Monograph:
Banks, S.P. (1985) Stabilizibility of Finite and Infinite Dimensional Bilinear Systems. Research Report. Acse Report 283 . Dept of Automatic Control and System Engineering. University of Sheffield

Stabilizability of finite and infinite dimensional bilinear systems

S. P. Banks

Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield. S1 3JD.

Research Report No 283

Abstract

The stabilizability of bilinear finite- and infinite-dimensional systems is studied together with the stabilizability of a class of non-linear finite-dimensional systems. A variable-structure approach is considered which does not require the linear part of the system to be dissipative.

1. Introduction

The theory of bilinear systems has proved, in the last decase, to be an important generalization of the familiar linear systems theory with important contributions from many authors. For example, in the area of feedback and optimal control see Longchamp (1980), Gutman (1981), Tzafestas et al (1984), Banks and Yew (1985a,b). In particular Gutman (1981) has developed a class of stabilizing controls for bilinear systems. Results on the stabilizability of bilinear distributed parameter systems have been obtained by Slemrod (1978), Ball and Slemrod (1979) and Ball et al (1982), and Ryan and Buckingham (1983) have strengthened the results of Slemrod (1978) in the finite dimensional case.

In this paper we shall consider the stabilizability of the bilinear system.

$$\dot{x} = Ax + uBx \tag{1.1}$$

where x belongs to $\mathbb{R}^n$ or a Hilbert space and A and B are appropriately defined operators. In Ball and Slemrod (1979), it is assumed that A satisfies

$$\langle Ax,x\rangle \leq 0 \quad \forall\, x \in \mathcal{D}(A) ,$$

and then, formally, we may differentiate $V \triangleq \| x \|^2$ along the trajextories of (1.1) to obtain

$$\dot{V} = 2\langle x,Bx\rangle u + 2\langle Ax,x\rangle$$
$$\leq -2\langle x,Bx\rangle^2$$

if we choose the control

$$u = -\langle x,Bx\rangle.$$

In this paper we shall use the control

$$u = \frac{-2\langle x,Ax\rangle-1}{2\langle x,Bx\rangle} \tag{1.2}$$

to obtain

$$\dot{V} = -1$$

apart from when $\langle x,Bx\rangle = 0$. We shall then show that the set of points $\langle x,Bx\rangle = 0$, with appropriate assumptions on A, is a sliding mode for the system.

We shall also consider nonlinear systems of the form

$$\dot{x} = g_o(x) + ug_r(x) + \dots + u^m g_m(x)$$

and obtain a stabilizing feedback controller, generalizing the bilinear case. This will be related to the sets of points (which are assumed to be submanifolds of $\mathbb{R}^n$) which satisfy one or more of the equations

$$\langle x, g_i(x)\rangle = 0 \ , \quad i=1,\dots,m.$$

In the case of distributed parameter systems, because of the difficulty of (1.2) being defined only when $x \in \mathcal{D}(A)$, we shall use the spectral theorem to write $A+A^*$ in the form of an unbounded stable operator and an bounded unstable operator and then define the control

$$u = \frac{-\langle x, P(A+A^*)x\rangle - 1}{2\langle x, Bx\rangle} \tag{1.3}$$

where P is the projection on the unstable subspace. Since $P(A+A^*)$ is bounded (1.3) is defined for all $x \in H$ apart from when $\langle x, Bx\rangle = 0$. We shall then use straightforward existence theory together with the Lyapunov function $\| x \|^2$ to show that (1.3) is a stabilizing control for (1.1) on H.

Finally we shall present a number of simple examples to illustrate the theory. In particular we shall consider a bilinear hyperbolic system with a compact (integral) perturbation.

2. Notation and Terminology

In this paper we shall use standard terminology for finite and infinite dimensional vector spaces. In particular, to be consistent with the infinite-dimensional case, we shall denote the inner product of two vectors $x,y\in\mathbb{R}^n$ by $\langle x,y\rangle$ rather than $x^T y$. However, for the transpose of a matrix A we shall use the usual notation A^T, while the dual operator of a densely defined operator A on a Hilbert space will be denoted by A^*.

If M is a differentiable manifold, then ∂M will denote the boundary of M. Moreover, we shall use the well-known result that if $f:\mathbb{R}^n\to\mathbb{R}$ is a differentiable map such that grad $f \neq 0$ at each point of $f^{-1}(0)$, then $f^{-1}(0)$ is an n-1 dimensional submanifold of $\mathbb{R}^n$.

For a matrix A which is simmetric we shall use the fact that A can be diagonalized by an orthogonal matric P of eigencectors of A so that

$$PAP^{-1} = \Lambda$$

where Λ contains the eigenvalues of A along the diagonal. If A is any matrix, Ker A will denote the kernel of A, i.e. the set of vectors x such that $Ax = 0$.

If $p(x)=a_m x^m+\ldots+a_o x^o$ is a polynomial we shall use the discriminant of p, defined as the determinant

$$\mathcal{D}_x(p) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_o & & & & \\ & a_m & a_{m-1} & \cdots & a_o & & & \\ & & a_m & \cdots & & a_o & & \\ & & & \cdots & \cdots & & & \\ & & & & a_m & \cdots & a_o & \\ ma_m & (m-1)a_{m-1} & \cdots & a_2 & a_1 & & & \\ & ma_m & \cdots & & a_2 & a_1 & & \\ & & \cdots & \cdots & & & & \\ & & & & ma_m & (m-1)a_{m-1} & \cdots & a_1 \end{vmatrix}$$

and the discriminant variety of the points $y \in \mathbb{R}^n$ such that $\mathcal{D}_x(p) = 0$ in the case when the coefficients a_i depend on y.

Finally, in the case of distributed systems, we shall denote the Hilbert space of square integrable functions on $[0,1]$ by $L^2(0,1)$ and the corresponding Soboler spaces by $H^i(0,1)$, $H^i_0(0,1)$.

3. Stabilizability of Bilinear Systems

In this section we shall consider the bilinear system

$$\dot{x} = Ax + uBx \tag{3.1}$$

where $x \in \mathbb{R}^n$ and u is a scalar control. (A and B are, of course, nxn (constant) matrices). Let V be the usual scalar function

$$V = \langle x,x \rangle = ||x||^2$$

and differentiate V along the trajectories of (3.1). Then

$$\begin{aligned}\dot{V} &= \langle \dot{x},x \rangle + \langle x,\dot{x} \rangle \\ &= \langle Ax + uBx,x \rangle + \langle x,Ax + uBx \rangle \\ &= \langle (A + A^T)x,x \rangle + u \langle (B+B^T)x,x \rangle.\end{aligned}$$

Now let u be the control defined by

$$u = \frac{-\langle (A+A^T)x,x \rangle - 1}{\langle (B+B^T)x,x \rangle} \tag{3.2}$$

provided

$$Q(x) \triangleq \langle (B+B^T)\, x,x \rangle \neq 0. \tag{3.3}$$

and

$$u = 0, \tag{3.4}$$

if Q(x)=0. Then we have

$$\dot{V} = -1 \text{ if } Q(x) \neq 0 \tag{3.5}$$

Hence, if the quadratic form Q(x) is strictly positive (or strictly negative) definite, then the bilinear system (3.1) is globablly stabilizable with the control given by (3.2). In fact, since $\dot{V} = -1$ the origin is attained in finite time, depending, of course, on the initial condition.

In the case when Q(x) is not definite it is convenient to diagonalize Q(x) by introducing the transformation

$$y = Px$$

where P is an orthogonal matrix of eigenvectors of $B+B^T$. Then we have

$$\dot{y} = \tilde{A}y + u\tilde{B}y \tag{3.6}$$

where

$$\tilde{A} = PAP^{-1}, \quad \tilde{B} = PBP^{-1}$$

Let

$$\Lambda = \text{diag}\,\{\lambda_1,\ldots,\lambda_n\}$$

be the diagonal matrix of eigenvalues of $B+B^T$ (which are, of course, real). Assume these eigenvalues are ordered so that

$$\lambda_1,\ldots,\lambda_k > 0, \quad \lambda_{k+1},\ldots, \lambda_\ell < 0, \quad \lambda_{\ell+1},\ldots,\lambda_n = 0.$$

Then, in y-space, the bilinear form Q is given by

$$\begin{aligned} Q(y) &= \langle (\tilde{B} + \tilde{B}^T)y, y \rangle \\ &= \langle (PBP^{-1} + (P^{-1})^T B^T P^T)y, y \rangle \\ &= \langle P(B+B^T)P^{-1}y, y \rangle \\ &= \langle \Lambda y, y \rangle \end{aligned}$$

since P is an orthogonal matrix. Hence,

$$Q(y) = \lambda_1 y_1^2 + \ldots + \lambda_k y_k^2 - \lambda_{k+1} y_{k+1}^2 - \ldots - \lambda_\ell y_\ell^2 \tag{3.7}$$

Consider first the case when $n > k = \ell \geq 1$, so that

$$Q(y) = \lambda_1 y_1^2 + \ldots + \lambda_k y_k^2 \tag{3.8}$$

Then we define the control u by

$$u = \frac{-\langle(\tilde{A}+\tilde{A}^T)y, y\rangle - 1}{\sum_{i=1}^{k} \lambda_i y_i^2} \tag{3.9}$$

if $(y_1,\ldots,y_k) \neq (0,\ldots,0)$ and

$$u = 0$$

if $(y_1,\ldots,y_k) = (0,\ldots,0)$.

Now write $y = (y_p\ y_a)$ where

$$y_p = (y_1,\ldots,y_k), \quad y_a = (y_{k+1},\ldots,y_n)$$

then

$$\|y\|^2 = \|y_p\|^2 + \|y_a\|^2$$

and using the control (3.7) we have

$$\dot{V}(y) = \dot{\overline{\langle y,y\rangle}} = -1 , \quad (y_1,\ldots,y_k) \neq (0,\ldots,0)$$

It follows that $\| y_p \| \to 0$ in finite time and so we have

Lemma 3.1 If we partition $\tilde{A}$ corresponding to the partition (y_p, y_a) of y, i.e.

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{pp} & \tilde{A}_{pa} \\ A_{ap} & A_{aa} \end{pmatrix}$$

then the bilinear system

$$\dot{y} = \tilde{A}y + u\tilde{B}y$$

(and hence the system (3.1)) is stabilizable if the system

$$\dot{z} = \tilde{A}_{aa} z \tag{3.10}$$

is asymptotically stable. □

Next consider the case where

$$Q(y) = y_1^2 + \ldots + y_k^2 - y_{k+1}^2 - \ldots - y_\ell^2 \tag{3.11}$$

with $\ell > k > o$. (We have assumed, without loss of generality, that $\lambda_1 = \ldots = \lambda_\ell = 1$.) Since

$$\frac{\partial Q}{\partial y} = 2(y_1,\ldots,y_k,-y_{k+1},\ldots,-y_\ell)$$

we have $\frac{\partial Q}{\partial y_1} = 0$ only when $y_1 = 0$ and so

$$M \triangleq Q^{-1}(0) \setminus \{0_\ell \times \mathbb{R}^{n-\ell}\} \qquad (0_\ell = (0,\ldots,0) \in \mathbb{R}^\ell)$$

is a submanifold of $\mathbb{R}^n$ of dimension n-1 with two connected components given by

$$M_\pm : y_1 = \pm(y_{k+1}^2 + \ldots + y_\ell^2 - y_2^2 - \ldots - y_k^2)^{\frac{1}{2}} , \quad y_1 \neq 0 \tag{3.12}$$

for $y_{k+1}^2 + \ldots + y_\ell^2 > y_2^2 + \ldots + y_k^2$. Each connected part of M has the 'degenerate' boundary

$$\partial M = 0_\ell \times \mathbb{R}^{n-\ell} .$$

Lemma 3.2 If each of the two systems

$$\dot{z} = \tilde{A}_\pm z \quad , \quad z \in \mathbb{R}^{n-1}$$

is asymptotically stable, where $\tilde{A}_{\pm}$ is the projection of the vector field $y \to \tilde{A}y$ along the submanifold $\overline{M}_{\pm} = M_{\pm} \cup \partial M$ given by (3.12), then the bilinear system

$$\dot{y} = \tilde{A}y + u\tilde{B}y$$

is stabilizable.

Proof As in lemma 3.1, we can choose a control to drive the system to the submanifold $M_{\pm}$ in finite time. Once on $\overline{M}_{\pm}$ we switch the control off and follow the projected dynamics along $\overline{M}_{\pm}$, which is assumed to be a stable submanifold for the flow. □

In order to evaluate the projection $\tilde{A}_{\pm}z$ for the vector field $\tilde{A}y$ along the submanifold $\overline{M}_{\pm}$, in terms of the coordinates $y_2,\dots,y_n$ note that the normal vector to the level surfaces $Q(y) = c$ of Q is given by grad $Q(y)$ evaluated on the level surface. Now $Q(y)$ is given by (3.11) and

$$\text{grad } Q(y) = 2(y_1,\dots,y_k,-y_{k+1},\dots,-y_\ell) \tag{3.14}$$

so that the normal vector to $M_{\pm}$ is given by (3.14) with y_1 replaced by the right hand side of (3.12). In terms of $\tilde{B}$ we have

$$\text{grad } Q(y) = 2(\tilde{B}+\tilde{B}^T)y$$

and the submanifold ∂M is just $\text{Ker}(\tilde{B}+\tilde{B}^T)$. Hence the projection of $\tilde{A}y$ along $\overline{M}_{\pm}$ is given by

$$\tilde{A}_{\pm}z = \left.\left(\tilde{A}y - \langle \tilde{A}y, (\tilde{B}+\tilde{B}^T)y\rangle \frac{(\tilde{B}+\tilde{B}^T)y}{\|(\tilde{B}+\tilde{B}^T)y\|^2}\right)\right|_{M_{\pm}} \tag{3.16}$$

if $y \notin \text{Ker}(\tilde{B}+\tilde{B}^T)$ and

$$\tilde{A}_{\pm}z = \tilde{A}_{aa}y \quad , \quad y \in \text{Ker}(\tilde{B}+\tilde{B}^T) \tag{3.17}$$

where $\tilde{A}_{aa}$ is the submatrix of $\tilde{A}_{aa}$ corresponding to the partition (y_p,y_a) of y where $y_p=(y_1,\dots,y_\ell)$, $y_a=(y_{\ell+1},\dots,y_k)$ as in lemma 3.1. In (3.16) and (3.17) $z = (y_2,\dots,y_n)$ and y_1 is given by (3.12).

We can of course, return to x-coordinates and then it follows that we have

Theorem 3.3 Consider the bilinear system

$$\dot{x} = Ax+uBx \tag{3.18}$$

and let

$$B_s = B+B^T, \quad A_s = A+A^T$$

Let $\overline{M}$ be the union of the submanifolds $M_\pm$, ∂M defined by

$$<B_s x,x> = 0 \quad ,$$

as above. Then if $A_a z$ ($z \varepsilon M$) is the vector field given by

$$A_a z = Az - <Az,B_s z> \frac{B_s z}{\| B_s z\|^2} \quad , \quad z \notin \mathrm{Ker} B_s z$$

and

$$A_a z = P_{\partial m} Az \quad , \quad z \varepsilon \mathrm{Ker} B_s z$$

where $P_{\partial m}$ is the projection on ∂M , then the system (3.18) is stabilizable. A stabilizing controller is given by

$$u = \begin{cases} -(<A_s x,x>+1)/<B_s x,x> \quad , & x \notin \overline{M} \\ 0 \quad , & x \varepsilon \overline{M} \end{cases} \quad . \square$$

<u>Example 3.4</u> A simple example will illustrate the importance of the stability of the projected 'Ax dynamics' on the $<B_s x,x> = 0$ manifold. In fact, consider the system

$$\dot{x}_1 = A_1 x_1 + u x_1 \quad , \quad x_1 \varepsilon \mathbb{R}^k \quad , \quad x_2 \varepsilon \mathbb{R}^{n-k}$$
$$\dot{x}_2 = A_2 x_2$$

which is already in the 'cononical form' specified above, with

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \qquad B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

In this case

$$Q(x) \triangleq <(B+B^T)x, x> = 2\| x_1 \|^2$$

and M is the submanifold defined by $x_1 = 0$. The control

$$u = -(<A_{1s} x_1,x_1> + <A_{2s} x_2,x_2>+1)/\| x_1\|^2 \ , \ x_1 \neq 0 \qquad (3.19)$$

(with $A_{is} = A_i + A_i^T$, $i=1,2$) will drive the system to M in finite time. However, u can never affect x_2 and so the stability of A_2 is necessary in order that the system be stabilizable. (Of course, in (3.19) the term $<A_{2s} x_2,x_2>$ is irrelevant and can be omitted.)

This example shows that there is at least one class of systems for which the above conditions for stabilizability are also necessary. Namely, those for which $[A,B] = 0$ (i.e. A and B commute) and A,B diagonalizable with the eigenvalues of $B+B^T$ equal in sign (i.e. all ≥ 0 or all ≤ 0). For then A and B are simultaneously diagonalizable and we can then reduce the system to one of the form considered in example 3.4.

The above results can even be generalized to so-called linear-analytic systems of the form

$$\dot{x} = A(x) + uB(x) \tag{3.20}$$

where A(x) and B(x) are (nonlinear) analytic vector fields on $\mathbb{R}^n$. In this case we must consider the set of points defined by

$$f(x) \triangleq \langle B(x),x\rangle = 0$$

The set $f^{-1}(0)$ is a submanifold if grad f(x) has rank 1 at each point of $f^{-1}(0)$. Now

$$\text{grad } f(x) = B(x) + \left(\frac{\partial B(x)}{\partial x}\right)^T x .$$

Suppose that f^{-1} is the union of a finite number of submanifolds of dimension <n. Write $\overline{M}$ for this union as above. Of course, if grad $f(x) \neq 0$ for $x \in f^{-1}(0)$, then $f^{-1}(0)$ is a submanifold of dimension n-1. We are therefore assuming that the points x satisfying

$$\text{grad } f(x) = B(x) + \left(\frac{\partial B(x)}{\partial x}\right)^T x = 0$$

form a submanifold of dimension <n. As before, let $A_a(z)$ $(z \in \overline{M})$ denote the projection of the vector field A(x) along $\overline{M}$. Then, if the system

$$\dot{z} = A_a(z) \quad , \quad z \in \overline{M}$$

is asymptotically stable then the linear-analytic system (3.20) is stabilizable using the control

$$u = \frac{-2\langle A(x),x\rangle - 1}{2\langle B(x),x\rangle} \quad , \quad x \notin \overline{M}$$

$$u = 0 \quad , \quad x \in \overline{M}$$

4. Stabilizability of Nonlinear Systems

We shall now extend the theory of section 3 to general nonlinear systems of the form

$$\dot{x} = f(x,u). \qquad (4.1)$$

Theorem 3.3 can be generalized in the following way:

Lemma 4.1 Suppose that we can solve the equation

$$\langle x,f(x,u)\rangle = -1$$

for u, for all x apart from on a union M of submanifolds of $\mathbb{R}^n$, each of dimension $\leq n-1$. If the system (4.1) projected onto M is stable, then the system (4.1) is stabizable. (The projection of f on M is the component of f in the tangent bundle of M, which can be obtained from f by subtracting from f its projection along the normal bundle of M.)

Proof As in section 3 of the proof follows from the relation

$$\dot{V} = \langle x,f(x,u)\rangle = -1$$

where $V = \langle x,x\rangle$ which shows that we must hit the manifold M in finite time. □

The structure of M in lemma 4.1 is difficult to obtain in the general situation however, and so we shall now restrict attention to the case where f is a polynomial function in u. Then we can write

$$f(x,u) = f(x,0) + f'(x,0)u+\ldots+f^m(x,0)u^m/m!$$

for some m>0 , or

$$f(x,u) = g_o(x) + g_1(x)u +\ldots+ g_m(x)u^m$$

for some functions g_i , $o\leq i\leq m$.

Defining, as before, $V = \langle x,x\rangle$, we have

$$\dot{V} = 2\left[\langle x,g_o(x)\rangle +u\langle x,g_1(x)\rangle+\ldots+ u^m\langle x,g_m(x)\rangle\right] \qquad (4.2)$$

If each $g_i \varepsilon C^\infty(\mathbb{R}^n)$, then $\dot{V}$ is a polynomial in u with coefficients in $C^\infty(\mathbb{R}^n;\mathbb{R})$, i.e. $\dot{V}\varepsilon C^\infty(\mathbb{R}^n)[u]$. We can define the discriminant set of the polynomial $\dot{V}$ + 1 in the usual way, i.e. as the determinant

$$\mathcal{D}_u(\dot{V}+1) = \left| \begin{array}{cccccc} a_m & a_{m-1} & a_{m-2} & \cdots & a_o & \\ & a_m & a_{m-1} & \cdots & & a_o \\ & & a_m & \cdots & & \\ & & & \ddots & & \ddots \\ & & & a_m & a_{m-1} \cdots & a_o \\ ma_m & (m-1)a_{m-1} & \cdots & a_2 & a_1 & \\ & ma_m & \cdots & a_2 & a_1 & \\ & & \ddots & & \ddots & \\ & & & ma_m & (m-1)a_{m-1} \cdots & a_1 \end{array} \right| \qquad (4.3)$$

where there are m-1 rows containing $(a_o, a_1, \ldots, a_m)$ and m rows containing $(a_1, 2a_2, \ldots, (m-1)a_{m-1}, ma_m)$, and

$$a_i = 2\langle x, g_i(x)\rangle \quad , \ i>o, \ a_o = 2\langle x, g_o(x)\rangle + 1 \qquad (4.4)$$

It is well-known that $\dot{V} = -1$ has less than m solutions if and only if $\mathcal{D}_u(\dot{V}+1) = 0$. Of course, $\mathcal{D}_u(\dot{V}+1)$ is an element of $C^\infty(\mathbb{R}^n;\mathbb{R})$ and so we may consider the set of points $x \in \mathbb{R}^n$ such that

$$\mathcal{D}_u(\dot{V}+1) = 0. \qquad (4.5)$$

Call this set M_m. Also let R_m denote the set of points $x \in \mathbb{R}^n$ such that $\dot{V}+1$ has no real toots. (If m is odd then $R_m = \phi$.) We shall assume that $M_m \cup R_m$ is the union of a collection of submanifolds (containing the origin) of $\mathbb{R}^n$ each of dimension $\leq n-1$. Then we have

Theorem 4.2 Suppose that the projection of the system $\dot{x} = g_o(x)$ on each of the manifolds in $M_m \cup R_m$ is stable. Then (4.1) is stabilizable and off $M_m \cup R_m$ we can choose a control which is a real solution of $\dot{V}+1=0$ where $\dot{V}$ is defined by (4.2)

Proof The proof is the same as that for lemma 4.1. □

Corollary 4.3 If m is odd and the projection of $\dot{x}=g_o(x)$ along the discriminant variety is stable, then the system itself is stabilizable. □

Of course, we may not have to require stability of the projected dynamics along the whole discriminant variety. For, note that the set

$$Z_m = \{x\varepsilon\mathbb{R}^n : a_m = <x,g_m(x)>= 0 \}$$

is clearly contained in M_m. It is then clearly sufficient for the projected dynamics along $Z_m \cup R_m$ to be stable. Now, on Z_m we have

$$\dot{V} = 2\left[<x,g_o(x)>+u<x,g_1(x)>+\ldots+u^{m-1}<x,g_{m-1}(x)>\right]. \tag{4.6}$$

Define Z_{m-1} and R_{m-1} for this polynomial in u of order m-1, and we did for Z_m,R_m with respect to the polynomial in (4.2). Then it is sufficient for stabilizability that the projected dynamics along $R_m \cup (Z_m \cap (Z_{m-1} \cup R_{m-1}))$ are stable. Continuing the argument in this way we have

Theorem 4.3 Consider the nonlinear system

$$\dot{x} = g_o(x)+ug_1(x)+\ldots+u^m g_m(x) \ , \tag{4.7}$$

and assume that the sets $Z_m,Z_{m-1},\ldots,Z_1$ are unions of submanifolds of $\mathbb{R}^n$ each containing the origin, and that the same is true of the sets

$$R_{2[m/2]} \ , \quad R_{2\left[\frac{m-2}{2}\right]}, \ldots, \ R_2$$

where Z_i,R_i are as defined above. (Since $R_i=\phi$ if i is odd we need consider only the sets R_i for i even). Then the system (4.7) is stabilizable if the system defined by the projection of the vector field $x\to g_o(x)$ along the submanifolds in

$$R_m \cup (Z_m \cap (\ldots(R_4 \cup (Z_4 \cap Z_3 \cap (R_2 \cup (Z_2 \cap Z_1))))\ldots)) \quad , \quad m \text{ even}$$

$$Z_m \cap (\ldots(R_4 \cup (Z_4 \cap Z_3 \cap (R_2 \cup (Z_2 \cap Z_1))))\ldots) \quad , \quad m \text{ odd}$$

Moreover, we may choose a sequence of controls as follows:

if $x_o \notin R_m \cup Z_m$ let $u(t)$ be a real solution of $\dot{V} = -1$, until

$x(t) \in R_m \cup Z_m$ for $t=t_1$, say

if $x(t_1) \notin R_{m-1} \cup Z_{m-1}$ let $u(t)$ be a real solution of $\dot{V} = -1$, where

$\dot{V}$ is given by (4.6) until $x(t) \in R_{m-1} \cup Z_{m-1}$ for $t=t_2$, say.

. . . .

if $x(t_{m-1}) \notin Z_m$ choose $u(t) = \dfrac{-2<x,g_o(x)>-1}{2<x,g_1(x)>}$. □

Corollary 4.4 If each R_i is empty (so that each polynomial in u of order 1 to m has a real root) then for stabilizability of (4.7) it is sufficient that $\bigcap_{i=1}^{m} Z_i$ is a union of submanifolds (each containing zero) and that along each of these submanifolds, the projection of the vector field $\vec{g}_o(x)$ defines a stable system. □

We can consider the quadratic case in detail. Hence suppose that x satisfies the equation

$$\dot{x} = g_o(x) + ug_1(x) + u^2 g_2(x) \; . \tag{4.8}$$

Then if $V = <x,x>$ we have

$$\dot{V} = 2\{<g_o(x),x> + u<g_1(x),x> + u^2<g_2(x),x>\}$$

The discriminant of the polynomial $a_2u^2+a_1u+a_o$ is just the familiar form $-a_2(a_1^2-4a_oa_2)$ and so $\dot{V}+1$ has less than two roots if and only if

$$<g_2(x),x> = 0 \quad \text{or} \quad <g_i(x),x>^2 = 4(<g_o(x),x>+\tfrac{1}{2})<g_2(x),x>$$

Suppose that when $<g_2(x,x)> \neq 0$ we have

$$<g_1(x),x>^2 \geq 4(<g_o(x),x>+\tfrac{1}{2})<g_2(x),x>$$

(Of course, the number 1 in $\dot{V}+1$ is arbitrary and can be replaced by any $\varepsilon>0$). Then, off the set $<g_2(x),x>= 0$, we can choose either of the controls

$$u = \frac{-<g_1(x,x)>\pm\sqrt{<g_1(x),x>^2-4(<g_o(x),x>+\frac{1}{2})<g_2(x),x>}}{2<g_2(x),x>}$$

On the set $(\langle g_2(x),x\rangle = 0) \cap (g_1(x),x\rangle \neq 0)$ we can choose the control

$$u = \frac{-2\langle g_1(x,x)\rangle - 1}{2\langle g_1(x),x\rangle}$$

Finally if the set $M = (\langle g_2(x),x\rangle = 0) \cap (\langle g_1(x),x\rangle = 0)$ is a union of submanifolds, each containing the origin and the projection of the system $\dot{x} = g_o(x)$ along M is stable, then the system (4.8) is stabilizable.

5. Infinite Dimensional Systems

In this section we would like to extend lemma 3.1 obtained above to certain classes of distributed parameter systems. Since we have defined a discontinuous feedback control which produces a sliding mode in the finite dimensional case, we come to a basic problem in the infinite dimensional case - that is, the existence of solutions of systems with discontinuous right hand sides. Before discussing this problem we shall consider first an extension of the spectral results of section 3.

Let

$$\dot{x} = Ax + uBx \tag{5.1}$$

be a system defined on a Hilbert space and assume that A is an (unbounded) self-adjoint operator from $\mathcal{D}(A)$ into H and $B \in \mathcal{B}(H)$. Proceeding formally, as above, we would like to define a Lyapunov function

$$V = ||x||^2 \tag{5.2}$$

and differentiate V along the trajectories of (5.1). Then

$$\begin{aligned} \dot{V} &= \langle x,\dot{x}\rangle + \langle\dot{x},x\rangle \\ &= \langle x,Ax + uBx\rangle + \langle Ax + uBx,x\rangle \\ &= \langle (A + A^*)x,x\rangle + u\langle (B + B^*)x,x\rangle \end{aligned}$$

and then we define

$$u = \frac{-\langle (A + A^*)x,x\rangle - 1}{\langle (B + B^*)x,x\rangle}$$

as before. The first point to notice is that we must have $x \in \mathcal{D}(A)$ for (5.3) to be defined.

To be more specific let us first recall the spectral theorem for an (unbounded) self-adjoint operator (see, for example, Helmberg, Yosida, 1974, or Dunford and Schwartz, 1963).

Theorem 5.1 (The Spectral Theorem). Let C be a selfadjoint operator on a Hilbert space H. Then there exists a family of projections $\{P(\lambda):\lambda\in\mathbb{R}\}$, such that

(i) $P(\lambda) \leq P(\lambda')$, for $\lambda\leq\lambda'$

(ii) $(s)\lim_{\lambda\to-\infty} P(\lambda) = 0$, $(s)\lim_{\lambda\to+\infty} P(\lambda) = I$

(iii) $P(\lambda+0) = P(\lambda)$, for all $\lambda\in\mathbb{R}$

(iv) $h\in\mathcal{D}(C)$ if and only if $\int_{-\infty}^{\infty} \lambda^2 d\|P(\lambda)h\|^2 <\infty$,

and then

$$Ch = \int_{-\infty}^{\infty} \lambda dP(\lambda)h \qquad \text{for all } h\in\mathcal{D}(C). \ \square$$

Since A is self-adjoint we can find such a family of projections for A which we denote by $P_A(\lambda)$, $\lambda\in(-\infty,\infty)$ and we can also find a family of projections $P_B(\lambda)$ for the 'symmetrized' form $B+B^*$ of B. We shall assume that $B+B^*$ is a positive operator, so that the (real) spectrum of $B+B^*$ is an interval $[\alpha,\beta]\subseteq\mathbb{R}$, $0<\alpha\leq\beta<\infty$. Now suppose that the spectrum of A is separated so that

$$\sigma(A)\subseteq(-\infty, -\varepsilon)\cup[\gamma,\delta]$$

where $\varepsilon>0$, $0<\gamma\leq\delta<\infty$. Then we can write

$$Ah = \int_{-\infty}^{-\varepsilon} \lambda dP_A(\lambda)h + \int_{\gamma-0}^{\delta} \lambda dP_A(\lambda)h , \qquad h\in\mathcal{D}(A) .$$

Note that $P_A(\lambda)$ is constant on $[-\varepsilon,\gamma)$ (with a jump at γ). Let P_ε^A denote this constant projection. Similarly we write

$$(B + B^*)h = \int_{\alpha-0}^{\beta} \lambda dP_B(\lambda)h , \qquad h\in H$$

and we let $P_\alpha^B = P_B(\alpha)$. Assume that

$$I - P_\varepsilon^A \leq P_\alpha^B . \tag{5.4}$$

Now, in (5.3) the denominator is given by

$$<(B + B^*)x,x> = \int_{\alpha-o}^{\beta} \lambda d\langle P_B(\lambda)x,x\rangle$$

$$\geq <P_\alpha^B x,x> , \qquad (5.5)$$

and so we consider the following orthogonal splitting of the state:

$$x = x_1 + x_2 \quad ,$$

where

$$x_1 = \bar{P}_\varepsilon^A x \triangleq (I-P_\varepsilon^A)x , \quad x_2 = P_\varepsilon^A x .$$

Since the integrals in

$$Ax = \int_{-\infty}^{-\varepsilon} \lambda dP_A(\lambda)x + \int_{\gamma-o}^{\delta} \lambda dP_A(\lambda)x \quad , \quad x\varepsilon\mathcal{D}(A)$$

are given by Riemann-Stieltjes sums, as simple limit argument, together with property (i) in theorem (5.1) shows that

$$P_\varepsilon^A Ax = \int_{-\infty}^{-\varepsilon} \lambda dP_A(\lambda)x \quad , \quad x\varepsilon\mathcal{D}(A)$$

and so

$$\bar{P}_\varepsilon^A Ax = \int_{\gamma-o}^{\delta} \lambda dP_A(\lambda)x \quad , \quad x\varepsilon\mathcal{D}(A)$$

Thus, from (5.1)

$$\begin{aligned} \dot{x}_1 &= \bar{P}_\varepsilon^A Ax+u \bar{P}_\varepsilon^A Bx \\ \dot{x}_2 &= P_\varepsilon^A Ax+u P_\varepsilon^A Bx \end{aligned} \qquad (5.6)$$

Now choose the control u to be given by (5.3) on $\mathcal{R}(\bar{P}_\varepsilon^A)$ and to be zero on $\mathcal{R}(P_\varepsilon^A)$. Informally, as in section 3, we see that x approaches the subspace

$$<(B + B^*)x,x> = 0 \qquad (5.7)$$

in finite time. By (5.5),

$$\begin{aligned} <(B + +^*)x,x> &\geq ||P_\alpha^B x||^2 \\ &\geq ||\bar{P}_\varepsilon^A x||^2 , \end{aligned}$$

by (5.4). Hence on the subspace (5.7) we have $x_1 = 0$, and so on this subspace (with zero control), (5.6) implies that

$$x_1 = 0$$

$$\dot{x}_2 = P^A_\varepsilon Ax = AP^A_\varepsilon x = Ax_2$$

The latter equation can be written

$$\dot{x}_2 = \int_{-\infty}^{-\varepsilon} \lambda dP_A(\lambda)x_2 \, . \tag{5.8}$$

If we now assume that A gnerates a semigroup which satisfies the spectrum determined growth assumption (see, Banks, 1983, Curtain and Pritchard, 1978) i.e. the semigroup T(t) generated by A is stable if and only if the spectrum of A is in the left half plane, then the system (5.1), under the above assumptions, is stabilizable.

We must now justify this result by showing that the system

$$\dot{x} = Ax - \left\{ \frac{2\langle Ax,x\rangle+1}{\langle (B+B^*)x,x\rangle} \right\} Bx \quad \text{for} \quad \langle (B+B^*)x,x\rangle \neq 0$$

$$\dot{x} = Ax \quad \text{for} \quad \langle (B+B^*)x,x\rangle = 0$$

has a solution, with $x(t)\varepsilon\mathcal{D}(A)$ for all t. However, to prove that this system has a solution is not particularly easy because of the term $\langle Ax,x\rangle$ in the control and so we shall define the control to be given by, instead of (5.3), the following:

$$u = -\left[\frac{2\langle \overline{P}^A_\varepsilon Ax,x\rangle + 1}{\langle (B+B^*)x,x\rangle} \right] \quad , \quad \text{for } \langle (B+B^*)x,x\rangle \neq 0$$

and u = o otherwise. Here,

$$\overline{P}^A_\varepsilon Ax = \int_{\gamma-o}^{\delta} \lambda dP_A(\lambda)x, \qquad x\varepsilon\mathcal{D}(A) \; ,$$

and the right hand side defines a bounded operator and so is, in fact, valid for all $x\varepsilon H$. Using this control we obtain

$$\dot{V} = \overline{\dot{<x,x>}} = 2<\int_{-\infty}^{-\varepsilon}\lambda dP_A(\lambda)x,x> -1$$

$$\leq -1 ,$$

and the subspace $<(B+B^*)x,x>$ is still attracting in finite time

Definition A function $x\varepsilon C([0,t_1];H)$ is a weak solution of the equation

$$\dot{x} = Ax + f(x,t)$$

on $[o,t_1]$ if $f(x(.),.)\ \varepsilon L^1(o,t_1;H)$ and if for all $h\varepsilon\mathcal{D}(A^*)$ the function $<x(t),h>$ is absolutely continuous on $[o,t_1]$ and satisfies

$$(d/dt)<x(t),h> = <x(t),A^*h> + <f(x(t),x),h>,$$

for almost all $t\varepsilon[o,t_1]$.

Theorem 5.2 If B is a bounded operator and A satisfies the above assumptions and, moreover, generates a semigroup T(t) on H, then the system

$$\dot{x} = Ax - \left\{\frac{2<\overline{P}_\varepsilon^A Ax,x>+1}{<(B+B^*)x,x>}\right\} Bx \quad , \quad <(B+B^*)x,x> \neq 0 \qquad (5.9)$$

$$\dot{x} = Ax \quad , \quad <(B+B^*)x,x> = 0$$

has a unique weak solution which converges to 0 as $t\to\infty$.

Proof For $x_o \underline{\underline{\Delta}} x(0)\varepsilon \mathcal{M}$, where $\mathcal{M} = \{x\varepsilon H:<(B+B^*)x,x>=0\}$, the result is clear, by using (5.8).

Consider the case when $x(0)\notin \mathcal{M}$. If $x(t)\notin\mathcal{M}$ for $t\varepsilon[o,\tau]$, then we may write (5.9) in the 'mild form':

$$x(t) = T(t)x_o - \int_o^t T(t-s)\left\{\frac{2<\overline{P}_\varepsilon^A Ax(s),x(s)>+1}{<(B+B^*)x(s),x(s)>}\right\} Bx(s)ds. \qquad (5.10)$$

An elementary limit argument shows that (Ball, 1978)

$$\| x(t) \|^2 \leq \| x_o\|^2 - t \qquad (5.11)$$

and so $x\to\mathcal{M}$ in finite time provided the solution exists. Since $\overline{P}_\varepsilon^A A$ is a bounded operator, and since $\| x(t) \|^2$ is decreasing,

$$f(x(.)) = \frac{\langle \overline{P}^A_\varepsilon Ax, x\rangle + 1}{\langle (B+B^*)x, x\rangle} \tag{5.12}$$

is in $L^1[o,\tau]$ and so a function $x(.)$ is a weak solution of (5.9) on $[o,\tau]$ if and only if it is a mild solution of (5.10) (see Ball, 1978, Balakrishan, 1976).

Now it is easy to check that, for $x \notin M$, the map f in (5.12) is locally Lipschitz and so by a standard result the system (5.10) has a unique solution on any interval $[o,\tau]$ such that $x \notin M$. Moreover, $x \varepsilon C([o,\tau];H)$. From (5.11) it follows that there must be some minimal time τ_m, say, such that $x \varepsilon M$ when $t=\tau_m$ and $x\varepsilon C([o,\tau];H)$ for any $t<\tau_m$. Let $t_1, t_2, \ldots$ be any sequence such that $t_i \to \tau_m$. Then, by (5.11), $x(t_i)$ converges weakly to x_m. If we define $x(\tau_m) = x_m$ then the result follows. □

Remark (a) If the operator A splits in the form

$$Ax = \int_{-\infty}^{o} \lambda dP_A(\lambda) + \int_{\gamma-o}^{\delta} \lambda dP_A(\lambda) ,$$

i.e. with $\varepsilon = o$ then we would have to use a variant of the invariance principal in the proof of theorem 5.2, as in Ball 1978, Ball and Slemrod (1979).

(b) Theorem 5.2 is also valid if A is not necessarily self-adjoint - we then simply use the spectral respresentation of A+A* as in the finite dimensional case.

6. Examples

In this section we shall present some simple examples to illustrate the above theory.

Example 6.1 Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} x + u \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} x = Ax + uBx \tag{6.1}$$

Then A and B commute and are diagonalizable and hence can be diagonalized simultaneously. Thus, a simple change of coordinates produces the system

$$\dot{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y + \quad u \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} y = A'y + uB'y, \; y = Px, \text{ some } P. \tag{6.2}$$

Define the control by

$$\begin{aligned} u &= \frac{-2\langle A'y,y\rangle - 1}{2\langle B'y,y\rangle} \\ &= \frac{-2(y_1^2 - y_2^2) - 1}{2(4y_1^2 - y_2^2)} \end{aligned} \tag{6.3}$$

if $4y_1^2 - y_2^2 \neq 0$, and

$$u = 0 \tag{6.4}$$

if $2y_1 = \pm y_2$. Then set

$$\{(y_1,y_2)\in\mathbb{R}^2 : 2y_1 = \pm y_2\}$$

consists of the two submanifolds with $2y_1=y_2$, $2y_2=-y_2$ and it is easy to check that the projection of the vector field $y\to A'y$ on each of these submanifolds is stable. Each of the submanifolds is a sliding mode for the system defined by (6.2), (6.3) and (6.4). Translating back to x- coordinates gives the control

$$u = \frac{-2\langle PAP^{-1}x,x\rangle - 1}{2\langle PAP^{-1}x,x\rangle}$$

if $\langle PAP^{-1}x,x\rangle \neq 0$ and

$$u = 0$$

otherwise.

<u>Remark</u> We can obviate the difficulty of the unbounded control (i.e. $u\to\infty$ as x approaches the switching manifolds) by replacing the two switching manifolds $2y_1=\pm y_2$ by the four manifolds

$$\{(y_1,y_2)\in\mathbb{R}^2 : 2y_1=\pm(1\pm\varepsilon)y_2\}$$

for small ε. In the four regions where

$$2y_1(1+\varepsilon)^{-1} < y_2 < 2y_1(1-\varepsilon)^{-1} \quad , \quad y_1>o, y_2>o$$
$$2y_1(1-\varepsilon)^{-1} < y_2 < 2y_1(1+\varepsilon)^{-1} \quad , \quad y_1<o, y_2<o$$
$$-2y_1(1+\varepsilon)^{-1} < y_2 < -2y_1(1-\varepsilon)^{-1} \quad , \quad y_1<o, y_2>o$$
$$-2y_1(1-\varepsilon)^{-1} < y_2 < -2y_1(1+\varepsilon)^{-1} \quad , \quad y_1>o, y_2<o$$

we turn off the control and follow the linear trajectories (which are decaying in these regions).

Example 6.2 Suppose that in the system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x + u \begin{pmatrix} 0 \\ x_2 - g(x_1) \end{pmatrix} = Ax + uB(x) \qquad (6.5)$$

the graph of $g(x_2)$ in the (x_1, x_2) plane lies in the cone

$$\{(x_1,x_2) : (x_1,x_2) = 0 \text{ or } o<x_2<x_1(1-\varepsilon) \text{ or } x_1(1-\varepsilon)<x_2<o\}$$

Then we can define the control

$$u = \frac{-2(-x_1^2+x_2^2) - 1}{2(x_2-g(x_1))x_2} \qquad (6.6)$$

if $(x_1-g(x_2))x_2 \neq 0$ and

$$u = 0$$

if $x_2 = 0$ or $x_2 = g(x_1)$. Again the projection of $x \to Ax$ on the submanifolds

$$M_1 = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\}, \quad M_2 = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = g(x_1)\}$$

is stable and so the system (6.5) is stabilizable with the control (6.6). As in the above remark, by perturbing M_1 and M_2 we can use bounded controls.

Example 6.3 Consider the system

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + u \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} x + u^3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x$$

This is similar to the system (6.0) with an extra term in u^3. In this case we can define the control to be any real solution of

$$2(x_1^2 - x_2^2) + 2u(4x_1^2 - x_2^2) + 2u^3x_2^2 = -1$$

if $x_2 \neq 0$. This will produce a sliding mode on the x_1 axis, along which we can define the control

$$u = \frac{-2x_1^2 - 1}{8x_1^2}$$

(as in (6.3) with $x_2 = 0$).

Example 6.4 An example for the distributed parameter case can be given for the following hyperbolic equation:

$$\frac{\partial^2\phi(x,t)}{\partial t^2} = \frac{\partial^2\phi(x,t)}{\partial x^2} - \alpha \cdot \frac{\partial\phi(x,t)}{\partial t} + \int_o^1 k(x,y)\phi(y,t)dy + u\phi(x,t) \qquad (6.7)$$

$$\phi(o) = \phi(1) = 0$$

Then if $\Phi = (\phi, \partial\phi/\partial t)$, we can write this equation in the form

$$\frac{d\Phi}{dt} = \begin{pmatrix} 0 & I \\ A+K & -\alpha \end{pmatrix}\Phi + u\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\Phi \qquad (6.8)$$

where

$$(K\phi)(x) = \int_o^1 k(x,y)\phi(y,t)dy$$

Equation (6.8) is defined on the Hilbert space $H=H_o^1(0,1)\oplus L^2(0,1)$ with the inner product

$$<(\phi_1,\psi_1),(\phi_2,\psi_2)>_H = <(-A)^{\frac{1}{2}}\phi_1,(-A)^{\frac{1}{2}}\phi_2>_{L^2} + <\psi_1,\psi_2>_{L^2} .$$

If

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & -\alpha \end{pmatrix}$$

then $\mathcal{D}(\mathcal{A}) = (H_o^1(0,1) \cap H^2(0,1)) \oplus H_o^1(0,1)$. Also

$$<\Phi,\mathcal{A}\Phi>_H = -\alpha||\psi||^2 \qquad (6.9)$$

$$<\Phi,\mathcal{A}^*\Phi>_H = -\alpha||\psi||^2$$

where $\Phi = (\phi, \psi) \varepsilon \mathcal{D}(\mathcal{A})$. As is well-known (Banks, 1983) it follows that generates a stable semigroup. Moreover the operator $\mathring{\mathcal{A}} + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ is a bounded perturbation of $\mathcal{A}$ and so it too generates a (not necessarily) stable semigroup. The dual of $\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 \\ K^* & 0 \end{pmatrix}$ and so condition (5.4) holds trivially in this case since $P^B_\alpha = (0 \quad I)^T$ (with $\alpha = 1$). Hence by theorem 5.2 and (6.9) the system (6.7) is stabilizable.

7. Conclusions

In this paper we have considered the stabilizability of a general bilinear system

$$\dot{x} = Ax + uBx$$

in finite and infinite-dimensional spaces. The stabilizing feedback controller has been defined in such a way that the resulting system is of the variable-structure type with a stable sliding mode on the subspace of the state space defined by

$$<(B+B^*)x,x> = 0.$$

It has been seen that this leads to unbounded controls in the neighbourhood of this set, but in many cases a simple perturbation of the switching manifolds leads to a bounded controller.

In the finite-dimensional case we have also discussed the nonlinear control system of the form

$$\dot{x} = g_o(x) + ug_1(x) + \ldots + u^m g_m(x) \tag{7.1}$$

and have shown the existence of a number of switching manifolds defined by the solutions of polynomial equations in u. These polynomials are defined by

$$\sum_{i=o}^{k} 2<g_i(x),x>u^i = -1 \tag{7.2}$$

where k=m,m-1,...,1. In the submanifolds where no real solutions of these polynomials exist, we must have the projection of the unforced system $\dot{x}=g_o(x)$ along the tangent spaces of these submanifolds being stable.

In the distributed-parameter case we have used the spectral theorem for self-adjoint operators to reduce A+A* to a part which is asymptotically stable and a bounded, not necessarily stable, part whose total spectral subspace is contained in the minimum spectral subspace of B+B*. The control term uB can then be used to 'cancel out' the unstable part of A.

Finally a number of simple examples is given to illustrate the theory. The advantage of this approach is that the feedback control can be written down directly in terms of A and B, apart from systems of the form (7.1), where numerical evaluation of the roots of equations (7.2) must be applied if k>4.

8. References

1. Balakrishnan, A.V., Applied Functional Analysis, Applications of Mathematics, vol.3, Springer, New York, 1976.

2. Ball, J.M., On the asymptotic behaviour of generalised processes, with applications to nonlinear evolution equations, J.Differential Equations, 27, 224-265 (1978).

3. Ball, J.M, Marsden, J.E. and Slemrod, M. Controllability for distributed bilinear systems, SIAM J.Control, 20, 575-597 (1982).

4. Ball, J.M. and Slemrod, M., Feedback stabilization of distributed semi-linear control systems, App.Math.Optim.,5, 169-179 (.979).

5. Banks, S.P. State-space and frequency domain methods in the control of distributed parameter systems, Peter Peregrinus, Stevenage, 1983.

6. Banks, S.P. and Yew, M.K., On a class of suboptimal control for infinite dimensional bilinear systems, Sys.Control Letts, 5, 327-333 (1985).

7. Banks, S.P. and Yew, M.K., On the optimal control of bilinear systems and its relation to Lie algebras, to be published in Int.J.Control (1985b).

8. Curtain, R.F. and Pritchard, A.J., Infinite dimensional linear systems theory, Lecture notes in control and information sciences, vol.8. Springer, Berlin/Heidelberg, 1978.

9. Dunford, N. and Schwartz, J.T., Linear Operators, Part II, Interscience, New York, 1963.

10. Gutman, P.O., Stabilizing control for bilinear systems, IEEE Trans. Automat. Contr., vol.Ac-26, 917-922 (1981).