}^{j_1 \dots j_k} = \sum_{j_1 \geq 0} \dots \sum_{j_k \geq 0} v_{i_1 j_1}^{l_1, i_1} \dots v_{i_k j_k}^{l_{i_k}, i_k} \langle v_{k i_1 \dots i_k}^{*j_1 \dots j_k}, e_j \rangle$$

5. Realization theory.

In this section we shall consider the problem of the realizability and the state space realization of an analytic map $s: \ell^2 \rightarrow \ell^2$, which defines a generalized frequency response. We shall again study the problem in the linear, bilinear and general nonlinear situations and the results will be expressed in the form of conditions on the multi-dimensional Laplace transforms of various kernel functions associated with S .

5A: Linear Systems.

Theorem 1:

A necessary and sufficient condition for a sequence $\{G_{lj}\}_{l, j \geq 0}$ of numbers to be the 'generalized frequency response' of a linear system with zero initial condition (with respect to a given basis $\{e_k\}_{k \geq 0}$ of $L^2[0, \infty]$) is that there exists a strictly proper rational function $G(s)$ such that

$$\sum_{l \geq 0} G_{lj} E_l(s) = G(s) E_j(s) \quad (5.1)$$

for all $j \geq 0$. $G(s)$ is then the transfer function of the linear system.

Proof:

In section 4, we derived the 'generalized frequency response' s_{x_0} of linear systems. Taking $x_0 = 0$, this reduces to

$$y_\ell = \sum_{j \geq 0} G_{\ell j} u_j$$

where

$$y(t) = \sum_{\ell \geq 0} y_\ell e_\ell(t), \quad t \in [0, \infty)$$

and

$$\sum_{\ell \geq 0} G_{\ell j} e_\ell(t) = \int_0^t C e^{A(t-\tau)} B e_j(\tau) d\tau$$

Taking Laplace transform, we obtain

$$\sum_{\ell \geq 0} G_{\ell j} E_\ell(s) = C(sI - A)^{-1} B \cdot E_j(s)$$

that is

$$\sum_{\ell \geq 0} G_{\ell j} E_\ell(s) = G(s) E_j(s) .$$

5.B: Bilinear Systems:

Let $E_k(s)$ have only simple poles $\alpha_k^{j_k}$ $j_k=1, \dots, r_k$. The corresponding residues are denoted $E_k^{j_k}$, $k \geq 0$.

Theorem 2:

A necessary and sufficient condition for a sequence

$$\{V_{j_1 k_1 \dots j_\ell k_\ell} \}_{j_1 \geq 1, k_1 \dots j_\ell \geq 1, k_\ell \geq 0}$$

of numbers to be the 'generalized frequency response' of a bilinear system with zero initial condition (with respect

to a given basis $\{e_k\}_{k \geq 0}$ of $L^2[0, \infty]$) is that there exist four matrices $G_0(s)$,

$G_1(s)$, $G_2(s)$, $G_3(s)$ with dimensions respectively $l \times l$, $l \times m$, $m \times m$, $m \times l$ of strictly

proper rational functions such that

$$(i) \quad \sum_{\ell \geq 0} V_{1k_1 \ell} E_\ell(s) = G_0(s) E_{k_1}(s) \tag{5.2}$$

$$\begin{aligned}
 \text{(ii)} \quad \sum_{\ell \geq 0} V_{jk_1 \dots k_j \ell} E_\ell(s) &= \sum_{\ell_{k_j}=1}^{r_{k_j}} \dots \sum_{\ell_{k_2}=1}^{r_{k_2}} \\
 &E_{k_j}^{\ell_{k_j}} \dots E_{k_2}^{\ell_{k_2}} G_1(s) G_2(s - \alpha_{k_j}^{\ell_{k_j}}) \dots \\
 &\dots G_2(s - \alpha_{k_j}^{\ell_{k_j}} - \dots - \alpha_{k_3}^{\ell_{k_3}}) \cdot G_3(s - \alpha_{k_j}^{\ell_{k_j}} - \dots - \alpha_{k_2}^{\ell_{k_2}}) \\
 &\cdot E_{k_1}^{\ell_{k_1}}(s - \alpha_{k_j}^{\ell_{k_j}} - \dots - \alpha_{k_2}^{\ell_{k_2}})
 \end{aligned} \tag{5.3}$$

for all $k_1, \dots, k_j \geq 0, j > 1$.

Proof:

For $x_0 = 0, s_{x_0}$ given in (4.15) reduces to

$$y_\ell = \sum_{j \geq 1} \sum_{k_1 \geq 0} \dots \sum_{k_j \geq 0} u_{k_1} \dots u_{k_j} V_{jk_1 \dots k_j \ell}$$

where

$$y(t) = \sum_{\ell \geq 0} y_\ell e_\ell(t) \tag{5.4}$$

$$\text{and } V_{jk_1 \dots k_j}(t) = \sum_{\ell \geq 0} V_{jk_1 \dots k_j \ell} e_\ell(t)$$

but (4.7) and (4.13) yield

$$V_{1k_1}(t) = \int_0^t C e^{A(t-\sigma_1)} B e_{k_1}(\sigma_1) d\sigma_1 \tag{5.5}$$

and

$$\begin{aligned}
 V_{jk_1 \dots k_j}(t) &= \int_0^t \int_0^{\sigma_j} \dots \int_0^{\sigma_2} C e^{A(t-\sigma_j)} N e^{A(\sigma_j - \sigma_{j-1})} N \dots \\
 &\dots N e^{A(\sigma_2 - \sigma_1)} B e_{k_1}(\sigma_1) \dots e_{k_j}(\sigma_j) \cdot \\
 &\cdot d\sigma_1 \dots d\sigma_j, \quad j > 1
 \end{aligned} \tag{5.6}$$

Taking the Laplace transform of (5.4) we obtain

$$\sum_{\ell \geq 0} V_{jk_1 \dots k_j \ell} E_\ell(s) = V_{jk_1 \dots k_j}(s) \tag{5.7}$$

whereas the Laplace transform of (5.5) yield:

$$V_{1k_1}(s) = C \{sI - A\}^{-1} B \cdot E_{k_1}(s)$$

that is

$$\sum_{\ell \geq 0} V_{1k_1 \ell} E_{\ell}(s) = G_0(s) E_{k_1}(s) \quad (5.8)$$

where $G_0(s) = C\{sI-A\}^{-1}B$ (5.9)

In order to find the Laplace transform of (5.6) we are going to write it in a different form, more suitable to the introduction of the multidimensional Laplace transform [5]. Consider the change of variables :

$$\sigma_1 = t - \tau_1, \dots, \sigma_j = t - \tau_j$$

we obtain,

$$V_{jk_1 \dots k_j}(t) = \int_0^{\infty} \int_0^{\infty} \dots \int_0^{\infty} C e^{A\tau_j} N e^{A(\tau_{j-1} - \tau_j)} N \dots N e^{A(\tau_1 - \tau_2)} B \cdot e_{k_1}(t - \tau_1) \dots e_{k_j}(t - \tau_j) \delta_{-1}(\tau_{j-1} - \tau_j) \dots \delta_{-1}(\tau_1 - \tau_2) \cdot d\tau_1 \dots d\tau_j \quad (5.10)$$

where δ_{-1} is the unit step, and the e_k 's being zero for negative arguments. We shall introduce functions $\hat{V}_{jk_1 \dots k_j}(t_1, \dots, t_j)$ defined by

$$\hat{V}_{jk_1 \dots k_j}(t_1, \dots, t_j) = \int_0^{\infty} \dots \int_0^{\infty} C e^{A\tau_j} N e^{A(\tau_{j-1} - \tau_j)} N \dots N e^{A(\tau_1 - \tau_2)} B \cdot e_{k_1}(t_1 - \tau_1) \dots e_{k_j}(t_j - \tau_j) \cdot \delta_{-1}(\tau_{j-1} - \tau_j) \dots \delta_{-1}(\tau_1 - \tau_2) \cdot d\tau_1 \dots d\tau_j \quad (5.11)$$

Then, taking the j -dimensional Laplace transform we obtain

$$\hat{V}_{jk_1 \dots k_j}(s_1, \dots, s_j) = \bar{V}_j(s_1, \dots, s_j) E_{k_1}(s_1) \dots E_{k_j}(s_j) \quad (5.12)$$

where

$$\bar{V}_j(s_1, \dots, s_j) = \int_0^{\infty} \dots \int_0^{\infty} C e^{A\tau_j} N e^{A(\tau_{j-1} - \tau_j)} N \dots N e^{A(\tau_1 - \tau_2)} B \delta_{-1}(\tau_{j-1} - \tau_j) \dots \delta_{-1}(\tau_1 - \tau_2) \cdot e^{-(s_1 \tau_1 + \dots + s_j \tau_j)} d\tau_1 \dots d\tau_j \quad (5.13)$$

and

$$\hat{V}_{jk_1 \dots k_j}(s_1, \dots, s_j) = \int_0^{\infty} \dots \int_0^{\infty} \hat{V}_{jk_1 \dots k_j}(t_1, \dots, t_j) e^{-(s_1 t_1 + \dots + s_j t_j)} \cdot dt_1 \dots dt_j \quad (5.14)$$

We need the following lemma ([6]):

lemma 1:

The following relation holds between $V_{jk_1 \dots k_j}^{(s)}$ and $\hat{V}_{jk_1 \dots k_j}^{(s_1, \dots, s_j)}$

$$V_{jk_1 \dots k_j}^{(s)} = \frac{1}{(2\pi i)^{j-1}} \int_{p_1 - i\infty}^{p_1 + i\infty} \dots \int_{p_{j-1} - i\infty}^{p_{j-1} + i\infty} \hat{V}_{jk_1 \dots k_j}^{(s-s_1, \dots, s_{j-1}, s_1, s_2, \dots, s_{j-1})} ds_{j-1} \dots ds_1 \quad (5.15)$$

In the following, we shall be concerned with the evaluation of $V_{jk_1 \dots k_j}^{(s)}$.

From (5.8) we obtain

$$\bar{V}_j(s_1, \dots, s_j) = \int_0^\infty \dots \int_0^\infty C e^{A\sigma_j} N e^{A\sigma_{j-1}} \dots N e^{A\sigma_1} B \cdot e^{-\{s_1[\sigma_1 + \dots + \sigma_j] + s_2[\sigma_2 + \dots + \sigma_j] + \dots + s_j \sigma_j\}} \cdot d\sigma_1 \dots d\sigma_j$$

Therefore

$$\bar{V}_j(s_1, \dots, s_j) = C \{ (s_1 + \dots + s_j) I - A \}^{-1} N \{ (s_1 + \dots + s_{j-1}) I - A \}^{-1} \dots \cdot N \dots N \{ s_1 I - A \}^{-1} B \quad (5.16)$$

Now let $N = N_1 \cdot N_2$ where N_1 is $n \times m$ and N_2 is $m \times n$, and define $G_1(s)$, $G_2(s)$, $G_3(s)$ by

$$\begin{aligned} G_1(s) &= C \{ sI - A \}^{-1} N_1 \\ G_2(s) &= N_2 \{ sI - A \}^{-1} N_1 \\ G_3(s) &= N_2 \{ sI - A \}^{-1} B \end{aligned} \quad (5.17)$$

(5.11) becomes

$$\bar{V}_j(s_1, \dots, s_j) = G_1(s_1 + \dots + s_j) G_2(s_1 + \dots + s_{j-1}) \dots G_2(s_1 + s_2) G_3(s_1) \quad (5.18)$$

hence (5.7) and (5.10) yield

$$V_{jk_1 \dots k_j}^{(s)} = \frac{1}{(2\pi i)^{j-1}} \int_{p_1 - i\infty}^{p_1 + i\infty} \dots \int_{p_{j-1} - i\infty}^{p_{j-1} + i\infty} G_1(s) G_2(s-s_{j-1}) \dots$$

$$\begin{aligned} & \dots G_2(s-s_2 \dots -s_{j-1}) \cdot G_3(s-s_1 \dots -s_{j-1}) \cdot \\ & E_{k_1}(s-s_1 \dots -s_{j-1}) E_{k_2}(s_1) \dots E_{k_j}(s_{j-1}) \cdot \\ & ds_{j-1} \dots ds_1 \end{aligned} \tag{5.19}$$

therefore

$$\begin{aligned} V_{jk_1 \dots k_j}(s) &= G_1(s) \int_0^\infty \dots \int_0^\infty g_2(t_{j-1}) \dots g_2(t_2) g_3(t_1) e_{k_1}(t_0) \cdot \\ & \cdot \left\{ \frac{1}{(2\pi i)^{j-1}} \int_{p_1-i\infty}^{p_1+i\infty} \dots \int_{p_{j-1}-i\infty}^{p_{j-1}+i\infty} E_{k_2}(s_1) \dots E_{k_j}(s_{j-1}) \exp[-(s-s_{j-1})t_{j-1} + \right. \\ & \left. + \dots + (s-s_2 \dots -s_{j-1})t_2 + (s-s_1 \dots -s_{j-1})t_1 + \right. \\ & \left. + (s-s_1 \dots -s_{j-1})t_0] ds_{j-1} \dots ds_1 \right\} dt_{j-1} \dots dt_0 \end{aligned} \tag{5.20}$$

Thus

$$\begin{aligned} V_{jk_1 \dots k_j}(s) &= G_1(s) \int_0^\infty \dots \int_0^\infty g_2(t_{j-1}) \dots g_2(t_2) g_3(t_1) \cdot \\ & e_{k_1}(t_0) \left\{ \frac{1}{(2\pi i)^{j-1}} \int_{p_1-i\infty}^{p_1+i\infty} \dots \int_{p_{j-1}-i\infty}^{p_{j-1}+i\infty} E_{k_2}(s_1) \dots E_{k_j}(s_{j-1}) \right. \\ & \cdot \exp[s_{j-1}(t_{j-1} + \dots + t_0) + s_{j-2}(t_{j-2} + \dots + t_0) + \dots \\ & \left. + s_2(t_2 + t_1 + t_0) + s_1(t_1 + t_0)] ds_{j-1} \dots ds_1 \right\} \cdot \\ & \cdot \exp[-s(t_{j-1} + \dots + t_0)] dt_{j-1} \dots dt_0 \end{aligned} \tag{5.21}$$

Applying residue theory this will yield,

$$\begin{aligned} V_{jk_1 \dots k_j}(s) &= G_1(s) \int_0^\infty \dots \int_0^\infty g_2(t_{j-1}) \dots g_2(t_2) g_3(t_1) \cdot \\ & e_{k_1}(t_0) \left\{ \sum_{\ell_{k_j}=1}^{\ell_{k_j}} \dots \sum_{\ell_{k_2}=1}^{\ell_{k_2}} E_{k_j}^{-\ell_{k_j}} \dots E_{k_2}^{\ell_{k_2}} \cdot \exp[\alpha_{k_j}^{\ell_{k_j}} (t_{j-1} + \dots + t_0) + \right. \\ & \left. + \dots + \alpha_{k_2}^{\ell_{k_2}} (t_1 + t_0) - s(t_{j-1} + \dots + t_0)] \right\} dt_{j-1} \dots dt_0 \end{aligned} \tag{5.22}$$

hence we obtain (5.3).

Remark 1:

To obtain the actual matrices involved in the bilinear representation we proceed as follows [4]. Consider the matrix

$$G(s) = \begin{pmatrix} G_0(s) & G_1(s) \\ G_3(s) & G_2(s) \end{pmatrix} \quad (5.23)$$

Since all the elements in $G(s)$ are proper rational functions, $G(s)$ may be considered as the transfer function of a constant linear system of finite order with $m+1$ outputs and $m+1$ inputs. Therefore there exist three matrices A , R , S respectively $n \times n$, $n \times (m+1)$, $(m+1) \times n$ such that

$$S \{sI - A\}^{-1} R = G(s) \quad (5.24)$$

By partitioning S and R in the form

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, R = (R_1 \ R_2) \quad (5.25)$$

where S_1 is a $l \times n$ matrix and R_1 an $n \times 1$ matrix we obtain

$$\begin{aligned} G_0(s) &= S_1 \{sI - A\}^{-1} R_1 \\ G_1(s) &= S_1 \{sI - A\}^{-1} R_2 \\ G_2(s) &= S_2 \{sI - A\}^{-1} R_2 \\ G_3(s) &= S_2 \{sI - A\}^{-1} R_1 \end{aligned} \quad (5.26)$$

Now substitute $G_0(s), \dots, G_3(s)$ in (5.2) and (5.3), and let $C=S_1$, $B = R_1$, $N=R_2 S_2$. $\{A, N, B, C\}$ will constitute the bilinear realization.

Remark 2:

If we assume further that the basis functions have only single simple poles and that

$$\begin{aligned} \sum_{\ell \geq 0} V_{jk_1 \dots k_j \ell} E_\ell(s) / E_{k_j}^1 \dots E_{k_2}^1 E_1(s^{-\alpha_{k_j}^1} \dots -\alpha_{k_2}^1) \\ \text{and } \sum_{\ell \geq 0} V_{lk_1 \ell} E_\ell(s) / E_{k_1}^1(s) \end{aligned}$$

are independent of k_1 , $k_1 \geq 0$, we can obtain another characterization of a bilinear realization. For this, consider $\hat{V}(s) = \sum_{\ell \geq 0} V_{lk_1 \ell} E_\ell(s) / E_{k_1}^1(s)$ and

$$\begin{aligned} \tilde{V}_{k_2 \dots k_j}^{\sigma_1}(s, \sigma_1^{k_2}, \dots, \sigma_1^{k_j}) &= \sum_{\ell \geq 0} V_{jk_1 \dots k_j \ell} E_\ell(s) / E_{k_j}^1 \dots E_{k_2}^1 E_1(s^{-\alpha_{k_j}^1} \dots -\alpha_{k_2}^1), \quad j \geq 2 \end{aligned} \quad (5.27)$$

For fixed s , say \bar{s} , as k_2, \dots, k_j ranges from 0 to ∞ , $\tilde{v}_{k_2 \dots k_j}(\bar{s}, \alpha_{k_2}^1, \dots, \alpha_{k_j}^1)$ constitute an infinite sequence of complex numbers in $\bar{C} \cong P^1(C)$ (the Riemann sphere). Therefore it converges to $\tilde{v}_j(\bar{s}, s_2, \dots, s_j)$. Here $s_p = \lim_{k \rightarrow \infty} \alpha_{k_p} \in \bar{C}$.

Corollary:

A necessary and sufficient condition for a sequence $\{v_{j k_1 \dots k_j \ell}\}_{j \geq 1, k_1 \dots k_j \ell \geq 0}$ of numbers to be the 'generalized frequency response' of a bilinear system with zero initial condition (with respect to a given basis $\{e_k\}_{k \geq 0}$ of $L^2[0, \infty]$) is that

(i) $\hat{v}(s)$ is a strictly rational function

(ii) $\tilde{v}_j(s, s_3 - s_2, \dots, s_j - s_{j-1}, s - s_j)$

is a strictly proper rational recognizable function in s, s_2, \dots, s_j such that

$$\begin{aligned} \tilde{v}_j(s, s_3 - s_2, \dots, s_j - s_{j-1}, s - s_j) &= \\ &= G_1(s) G_2(s_3) \dots G_2(s_3) G_3(s_2), \quad j \geq 2 \end{aligned} \tag{5.28}$$

where $G_1(s), G_2(s), G_3(s)$ are matrices with dimensions respectively $l \times m, m \times m, m \times l$.

Proof:

(i) immediate

(ii) We have

$$\begin{aligned} \tilde{v}_{k_2 \dots k_j}(s, \alpha_{k_2}^1, \dots, \alpha_{k_j}^1) &= G_1(s) G_2(s - \alpha_{k_j}^1) \dots \\ &\dots G_2(s - \alpha_{k_j}^1 - \dots - \alpha_{k_3}^1) G_3(s - \alpha_{k_j}^1 - \dots - \alpha_{k_2}^1) \quad \text{for } j \geq 2. \end{aligned}$$

For fixed s , say \bar{s} , as k_2, \dots, k_j ranges from 0 to ∞ , the right and left side constitute sequences in \bar{C} , therefore they converge respectively to,

$$\begin{aligned} \tilde{v}_j(\bar{s}, s_2, \dots, s_j) \text{ and } G_1(\bar{s}) G_2(\bar{s} - s_j) \dots G_2(\bar{s} - s_j - \dots - s_3) \dots \\ \dots G_3(\bar{s} - s_j - \dots - s_2) \end{aligned}$$

hence

$$\begin{aligned} \tilde{v}_j(\bar{s}, s_2, \dots, s_j) &= G_1(\bar{s}) G_2(\bar{s} - s_j) \dots G_2(\bar{s} - s_j - \dots - s_3) \dots \\ &\dots G_3(\bar{s} - s_j - \dots - s_2) \end{aligned} \tag{5.29}$$

since \bar{s} was arbitrary,

$$\tilde{v}_j(s_1, s_2, \dots, s_j) = G_1(s) G_2(s-s_j) \dots G_2(s-s_j-\dots-s_3) \cdot G_3(s-s_j-\dots-s_2) \quad (5.30)$$

Therefore we have (5.28).

Remark 3:

Theorem 2 cannot, however, be extended to general non-linear system as the one given in (3.1) and (3.2) since we assumed the initial condition to be zero, whereas in (3.5) the initial tensor ϕ_0 is never the zero tensor even if the initial condition in the original system (3.1) and (3.2) is zero.

A similar result can be obtained however for bilinear systems when the initial state is arbitrary. The proof follows the same lines as the one given for the zero initial condition and shall be omitted. One remark can be made, however; there is a term involving t in the integrand, one can take it out and consider the Laplace transform of the product of two time functions, that is the frequency convolution of their Laplace transforms.

For simplicity we state the theorem for the case $B = 0$. The general case can be obtained *immediately by combining the following theorem with theorem 2:*

Theorem 3:

A necessary and sufficient condition for a sequence $\{W_{jk_1 \dots k_j \ell} \}_{j \geq 1, k_1 \dots k_j \ell \geq 0}$ of numbers, to be the 'generalized frequency response' of a bilinear system with non-zero initial condition (with respect to a given basis $\{e_k\}_{k \geq 0}$ of $L^2[0, \infty]$) is that there exist four matrices $G'_0(s)$, $G_1(s)$, $G_2(s)$, $G'_3(s)$ with dimensions respectively $l \times l$, $l \times m$, $m \times m$, $m \times l$ of strictly proper rational functions such that

$$(i) \quad \sum_{\ell \geq 0} W_{\ell} E_{\ell}(s) = G'_0(s) \quad (5.31)$$

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$$\begin{aligned}
 \text{(ii)} \quad \sum_{\ell \geq 0} W_{jk_1 \dots k_j} E_{\ell}^j(s) &= \sum_{k_j=1}^{r_{k_j}} \dots \sum_{k_1=1}^{r_{k_1}} E_{k_j}^{l_{k_j}} \dots E_{k_1}^{l_{k_1}} \\
 &\cdot G_1(s) G_2(s - \alpha_{k_1}^{l_{k_1}}) \dots G_2(s - \alpha_{k_1}^{l_{k_1}} - \dots - \alpha_{k_{j-1}}^{l_{k_{j-1}}}) \\
 &G_3(s - \alpha_{k_1}^{l_{k_1}} - \dots - \alpha_{k_j}^{l_{k_j}})
 \end{aligned} \tag{5.32}$$

5.C: Non-linear Systems:

Now, we can state a corollary for the general nonlinear system (3.1) and (3.2).

Corollary:

A necessary condition for a sequence $\{W_{ki_1 \dots i_k}^{l_{1,i_1} \dots l_{1,i_1} \dots l_{1,i_k} \dots l_{1,i_k}}\}_{k \geq 0}$ of numbers to be the 'generalized frequency response' of a general system of the form (3.1) and (3.2) (with respect to a given basis $\{e_k\}_{k \geq 0}$ of $L^2[0, \infty)$) is that there exist a tensor $G_0(s)$ and three sequences of infinite k-dimensional tensors $G_{1k}(s)$, $G_{2k}(s)$, $G_{3k}(s)$ whose components are analytic at infinity such that the following conditions hold for all indices:

$$\begin{aligned}
 \sum_{j \geq 0} W_{oj} E_j(s) &= G_0(s) \\
 \sum_{j \geq 0} W_{ki_1 \dots i_k}^{l_{1,i_1} \dots l_{1,i_1} \dots l_{1,i_k} \dots l_{1,i_k}} E_j(s) &= \\
 &= \sum_{j_1 \geq 0} \dots \sum_{j_k \geq 0} V_{i_1 j_1}^{l_{1,i_1} \dots l_{1,i_1}} \dots V_{i_k j_k}^{l_{1,i_k} \dots l_{1,i_k}} \\
 &\sum_{p_k=1}^{r_{k,j_k}} \dots \sum_{p_1=1}^{r_{1,j_1}} E_{j_k}^{p_k} \dots E_{j_1}^{p_1} G_{1k}^{k_1 \dots k_j}(s) G_{2k}^{k_1 \dots k_j}(s - \alpha_{j_1}^{p_1}) \dots \\
 &\dots G_{2k}^{k_1 \dots k_j}(s - \alpha_{j_1}^{p_1} - \dots - \alpha_{j_{k-1}}^{p_{k-1}}) G_{3k}^{k_1 \dots k_j}(s - \alpha_{j_1}^{p_1} - \dots - \alpha_{j_k}^{p_k})
 \end{aligned}$$

6. Conclusions:

In this paper we have introduced a generalized frequency response for a nonlinear system in terms of the response of the system to a set of standard inputs which

form a basis of $L^2[0, \infty]$. The theory has been illustrated in the linear, bilinear and general nonlinear cases. A realization theory has been presented in terms of the multidimensional Laplace transform. The method may be developed in a variety of ways and, in particular, we are now considering the application of these ideas in the design of nonlinear filters.

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