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On the Optimal Control of Bilinear Systems

and its relation to Lie algebras.

by

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Abstract

The optimal control of bilinear systems is considered and related to the Lie algebra generated by the system matrices. Interesting results obtain when this Lie algebra is nilpotent.

Key words: Bilinear Systems, Optimal Control, Lie algebras

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1. Introduction

Bilinear systems have been studied widely amd have been shown to be an important extension to linear systems (see, for example, Bruni, et al. 1984, Mohler, 1973, Isidori, 1973, Brockett, 1976, Gutman, 1981). The optimal control of bilinear systems has been considered by Tzafestas et al, 1984 and by Banks and Yew, 1985 - in the latter case the linear-quadratic regulator problem is extended to the bilinear-quadratic regulator problem. Optimal controls obtained in this way tend to be complicated, however. In this paper we shall study the simple problem

$$\dot{x} = Ax + uBx$$

$$\min_{u} \{J(u) \triangleq \int_{t_o}^{t_f} u^2 dt + x'(t_f)Fx(t_f).\}$$

We shall show that if A and B commute, i.e. $[A,B] = 0$ then the optimal control is constant. In the general case where $[A,B] \neq 0$ we shall show that, by considering the Lie algebra $\mathfrak{M}(A,B)$ generated by A and B, we can obtain a simple method for determining the optimal control.

Examples will be given in both the finite-and infinite-dimensional cases and it will be seen that an important special case arises when $\mathfrak{M}(A,B)$ is a nilpotent Lie algebra. For the elementary theory of Lie algebras, see Sagle and Walde, 1973.

2. Optimal Control of Bilinear Systems

In this section we shall consider the bilinear system

$$\dot{x} = Ax + uBx \quad , \; x(t_o) = x_o \tag{2.1}$$

where $x \in \mathbb{R}^n$ and u is a scalar control, and seek to minimise the simple cost function

$$J(u) = \int_{t_o}^{t_f} u^2 \, dt + x'(t_f) F \, x(t_f) \tag{2.2}$$

The Hamiltonian for the optimal control problem (2.2) subject to (2.1) is

$$H = u^2 + \lambda'(Ax+uBx)$$

and so we obtain the equations

$$\dot{\lambda}' = -(\lambda' A + u\lambda' B) \quad , \quad \lambda(t_f) = Fx(t_f) \qquad (a)$$

$$\dot{x} = Ax + uBx \qquad (b) \qquad (2.3)$$

and

$$2u - \lambda' Bx = 0 \qquad (c)$$

Hence

$$u = \tfrac{1}{2}\lambda^T Bx$$

Proposition 2.1

If $[B,A] \triangleq (BA-AB) = 0$ (i.e. if A and B commute), then the optimal control u* is a constant.

Proof From (2.3) (a) and (b) we have

$$\frac{d}{dt}(\lambda' Bx) = \dot{\lambda}' Bx + \lambda' B\dot{x}$$

$$= -(\lambda' A + u\lambda' B)Bx + \lambda' B(Ax+uBx)$$

$$= \lambda' [B,A] x \quad , \tag{2.4}$$

and so

$$\dot{u} = 0$$

if $[B,A] = 0$. □

Proposition 2.2 Under the assumptions of proposition 2.1, the (constant) optimal control u* is the solution of the equation

$$2u+x_o' e^{(A'+uB')T}\{F,B\}\, e^{(A+uB)T} x_o = 0, \tag{2.5}$$

where $T = t_f - t_o$ and $\{F,B\} = FB+BF$ is the auticommutator of F and B.

Proof Since u* is a constant we have

$$J(u^*) = u^{*2}T + x'(t_f)Fx(t_f)$$

However,

$$\dot{x} = (A+u^*B)x$$

and so $$x(T) = e^{(A+u^*B)T}x_o .$$

Hence

$$J(u^*) = u^{*2}T + x_o' e^{(A'+u^*B')T} F\, e^{(A+u^*B)T} x_o$$

and

$$\frac{dJ(u^*)}{du^*} = 2u^*T + x_o' e^{(A'+u^*B')T}\{F,B\}\, e^{(A+u^*B)T} x_o ,$$

since A and B commute . □

The condition that $[B,A] = 0$ is clearly very strong and so it is natural to seek similar conditions on the control to those above when $[B,A] \neq 0$. Of course, we no longer expect u to be constant. In this direction we note the following result, which generalises the equation (2.4) in proposition 2.1.

Lemma 2.3. For any (n×n) matrix X we have

$$\frac{d}{dt}(\lambda' Xx) = \lambda' [X, A+uB]x \tag{2.6}$$

Proof From (2.3)(a),(b) we have

$$\dot{\lambda}' Xx = -(\lambda' A+u\lambda' B)Xx = -\lambda'(A+uB)Xx$$

and

$$\lambda' X\dot{x} = \lambda' X(Ax+uBx) = \lambda' X(A+uB)x.$$

Hence

$$\frac{d(\lambda' Xx)}{dt} = \lambda' [X, A+uB]x .\ \square$$

Of course, if X=B then we obtain (2.4).

Consider now the Lie algebra M of all nxn matrices with the bracket $[L,M] = LM-ML$, and let M(A,B) denote the Lie subalgebra generated by A and B. Thus, M(A,B) consists of A and B and all possible brackets generated by A and B and their linear combinations. Since M is a finite-dimensional Lie algebra (of dimension n^2), M(A,B) must be finite-dimensional with dimension $m \leq n^2$.

Let $X_1, \ldots, X_m$ be a basis of M(A,B), and write

$$\begin{aligned} u_1 &= 2u = \lambda' Bx \\ u_2 &= \lambda' X_1 x \\ u_3 &= \lambda' X_2 x \\ &\vdots \\ u_m &= \lambda' X_{m-1} x \\ u_{m+1} &= \lambda' X_m x \end{aligned} \tag{2.7}$$

Then,

$$\begin{aligned} \dot{u}_1 &= \lambda' [B, A]\, x \\ \dot{u}_2 &= \lambda' [X_1, A+uB]\, x \\ \dot{u}_3 &= \lambda' [X_2, A+uB]\, x \\ &\vdots \\ \dot{u}_m &= \lambda' [X_{m-1}, A+uB]\, x \\ \dot{u}_{m+1} &= \lambda' [X_m, A+uB]\, x \end{aligned} \tag{2.8}$$

However, each term of the form $[X_i, A]$ or $[X_i, B]$ belongs to M(A,B) and so we may write

$$[X_i, A] = \sum_{j=1}^{m} \alpha_{ij} X_j \quad , \quad [X_i, B] = \sum_{j=1}^{m} \beta_{ij} X_j \tag{2.9}$$

for some constants α_{ij}, β_{ij}. Similarly, $[B, A] \in$ M(A,B) and so

$$[B, A] = \sum_{j=1}^{m} b_j X_j \tag{2.10}$$

for some constants b_j. Substituting (2.9) and (2.10) into (2.8) we have

$$\left.\dot{u}_1 = \sum_{j=1}^{m} b_j u_{j+1} \right\} \tag{2.11}$$

$$\dot{u}_i = \sum_{j=1}^{m} \alpha_{ij} u_{j+1} + \frac{u_1}{2} \sum_{j=1}^{m} \beta_{ij} u_{j+1} \Bigg\} , \qquad 2 \leq i \leq m+1$$

The equations (2.11) may be written in the form

$$\dot{u} = f(u) \tag{2.12}$$

for some (nonlinear) function f of $u = (u_1, \ldots, u_{m+1})$. We can solve (numerically) (2.12) for u in terms of t_o, t_f and some initial value u_o and so we can write

$$u(t) = U(t;\ t_o,\ t_f,\ u_o) \tag{2.13}$$

where

$$u_o = (u_1(t_o), \ldots, u_{m+1}(t_o)).$$

Substituting u(t) into (2.1), we have

$$\dot{x}(t) = Ax(t) + U(t;t_o,t_f,u_o)Bx(t) \ , \ x(t_o)=x_o \tag{2.14}$$

and solving (2.14) (again, numerically) we have

$$x(t) = \xi(t;t_o,t_f,u_o) \tag{2.15}$$

for some function ξ. Finally, substituting (2.13), (2.15) into the cost functional, we have

$$J(u_o) = \int_{t_o}^{t_f} U^2(t;t_o,t_f,u_o)dt + \xi'(t;t_o t_f u_o)F\xi(t;t_o,t_f u_o) \tag{2.16}$$

and we minimise J with respect to u_o to obtain the optimal control initial value, $u_1^*(t_o)/2$.

The above results can be simplified if M(A,B) is a nilpotent Lie algebra. Recall that a Lie algebra L is nilpotent if the sequence

$$[L, L], [[L,L], L], [[[L,L],L], L], \ldots \text{ terminates, i.e.}$$
$$(\mathrm{Ad}L)^k = 0$$

for some k>o. Here,

$$[L,L] = \{[X,Y] \ : X,Y \in L\}$$

and

$$(\mathrm{Ad}L)\ X = [L,X] \ , \quad X \in L.$$

Proposition 2.4. If $X \in (\mathrm{Ad}M(A,B))^{\ell} B$ then $\frac{d}{dt}(\lambda' Xx) = \lambda' Yx + u\lambda' Zx$, where

$Y, Z \in (\mathrm{AdM}(A,B))^{\ell+1}B.$

Proof. This follows from lemma 2.3, since

$$\frac{d}{dt}(\lambda' Xx) = \lambda'[X, A+uB]x$$
$$= -\lambda'(\mathrm{AdA})Xx \quad -u\lambda'(\mathrm{AdB})Xx \; . \square$$

Corollary 2.5 If M(A,B) is nilpotent and $(\mathrm{AdM}(A,B))^k = 0$, then

$$\lambda' Xx = \text{constant}$$

for any $X \in (\mathrm{AdM}(A,B))^{k-1}$. $\square$

Now recall that, for a nilpotent Lie algebra L, we have the descending

$$L \supseteq (\mathrm{AdL})L \supseteq (\mathrm{AdL})^2 L \supseteq \dots \supseteq (\mathrm{AdL})^k L = 0 \tag{2.17}$$

where $(\mathrm{AdL})^{k-1} L \neq 0$. Note that $(\mathrm{AdL})^p \neq (\mathrm{AdL})^{p+1}$ for any $p \in \{o, \dots k-1\}$. For, in the contrary case,

$$(\mathrm{AdL})^p = (\mathrm{AdL})^{p+1} = (\mathrm{AdL})^{p+2} = \dots = (\mathrm{AdL})^k = 0$$

and so we may replace k by p. Hence applying (2.17) to L = M(A,B) we can choose the basis $X_1, \dots, X_m$ as follows:

Partition $\Xi = \{X_1, \dots, X_m\}$ into k subsets

$$\Xi_1 = \{X_1, \dots, X_{\ell_1}\}$$
$$\Xi_2 = \{X_{\ell_1+1}, \dots, X_{\ell_1+\ell_2}\}$$
$$\dots\dots$$
$$\Xi_k = \{X_{\ell_1+\ell_2+\dots+\ell_{k-1}+1}, \dots, X_{\ell_1+\ell_2+\dots+\ell_k}\}$$

where $\ell_1 + \ell_2 \dots + \ell_k = m$ and Ξ_i is a basis of

$$[(\mathrm{AdM}(A,B))^{i-1} M(A,B)] \setminus [(\mathrm{AdM}(A,B))^{i} M(A,B)] \, ,$$

(which, as stated above, is nonempty).

Using this basis of M(A,B) it is easy to see that (2.11) takes the form

$$\begin{aligned} \dot{u}_1 &= b'u \\ \dot{u} &= \Gamma u + \frac{u_1}{2} \Delta u \end{aligned} \tag{2.18}$$

where Γ and Δ are nilpotent matrices, and $b = (b_1, \dots, b_m)'$.

We have therefore proved

Theorem 2.6 For the system (2.1), if M(A,B) is nilpotent Lie algebra, then the control is given as a linear feedback of a bilinear system (equation (2.18)) with nilpotent defining matrices. $\square$

Two special cases of theorem 2.6 are important.

Corollary 2.7. If M(A,B) is nilpotent with $(\mathrm{Ad}\, M(A,B))^2 = 0$, then the optimal control of the problem (2.1), (2.2) is of the form

$$u^* = c_1 + c_2 t$$

for some constants c_1, c_2.

Proof. This follows from (2.18) but can be seen directly as follows:

$$2\dot{u} = \lambda' [B,A]\, x \quad , \text{ by } (2.4)$$

and so

$$2\ddot{u} = \lambda' \big[[B,A], A+uB\big] x$$

by lemma 2.3. Hence $\ddot{u} = 0$ if $(\mathrm{Ad} M(A,B))^2 = 0$. □

Similarly we have

Corollary 2.8. If $(\mathrm{Ad} M(A,B))^3 = 0$, then the optimal control is given by

$$u^* = c_1 e^{c_3 t} + c_2 e^{-c_3 t} + c_4 ,$$

for some constants c_1, c_2, c_3, c_4.

Proof. Again we can see directly that

$$2\ddot{u} = \lambda' \big[[B,A],A\big] x + u\lambda' \big[[B,A],B\big] x,$$

while

$$\frac{d}{dt}\{\lambda' \big[[B,A],A\big] x\} = \lambda' \Big[\big[[B,A],A\big],A\Big] x + u\lambda' \Big[\big[[B,A],A\big],B\Big] x$$

and

$$\frac{d}{dt}\{\lambda' \big[[B,A],B\big] x\} = \lambda' \Big[\big[[B,A],B\big],A\Big] x + u\lambda' \Big[\big[[B,A],B\big],B\Big] x.$$

The right hand sides of both of the above expressions are zero, by assumption, and so

$$\ddot{u} = \alpha u + \beta,$$

for some constants α, β. The result now follows. □

Remark 2.9. The above results require only that M(A,B) is a finite dimensional Lie algebra and so will apply even to infinite dimensional systems for which M(A,B) is finite dimensional. (However equation (2.8) may have to be interpreted in a 'mild' sense.)

3. Examples

(3.1) Consider the system

$$\dot{x} = Ax + uBx \quad , \quad J(u) = \int_0^T u^2 dt + x'x$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Then,

$$[B,A] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[[B,A],A] = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$[[B,A],B] = -A$$

Now, $A, B, [B,A]$ and $[[B,A],A]$ are linearly independent, so

$$\dim (M(A,B)) = 4.$$

Hence if we write

$$X_1 = B,\ X_2 = A,\ X_3 = [B,A],\ X_4 = [[B,A],A]$$

and

$$\begin{aligned} u_1 &= u = \lambda' X_1 x \\ u_2 &= \lambda' X_2 x \\ u_3 &= \lambda' X_3 x \\ u_4 &= \lambda' X_4 x \end{aligned} \tag{3.1}$$

we have

$$\begin{aligned} \dot{u}_1 &= \lambda' X_3 x = u_3 \\ \dot{u}_2 &= \lambda' [X_2, A+uB] x = u\lambda' [X_2, B] x = -\frac{u_1}{2} \cdot u_3 \\ \dot{u}_3 &= \lambda' [X_3, A+uB] x = u_4 - \frac{u_1}{2} \cdot u_2 \\ \dot{u}_4 &= \lambda' [X_4, A+uB] x = -4u_3 \end{aligned} \tag{3.2}$$

since $[[[B,A],A],A] = -4[B,A]$ and $[[[B,A],A],B] = 0$.

(Note the difference between (2.8) and (3.1); we are now including B explicitly in the basis of M(A,B).) It is an elementary exercise to show that (3.2) implies

$$\dot{u}_1 = \pm\sqrt{\left\{\frac{u_1^4}{16} - \frac{c_2 u_1^2}{4} + c_2^2 - \frac{1}{4}(16u_1^2 - 8u_1 c_4 + c_4^2) + c_3\right\}} \tag{3.3}$$

for some constants c_2, c_3, c_4. We can solve (3.3) numerically, but as an example suppose we apply Euler's method of integration on the interval $[0,T]$ with only one step, for simplicity. Then, if $u_1(o) = c_1$, we have

$$u_1 = c_1 \quad , \qquad t\in[0, {}^T/_2)$$

$$u_1 = c_1 + \frac{T}{2}\sqrt{\left\{\frac{c_1^4}{16} - \frac{c_2 c_1^2}{4} + c_2^2 - \frac{1}{4}(16c_1^2 - 8c_1 c_4 + c_4^2) + c_3\right\}}, t\in[{}^T/_2, T]$$

$$= c_1' \quad \text{say}$$

Then

$$x(T) = e^{(A+c_1'B)T/2} \; e^{(A+c_1B)T/2} \; x_o$$

and

$$J(u) = \int_o^T u^2 dt + x'(T)x(T)$$

$$= \frac{T}{2}(c_1^2 + c_1'^2) + x_o' \, e^{(A'+c_1B')\frac{T}{2}} \; e^{(A'+c_1B')\frac{T}{2}} \; e^{(A+c_1'B)\frac{T}{2}} \, e^{(A+c_1B)\frac{T}{2}} \, x_o$$

Optimizing J with respect to c_1, c_2, c_3, c_4 will then give the optimal control.

(3.2) Consider the system

$$\dot{\phi} = A\phi + uB\phi \, , \qquad J(u) = \int_o^T u^2 + \phi'(T)\phi(T)$$

where $\phi \in \ell^2$ (the space of (infinite) square summable sequences), A is the left shift operator, and B = I, then

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 0 & & & & \ddots \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

in the usual basis. Now $[B,A] = 0$, so the optimal control is constant, say $u = c$. Hence

$$\dot{\phi} = (A+cB)\phi$$

and

$$\phi(T) = e^{(A+cB)T}\phi(0).$$

However, $[B,A] = 0$, so by the Campbell-Hausdorff formula, (which is clearly valid here), we have

$$e^{(A+cB)T} = e^{AT}e^{cBT}$$

and

$$e^{AT} = \begin{pmatrix} 1 & T & T^2/2! & T^3/3! & & \cdot & \\ & 1 & T & T^2/2! & & \cdot & \cdot \\ & & 1 & T & T^2/2! & \cdot & \cdot \\ & & & 1 & T & \cdot & \cdot \\ 0 & & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \end{pmatrix}, \quad e^{cBT} = \begin{pmatrix} e^{cT} & & & & 0 \\ & e^{cT} & & & \\ & & e^{cT} & & \\ & & & \cdot & \\ 0 & & & & \cdot \\ & & & & & \cdot \end{pmatrix}$$

Hence,

$$e^{(A+cB)T} = e^{cT}\ e^{AT}$$

Thus,

$$J(u) = c^2T + e^{2cT}\psi(T)$$

where

$$\psi(T) = \sum_{i=1}^{\infty} \sum_{j=o}^{\infty} \left(\phi_{j+i}(o) \frac{T^j}{j!} \right)^2$$

The optimal control is therefore the solution of the equation

$$2cT + 2Te^{2cT} \cdot \psi(T) = 0$$

(3,3) As a final example we consider briefly the system

$$\dot{\phi} = A + uB\phi,$$

where

$$A = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 2 & & \\ & & 0 & 3 & \\ 0 & & & 0 & 4 \\ & & & & \cdot \ \cdot \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & \\ & 1 & 0 & \\ & & 1 & 0 \\ 0 & & & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix}.$$

A is unbounded but we may evaluate e^{At} as before. In this case

$$[B,A] = -I$$

and so M(A,B) is nilpotent with $(\text{Ad } M(A,B))^2 = 0$. Hence the optimal control is $u = c_1 + c_2 t$

for some constants c_1, c_2. By the Campbell-Hausdorff formula,

$$e^{At} e^{uBt} = e^{\{At + uBt + \frac{u}{2}[A,B]t\}}$$

$$= (e^{At + uBt}) e^{-\frac{u}{2}t}$$

since $[A,B]$ commutes with A and B. Hence

$$e^{At + uBt} = e^{u/2t} e^{At} e^{uBt}$$

and we can proceed as in the above examples.

4. Conclusions

In this paper we have considered the 'minimum fuel' problem for a bilinear system and have shown how to obtain the optimal control by considering the Lie algebra generated by the system matrices. It should be noted that we have obtained an open-loop control depending on the initial value of the state x_o. However, using the principle of receding horizon control (Shaw, 1979, Banks, 1983) we can apply the control as if we were always starting an optimising interval of T seconds. We then just replace x_o by $x(t)$ and allow T to depend on the state to obtain a nonlinear feedback control. The details are the same as with linear systems.

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