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Boundary Stabilization of the Reaction-Diffusion Equation with Unilateral Conditions

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Abstract

The stabilization of linear parabolic systems of the reaction-diffusion type is considered, in the presence of unilateral boundary conditions.

1. Introduction

In this paper we shall consider the linearized stability of the reaction-diffusion system

$$\frac{\partial\phi}{\partial t} = d\Delta\phi + f(\phi,\psi) \tag{1.1}$$

$$\frac{\partial\psi}{\partial t} = \Delta\psi + g(\phi,\psi)$$

(defined in a domain $\Omega\subseteq\mathbb{R}^n$ with boundary $\partial\Omega$ which is sufficiently smooth) in a neighbourhood[†] of a stationary solution $(\overline{\phi},\overline{\psi})$ where

$$f(\overline{\phi},\overline{\psi}) = g(\overline{\phi},\overline{\psi}) = 0 \tag{1.2}$$

subject to the unilateral boundary conditions

$$\phi=\overline{\phi}\ ,\ \psi=\overline{\psi} \text{ on } \Gamma_D\ ,\ \frac{\partial\phi}{\partial n} = 0 \quad\text{on}\quad \Gamma_N,$$

$$\frac{\partial\psi}{\partial n} = u \text{ on } \Gamma_N\setminus\tilde{\Gamma}_N\ ,\quad \psi\geq\overline{\psi},\ \frac{\partial\psi}{\partial n}\geq 0,\ (\psi-\overline{\psi})\frac{\partial\psi}{\partial n} = 0 \quad\text{on}\quad \tilde{\Gamma}_N$$

where

$$\Gamma_D,\Gamma_N\subseteq\partial\Omega\ ,\quad \tilde{\Gamma}_N\subseteq\Gamma_N\ ,\quad \Gamma_D\cap\Gamma_N = \emptyset\ ,$$

$$\text{meas }(\tilde{\Gamma}_N) \neq 0\ ,\quad \text{meas }(\partial\Omega\setminus[\Gamma_D\cup\Gamma_N]) = 0\ ,$$

$$\text{meas }(\Gamma_D) \neq 0\ .$$

It is known (Drábek, Kucera, 1986) that unilateral conditions have a destabalizing effect on the linearized system, and so it is important to consider whether this can be overcome with the application of boundary control.

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† In an appropriate topology - see later

We shall write

$$F_{11} = \frac{\partial f}{\partial \phi}(\overline{\phi},\overline{\psi}) \quad , \quad F_{12} = \frac{\partial f}{\partial \psi}(\overline{\phi},\overline{\psi}) \quad ,$$

$$F_{21} = \frac{\partial g}{\partial \phi}(\overline{\phi},\overline{\psi}) \quad , \quad F_{22} = \frac{\partial g}{\partial \psi}(\overline{\phi},\overline{\psi}) \quad ,$$

and then we obtain the linearized form of (1.1) (writing Φ,Ψ in place of $\phi-\overline{\phi},\psi-\overline{\psi}$):

$$\frac{\partial \phi}{\partial t} = d\Delta\phi + F_{11}\phi + F_{12}\psi \quad , \tag{1.3a}$$

$$\frac{\partial \psi}{\partial t} = \Delta\psi + F_{21}\phi + F_{22}\psi \tag{1.3b}$$

together with the unilateral conditions

$$\phi = \psi = 0 \quad \text{on } \Gamma_D \, , \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_N \, ,$$

$$\frac{\partial \psi}{\partial n} = u \quad \text{on } \Gamma_N \setminus \tilde{\Gamma}_N \quad , \quad \psi \geq 0, \; \frac{\partial \psi}{\partial n} \geq 0, \quad \psi \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \tilde{\Gamma}_N \, . \tag{1.4}$$

2. Abstract Formulation

Let V be the Hilbert space defined by

$$V = \{ \phi \varepsilon H^1(\Omega) : \phi = 0 \text{ on } \Gamma_D \, , \text{ in the trace sense} \} \, ,$$

and let $H = L^2(\Omega)$. Then V has the inner product

$$<\phi,\psi>_V = \int_\Omega \sum_{i=1}^{n} \phi_{x_i} \psi_{x_i} dx \quad , \quad \phi,\psi \varepsilon V.$$

If $<\cdot,\cdot>_H$ is the usual inner product on H, then we can write

$$<A\phi,\psi>_V = <\phi,\psi>_H = \int_\Omega \phi\psi dx \, ,$$

for all $\phi,\psi \varepsilon H$, where A is a completely continuous, positive symmetric operator on V.

From (1.3a) we have, for $\phi \varepsilon V$ and any $w \varepsilon V$,

$$\left\langle \frac{\partial\phi(t)}{\partial t}, w \right\rangle_H - \langle d\Delta\phi(t) + F_{11}\phi(t) + F_{12}\psi(t) , w\rangle_H = 0$$

and so

$$\left\langle \frac{\partial\phi(t)}{\partial t}, w \right\rangle_H + \langle d\phi(t) - F_{11}A\phi(t) - F_{12}A\psi(t), w\rangle_V = 0 \tag{2.1a}$$

for almost all $t \geq o$, by Green's formula (see Lions, 1971).

Now let $K \subseteq V$ be the closed convex cone

$$K = \{\bar{w} \varepsilon V: \bar{w} \geq o \text{ on } \tilde{\Gamma}_N \text{, in the trace sense}\}$$

in V. Then, from (1.3b) we have, for $\psi \varepsilon K$ and all $\bar{w} \varepsilon K$,

$$\left\langle \frac{\partial\psi(t)}{\partial t} , \bar{w} - \psi(t) \right\rangle_H + \langle \psi(t) - F_{21}A\phi(t) - F_{22}A\psi(t), \bar{w} - \psi(t)\rangle_V$$

$$- \int_{\Gamma_N \setminus \tilde{\Gamma}_N} u(\bar{w} - \psi(t))d\Gamma \geq 0 \tag{2.1b}$$

for almost all $t \geq o$, provided $u \varepsilon H^{-\frac{1}{2}}(\Gamma_N \setminus \tilde{\Gamma}_N)$.

In the case where the inequalities in the boundary conditions (1.4) are replaced by equalities we have the following equations corresponding to (2.1a,b):

$$\left\langle \frac{\partial\phi(t)}{\partial t}, w \right\rangle_H + \langle d\phi(t) - F_{11}A\phi(t) - F_{12}A\psi(t), w\rangle_V = 0 \tag{2.2a}$$

$$\left\langle \frac{\partial\psi(t)}{\partial t}, \bar{w} \right\rangle_H + \langle \psi(t) - F_{21}A\phi(t) - F_{22}A\psi(t) , \bar{w}\rangle_V$$

$$- \int_{\Gamma_N \setminus \tilde{\Gamma}_N} u\bar{w}d\Gamma = 0 \tag{2.2b}$$

for $w, \bar{w} \varepsilon V$.

We can write equations (2.1) and (2.2) on the cross product spaces $\bar{V} = V x V$, $\bar{H} = H x H$ with the inner products

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle_{\bar{V}} = \langle \phi_1, \phi_2 \rangle_V + \langle \psi_1, \psi_2 \rangle_V$$

$$\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle_{\bar{H}} = \langle \phi_1, \phi_2 \rangle_H + \langle \psi_1, \psi_2 \rangle_H$$

in the following way. If $\bar{K} = V\times K \subseteq \bar{V}$ then (2.1) becomes

$$\left\langle \frac{\partial\Phi(t)}{\partial t}, \Omega-\Phi(t) \right\rangle_{\bar{H}} + \left\langle D\Phi(t) - F\bar{A}\Phi(t), \Omega-\Phi(t) \right\rangle_{\bar{V}} \geq 0 \tag{2.3}$$

for all $\Omega\varepsilon\bar{K}$ and almost all $t>0$, where $\Phi(t)\varepsilon\bar{K}$ and

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \qquad D(d) = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{A}\Phi = (A\phi \;\; A\psi) \qquad , \qquad \Phi = (\phi, \psi)\varepsilon\bar{V}$$

Similarly, (2.2) becomes

$$\left\langle \frac{\partial\Phi(t)}{\partial t}, \Omega \right\rangle_{\bar{H}} + \left\langle D(d)\Phi(t) - F\bar{A}\Phi(t), \Omega \right\rangle_{\bar{V}} = 0, \tag{2.4}$$

for all $\Omega\varepsilon\bar{V}$ and almost all $t>0$.

3. Eigenvalue Placement.

Since the operator A is completely continuous, symmetric and positive, it has a spectrum consisting of eigenvalues λ_i, $i=1, 2, \ldots.$ such that $\ldots.\geq\lambda_i\geq\lambda_{i+1}\geq\ldots.>0$ and $\lambda_i\to 0$ as $i\to\infty$. The corresponding eigenvectors $\{e_i\}$ form a complete orthonormal system in V. Hence we can write, for any $(\phi, \psi)\varepsilon\bar{V}$,

$$\phi = \sum_{i=1}^{\infty} \langle\phi, e_i\rangle e_i \qquad , \qquad \psi = \sum_{i=1}^{\infty} \langle\psi, e_i\rangle e_i .$$

Consider first the case of zero control $u=0$. Then the eigenvalue problem corresponding to equation (2.4), i.e.

$$F\bar{A}\Phi - D(d)\Phi = \mu\bar{A}\Phi$$

is equivalent to the system of equations

$$\langle\phi, e_i\rangle(d-F_{11}\lambda_i+\lambda_i\mu) - \langle\psi, e_i\rangle F_{12}\lambda_i = 0$$

$$\langle\phi, e_i\rangle F_{21}\lambda_i - \langle\psi, e_i\rangle(1-F_{22}\lambda_i+\lambda_i\mu) = 0.$$

Following Drábek and Kučera, 1986, we see that μ is an eigenvalue of the operator $F\bar{A}-D(d)I$ if and only if μ is a root of the quadratic equation

$$\lambda_i^2\mu^2 - \mu\left[(F_{11}+F_{12})\lambda_i-(d+1)\right]+(d-F_{11}\lambda_i)(1-F_{22}\lambda_i)-F_{12}F_{21}\lambda_i^2 = 0.$$

In particular, d is a critical point of (2.4) (i.e. $\mu=0$ is an eigenvalue) if

$$d = \frac{F_{12}F_{21}\lambda_i^2}{1-F_{22}\lambda_i} + F_{11}\lambda_i.$$

Then the following lemma is proved easily (Drábek and Kučera, 1986):

<u>Lemma 3.1.</u> Let the coefficients $F_{11}, \ldots, F_{22}$ satisfy

$$F_{11}>0,\ F_{12}<0,\ F_{21}>0,\ F_{22}<0,\ F_{11}+F_{22}<0,$$

and suppose that the greatest critical point d_o of (2.4) is simple (i.e. the geometric eigenspace has dimension 1). Then there exist continuous functions $\mu:(d_o-r,d_o] \to R$, $\Phi:(d_o-r, d_o]\to\bar{V}$ (for some $r>0$) such that $\mu(d)$ is an eigenvalue of (2.4) with the corresponding eigenvector $\Phi(d)$, and we have $\mu(d)>0$ for all $d\varepsilon(d_o-r, d_o)$ and $\mu(d_o) = 0$. Moreover, for any $d\varepsilon(d_o-r, d_o]$, d is the greatest number for which $\mu(d)$ is an eigenvalue of (2.4). □

It follows from this lemma that there exists $\rho>o$ such that for any $d_1\varepsilon(d_o-\rho, d_o|$ we have

(i) if $d>d_1$ then all real eigenvalues of $F\bar{A}-DI-\mu(d_1)\bar{A}$ are negative, and

(ii) if $d\varepsilon(d_1-\xi, d_1)$ (some $\xi = \xi(d_1)>0$) then there is one simple positive eigenvalue of $F\bar{A}-D(d)I-\mu(d_1)\bar{A}$ and the other real eigenvalues are negative. (Moreover, the complex eigenvalues all have negative real parts.)

Then, under the conditions of lemma 3.1, the following theorem can be proved (Drábek et al, 1985):

<u>Theorem 3.2</u> Suppose that $\mu_1\varepsilon(0, F_{11}]$ is a simple eigenvalue of (2.4) with some $d_1>0$ and assume that the associated set of solutions of $(F\bar{A}-D(d_1)I-\mu_1\bar{A})\Phi = 0$

has an element in $\overline{K}^{o}$. Then there exists $d'>d_1$ such that μ_1 is an eigenvalue of the inequality

$$\langle D\Phi - F\overline{A}\Phi + \mu_1\overline{A}\Phi,\ \Omega-\Phi\rangle_{\overline{V}} \geq 0 \quad \text{for all } \Omega\varepsilon\overline{K},$$

with $d=d'$ and such that the associated solution set of this inequality has an element in $\partial\overline{K}$, but the solution set of the equality $(F\overline{A}-D(d')I-\mu_1\overline{A})\Phi=0$ consists of 0 alone. □

It is easy to show that, for $d=d'$ the linear equality (2.4) is stable but the inequality (2.3) is unstable. From the proof of this theorem it is also clear that if the condition

$$\left\{\begin{array}{l} \text{if } d\varepsilon(d_1-\xi,\infty)\text{, for some fixed } \xi>0\text{, then the operator} \\ \qquad\qquad F\overline{A}-D(d)I-\mu(d_1)\overline{A} \\ \text{has only eigenvalues with negative real parts} \end{array}\right.$$

holds, then both the equality (2.4) and the inequality (2.3) are stable.

We shall need the following lemma:

<u>Lemma 3.3</u> Let H_1 and H_2 be Hilbert spaces and let $\mu:H_1\to H_2$ be a linear, continuous surjective mapping with kernel W. Also, let $\{e_i\}$ be a basis of H_1 such that $e_k\notin W$ for some k. Then there exists an element h of H_2 such that

$$\langle h,\mu(e_k)\rangle\neq 0, \quad \langle h,\mu(e_i)\rangle=0 \quad , \quad i\neq k$$

<u>Proof</u>. By the open mapping theorem, $\overline{\mu}:H_1/W\to H_2$ defined by

$$\mu(\overline{h}_1) = \mu(h_1) \quad , \quad h_1\varepsilon\overline{h}_1\varepsilon H_1/W$$

is a linear isomorphism. Now, since $e_k\notin W$ the set $\{\overline{e}_1,\ldots,\hat{\overline{e}}_k,\ldots\}$ does not generate H_1/W and so the set $\{\overline{\mu}(\overline{e}_1),\ldots,\hat{\overline{\mu}(\overline{e}_k)},\ldots\}$ does not generate H_2. The result now follows since the set $\{\overline{\mu}(\overline{e}_1),\ldots,\overline{\mu}(\overline{e}_k),\ldots\}$ does generate H_2. □

<u>Corollary 3.4</u> Let V be the space

$$V=\{\phi\varepsilon H^1(\Omega):\ \phi=0 \quad \text{on } \Gamma_D \text{ in the trace sense}\}$$

and let A be the operator considered above with basis eigenvectors $\{e_i\}$. Suppose that $e_k(x)\neq 0$ for x belonging to a subset of Γ_N of nonzero measure. Then there exists a function $\zeta\varepsilon H^{-\frac{1}{2}}(\Gamma_N)$ such that

$$\int_{\Gamma_N} \zeta(x)e_i(x)d\Gamma = \delta_{ik} . \tag{3.1}$$

Proof. Since $\Gamma=\Gamma_D\cup\Gamma_N$ is a disjoint union we have the trace map

$$\phi\mapsto\phi|_{\Gamma_N}$$

from V to $\{\psi\varepsilon H^{-\frac{1}{2}}(\Gamma) : \psi|_{\Gamma_D} = 0\}$. This map is linear, continuous and surjective (Lions-Magenes (1972)) and has kernel $H_o^1(\Omega)$. The result follows easily from lemma 3.3. □

Remark

If $e_k(x)\neq 0$ on (a subset of measure >0 of) $\Gamma_N\setminus\tilde{\Gamma}_N$, then we can replace (3.1) by

$$\int_{\Gamma_N\setminus\tilde{\Gamma}_N} \zeta(x)e_i(x)d\Gamma = \delta_{ik} . \tag{3.2}$$

We can now prove our main result:

Theorem 3.5 Suppose there exists $\rho>0$ such that, for any $d_1\varepsilon(d_o-\rho,d_o]$, the conditions (i) and (ii) preceding theorem 3.2 hold and let e_k be the eigenvector of A corresponding to the simple positive eigenvalue of the operator $F\bar{A}-D(d)I-\mu(d_1)\bar{A}$. Suppose that $e_k(x)\neq 0$ on a subset of measure >0 of $\Gamma_N\setminus\tilde{\Gamma}_N$. Then the systems (2.3) and (2.4) can be stabilized by boundary feedback.

Proof Let $\zeta\varepsilon H^{-\frac{1}{2}}(\Gamma_N\setminus\tilde{\Gamma}_N)$ be a function satisfying (3.2). Then we define the feedback control u by

$$u = \alpha\zeta<\phi,e_k> + \beta\zeta<\psi,e_k> \tag{3.3}$$

for some real α,β. Substituting u and $w=e_i, 1\leq i<\infty$ into (2.2) we have the eigenvalue equation

$$\mu^2\lambda_k^2 - \mu[(F_{11}+F_{22})\lambda_k-(d+1)\lambda_k+\alpha\lambda_k]$$

$$+(d-F_{11}\lambda_k)\alpha +(d-F_{11}\lambda_k)(1-F_{22}\lambda_k) - (F_{21}\lambda_k-\beta)F_{12}\lambda_k= 0 .$$

By appropriate choice of α and β we can therefore move the spectrum of $F\bar{A} - D(d)I - \mu(d_1)\bar{A}$ into the left half-plane and the result follows, since the spectrum for $i \neq k$ is unaffected. □

4. Example Consider the reaction-diffusion system

$$\frac{\partial\phi}{\partial t} = d\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \phi - \psi$$

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \phi - 2\psi$$

on the rectangle $[o,1] \times [o,1]$ with boundary conditions

$\phi = \psi = 0$ on $\Gamma_D \triangleq (\{o\} \cup \{1\}) \times [o,1] \cup ([o,1] \times \{o\})$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad \Gamma_N = [o,1] \times \{1\}$$

$$\frac{\partial\psi}{\partial n} = u \quad \text{on} \quad \Gamma_N \setminus \tilde{\Gamma}_N ,$$

$$\psi \geq o \;, \quad \frac{\partial\psi}{\partial n} \geq o \;, \quad \psi\frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \tilde{\Gamma}_N$$

where

$$\tilde{\Gamma}_N = [o,\tfrac{1}{2}] \times \{1\}.$$

In this case we have

$$F_{11}=1,\; F_{12}=-1,\; F_{21}=1,\; F_{22}=-2$$

and so the system satisfies the conditions of lemma 3.1. The spectrum of the operator A is easily seen to be given by

$$\lambda_{mn} = \frac{1}{m\pi + (2n+1)\pi/2} \;, \qquad m=1,2,\ldots; \quad n=o,1,2,\ldots$$

The equation for the greatest critical value of d is therefore

$$d_c = \max_{m,n} \left\{ \frac{F_{12}F_{21}\lambda_{mn}^2}{1-F_{22}\lambda_{mn}} + F_{11}\lambda_{mn} \right\}$$

$$= \max_{m,n} \left\{ \frac{-\lambda_{mn}^2}{1+\lambda_{mn}} + \lambda_{mn} \right\}$$

$$= \max_{m,n} \left\{ \frac{\lambda_{mn}}{1+\lambda_{mn}} \right\}$$

$$= \frac{2}{3\pi + 2} ,$$

corresponding to the eigenvalue λ_{10}. Since the eigenvector of A with this eigenvalue is simple and does not vanish anywhere on Γ_N, theorem 3.5 shows that we can shift λ_{10} by boundary feedback to be smaller tha $2/3\pi$ so that d_c is no longer a critical value of the equation. Hence the system is stabilizable.

5. Conclusions

In this paper we have shown that, although the introduction of unilateral conditions into a systems of parabolic equations can be destabilizing, we can stabilize such systems by the proper use of boundary control. In fact we can choose a control which is 'orthogonal' on the boundary to the eigenvalues which have negative real parts, so that we affect only the eigenvalues with positive real parts. Although the main result has only been proved for a single simple eigenvalue with positive real part, the theory can be extended to systems with a finite number of such eigenvalues.

6. References

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