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On the self-propulsion of an N-sphere micro-robot

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The aim of this paper is to describe the self-propulsion of a micro-robot (or micro-swimmer) consisting of N spheres moving along a fixed line. The spheres are linked to each other by arms with their lengths changing periodically. We use the asymptotic procedure containing the two-timing method and a distinguished limit. We show that self-propulsion velocity appears (in the main approximation) as a linear combination of velocities of all possible triplets of spheres. Velocities and efficiencies of three-, four- and five-sphere swimmers are calculated.

Key words: biological fluid dynamics, low-Reynolds-number flows, micro-/nano-fluid dynamics, propulsion

1. Introduction

Studies of micro-robots represent a flourishing modern research topic that strives to create a fundamental base for modern applications in medicine and technology, see Purcell ( ), Becker, Koelher & Ryder ( ), Najafi & Golestanian ( ), Dreyfus *et al.* ( ), Chang *et al.* ( ), Earl *et al.* ( ), Alouges, DeSimone & Lefebvre ( ), Golestanian & Ajdari (,), Leoni *et al.* ( ), Alexander, Pooley & Yeomans ( ), Gilbert *et al.* ( ) and Lauga ( ). The simplicity of the geometry represents the major advantage in studies of micro-robots (in contrast with the extreme complexity of self-swimming micro-organisms, e.g. Pedley & Kessler ( ), Vladimirov *et al.* ( ), Pedley ( ) and Polin *et al.* ( )); it allows us to describe the motions of micro-robots in greater depth. In this paper, we generalize the theory of the three-sphere micro-robot of Najafi & Golestanian ( ) and Golestanian & Ajdari ( ) to an N-sphere micro-robot. We employ the two-timing method and distinguished limit arguments, which lead to a simple and rigorous analytical procedure. Our calculation of the self-propulsion velocity of an N-sphere robot shows that it represents (in the main approximation) a linear combination of velocities due to all possible triplets of spheres. The velocities and Lighthill's swimming efficiencies of three-, four- and five-sphere robots are calculated as examples.

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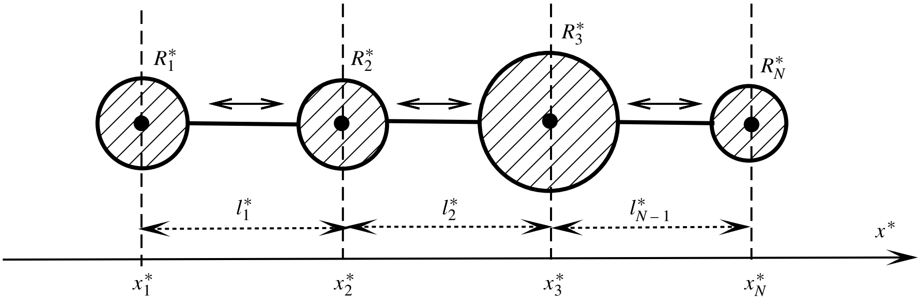

FIGURE 1. $N = 4$ spheres, linked by arms of periodically changing lengths.

2. Formulation of the problem

We consider a micro-robot consisting of N rigid spheres of radii R_i^*, $i = 1, 2, \ldots, N$ with their centres at the points $x_i^*(t^*)$ of the x^*-axis ($x_{i+1}^* > x_i^*$), where t^* is time, and asterisks mark dimensional variables and parameters. The spheres are connected by $N - 1$ arms/rods, such that the distances between the centres of neighbouring spheres are $l_\alpha^* = |x_{\alpha+1}^* - x_\alpha^*|$, $\alpha = 1, 2, \ldots, N-1$, see figure 1 (in this paper latin subscripts take values 0 to N, while greek subscripts take values 0 to $N - 1$). We accept Stokes's approximation where the masses of spheres and arms are zero; the arms are so thin (much thinner than any R_i^*) that their interactions with a fluid are negligible. The equations of motion can be written as

$$f_i^* + F_i^* = 0, \tag{2.1}$$

$$F_i^* = -\kappa_i^* \left\{ \dot{x}_i^* - \sum_{k \neq i} 3R_k^* \dot{x}_k^* / (2x_{ik}^*) \right\}, \quad x_{ik}^* \equiv |x_i^* - x_k^*|, \tag{2.2}$$

$$x_{\alpha+1}^* - x_\alpha^* = l_\alpha^*, \tag{2.3}$$

$$\sum_{i=1}^{N} f_i^* = 0, \tag{2.4}$$

where $\kappa_i^* \equiv 6\pi\eta R_i^*$, η is viscosity, dots above functions stand for $\mathrm{d}/\mathrm{d}t^*$, and the summation convention over repeating subscripts is not in use. The forces f_i^* are exerted by the arms on the ith sphere, while F_i^* represent viscous friction. In order to derive (2.2) we use a classical expression for Stokes's friction force as well as an explicit expression for fluid velocity for a sphere moving along the x^*-axis. The x^*-component u^* of this velocity at distance r^* along the x^*-axis is

$$u^* \simeq 3R^* U^* / (2r^*), \tag{2.5}$$

where R^* and U^* are the radius and velocity of a sphere, see Lamb ( ), Landau & Lifshitz ( ) and Moffatt ( ). Equality (2.4) follows from the fact that the external forces exerted on each arm are negligible. The geometrical configuration of a micro-robot is determined by given functions

$$l_\alpha^* = L_\alpha^* + \widetilde{l}_\alpha^*(\tau), \tag{2.6}$$

where L_α^* are mean values and $\widetilde{l}_\alpha^*(\tau)$ are oscillations, which represent 2π-periodic functions of a dimensionless variable $\tau \equiv \omega^* t^*$; ω^* is a constant. Since all l_α^* are given, then conditions (2.3) can be considered as geometrical constraints. Equalities (2.1)–(2.6) represent a system of $2N$ equations for $2N$ unknown functions:

$$\boldsymbol{x}^*(t^*) \equiv (x_1^*(t^*), x_2^*(t^*), \ldots, x_N^*(t^*)), \quad \boldsymbol{f}^*(t^*) \equiv (f_1^*(t^*), f_2^*(t^*), \ldots, f_N^*(t^*)). \tag{2.7}$$

These equations contain three characteristic lengths: distance L^* between neighbouring spheres, radius R^* of spheres, and amplitude λ^* of arm oscillations; the characteristic time scale is T^*:

$$R^* \equiv \sum_{i=1}^N R_i^*/N, \quad L^* \equiv \sum_{\alpha=1}^N L_\alpha^*/N, \quad \lambda^* \equiv \sum_{\alpha=1}^N \max \widetilde{l}_\alpha^*/N, \quad T^* \equiv 1/\omega^*. \tag{2.8}$$

The dimension of κ^* can be eliminated from the equations by division of (2.1) by κ^*, hence it does not play any role in scaling. We choose dimensionless variables and small parameters as

$$\boldsymbol{x}^* = L^*\boldsymbol{x}, \quad R_i^* = R^* R_i, \quad \widetilde{l}_\alpha^* = \lambda^* \widetilde{l}_\alpha, \quad t^* = T^* t, \quad f_i^* = 6\pi\eta R^* L^* f_i/T^*; \tag{2.9}$$

$$\varepsilon \equiv \lambda^*/L^* \ll 1, \quad \delta \equiv 3R^*/(2L^*) \ll 1. \tag{2.10}$$

Then the dimensionless form of (2.1)–(2.6) is

$$R_i x_{it} - \delta \sum_{k \neq i} R_{ik} x_{kt}/l_{ik} = f_i, \quad l_{ik} = L_{ik} + \varepsilon \widetilde{l}_{ik}, \tag{2.11}$$

$$x_{\alpha+1} - x_\alpha = l_\alpha, \tag{2.12}$$

$$\sum_i f_i = \boldsymbol{f} \cdot \boldsymbol{I} = 0, \quad \boldsymbol{I} \equiv (1, 1, \ldots, 1), \tag{2.13}$$

where $R_{ik} \equiv R_i R_k$, subscript t stands for $\mathrm{d}/\mathrm{d}t$,

$$l_{ik} \equiv \sum_{n=i}^{k-1} l_n \quad \text{for} \quad k \geqslant i+1 \quad \text{with } l_{ki} = l_{ik} \quad \text{otherwise}, \tag{2.14}$$

and definitions for L_{ik} and $\widetilde{l}_{ik}$ similar to (2.14) (for example, $L_{13} = L_{31} = L_1 + L_2$ etc.). Equation (2.11) can be rewritten in $(N \times N)$-matrix form:

$$\mathbb{A}\boldsymbol{x}_t = \boldsymbol{f} \quad \text{or} \quad \sum_{k=1}^N A_{ik} x_{kt} = f_i, \tag{2.15}$$

$$\mathbb{A} = A_{ik} = \begin{cases} R_i & \text{for } i = k, \\ -\delta R_{ik}/l_{ik} & \text{for } i \neq k. \end{cases} \tag{2.16}$$

3. Two-time method and asymptotic procedure

3.1. *Functions and notation*

The following dimensionless notation and definitions are in use.

(i) s and τ denote slow time and fast time; subscripts s and τ stand for related partial derivatives.

(ii) A dimensionless function, say $G = G(s, \tau)$, belongs to class $\mathscr{I}$ if $G = O(1)$ and all partial s- and τ-derivatives of G (required for our consideration) are also $O(1)$.

In this paper all functions belong to class $\mathscr{I}$, while all small parameters appear as explicit multipliers.

(iii) We consider only functions that are periodic in $\tau\{G \in \mathscr{P} : G(s,\tau) = G(s,\tau+2\pi)\}$, where s-dependence is not specified. Hence, all functions considered below belong to $\mathscr{P} \bigcap \mathscr{I}$.

(iv) For arbitrary $G \in \mathscr{P}$ the averaging operation is

$$\langle G \rangle \equiv \frac{1}{2\pi}\int_{\tau_0}^{\tau_0+2\pi} G(s,\tau)\,\mathrm{d}\tau \equiv \overline{G}(s), \quad \forall \tau_0. \tag{3.1}$$

(v) The tilde-functions (or purely oscillating functions) represent a special case of $\mathscr{P}$-functions with zero average $\langle \widetilde{G} \rangle = 0$. The bar-functions (or mean-functions) $\overline{G} = \overline{G}(s)$ do not depend on τ. A unique decomposition $G = \overline{G} + \widetilde{G}$ is valid.

3.2. *Asymptotic procedure*

The ε-dependence of l_{ik} (2.11) leads to the decomposition of matrix $\mathbb{A}$ (2.16) as a series in ε

$$\mathbb{A} = \overline{\mathbb{B}}_0 + \varepsilon\delta\widetilde{\mathbb{A}}'_0 + \cdots, \quad \overline{\mathbb{B}}_0 \equiv \overline{\mathbb{A}}_0 + \delta\overline{\mathbb{A}}_1, \tag{3.2}$$

$$\overline{\mathbb{A}}_0 \equiv \mathrm{diag}\{R_1, R_2, \ldots, R_N\}, \quad \widetilde{\mathbb{A}}'_0 \equiv \begin{cases} 0 & \text{for } i = k, \\ R_{ik}\widetilde{l}_{ik}/L^2_{ik} & \text{for } i \neq k, \end{cases} \tag{3.3}$$

where we do not present the expression for $\overline{\mathbb{A}}_1$ since it does not affect the final answer. In double series (with small parameters ε and δ) matrices $\overline{\mathbb{A}}_0$, $\overline{\mathbb{A}}_1$ and $\widetilde{\mathbb{A}}'_0$ appear in the terms of orders $\varepsilon^0\delta^0$, $\varepsilon^0\delta^1$, and $\varepsilon^1\delta^1$. The introduction of fast time variable τ and slow time variable s represents a crucial step in our asymptotic procedure. We choose

$$\tau = t, \quad s = \varepsilon^2 t. \tag{3.4}$$

This choice can be justified by the same distinguished limit arguments as in Vladimirov ( ). Here we present this choice without proof; however the most important part of this proof (that this choice leads to a valid asymptotic procedure) is given and used below. After the use of the chain rule

$$\mathrm{d}/\mathrm{d}t = \partial/\partial\tau + \varepsilon^2\partial/\partial s \tag{3.5}$$

we accept (temporarily) that τ and s represent two independent variables. The substitution of (3.5) and (3.2) into (2.15) gives its two-time form:

$$(\overline{\mathbb{B}}_0 + \varepsilon\delta\widetilde{\mathbb{A}}'_0 + \cdots)(\boldsymbol{x}_\tau + \varepsilon^2\boldsymbol{x}_s) = \boldsymbol{f}. \tag{3.6}$$

Unknown functions are taken as regular series:

$$\boldsymbol{x}(\tau,s) = \overline{\boldsymbol{x}}_0(s) + \varepsilon\boldsymbol{x}_1(\tau,s) + \cdots, \quad \boldsymbol{f}(\tau,s) = \boldsymbol{f}_0(\tau,s) + \varepsilon\boldsymbol{f}_1(\tau,s) + \cdots, \tag{3.7}$$

where $\widetilde{\boldsymbol{x}}_0 \equiv 0$, which means that long distances of self-swimming are caused by small oscillations: $|\overline{\boldsymbol{x}}_0| \gg |\varepsilon\widetilde{\boldsymbol{x}}_1(\tau,s)|$.

3.3. *Successive approximations*

The successive approximations of (3.6), (3.7) yield

terms of order $\varepsilon^0 = 1$: $\boldsymbol{f}_0 \equiv 0$;

terms of order ε^1 : $\overline{\mathbb{B}}_0\boldsymbol{x}_{0\tau} = \boldsymbol{f}_1$; its average gives $\overline{\boldsymbol{f}}_1 \equiv 0$ and the oscillating part is

$$\overline{\mathbb{B}}_0\widetilde{\boldsymbol{x}}_{1\tau} = \widetilde{\boldsymbol{f}}_1; \tag{3.8}$$

terms of order ε^2 : $\quad \overline{\mathbb{B}}_0\widetilde{\boldsymbol{x}}_{2\tau} + \delta\widetilde{\mathbb{A}}_0'\widetilde{\boldsymbol{x}}_{1\tau} + \overline{\mathbb{B}}_0\overline{\boldsymbol{x}}_{0s} = \boldsymbol{f}_2$; its averaged part is

$$\overline{\mathbb{B}}_0\overline{\boldsymbol{x}}_{0s} + \delta\langle\widetilde{\mathbb{A}}_0'\widetilde{\boldsymbol{x}}_{1\tau}\rangle = \overline{\boldsymbol{f}}_2. \tag{3.9}$$

Force $\overline{\boldsymbol{f}}_2$ can be excluded from (3.9), (2.13):

$$\boldsymbol{I}\cdot\overline{\mathbb{B}}_0\overline{\boldsymbol{x}}_{0s} + \delta\boldsymbol{I}\cdot\langle\widetilde{\mathbb{A}}_0'\widetilde{\boldsymbol{x}}_{1\tau}\rangle = 0. \tag{3.10}$$

The averaged self-propulsion motion means that the rate of change is $\overline{x}_{0is} = \overline{X}_{0s}$ with the same function $\overline{X}_0(s)$ for all spheres; therefore we write $\overline{\boldsymbol{x}}_{0s} = \overline{X}_{0s}\boldsymbol{I}$. Hence, (3.10) gives

$$\overline{X}_{0s} = -\delta\frac{\boldsymbol{I}\cdot\left\langle\widetilde{\mathbb{A}}_0'\widetilde{\boldsymbol{x}}_{1\tau}\right\rangle}{\boldsymbol{I}\cdot\overline{\mathbb{A}}_0\boldsymbol{I}} \tag{3.11}$$

where in the denominator matrix $\overline{\mathbb{B}}_0$ is replaced with $\overline{\mathbb{A}}_0$, since we consider only the main (linear in δ) term in (3.11). Expression (3.11) still contains unknown functions $\widetilde{\boldsymbol{x}}_{1\tau}$ which can be determined from (3.8) with the use of constraints (2.12), (2.13). Indeed, (3.8) (with the terms required for linear-in-δ precision in (3.11)) gives

$$\widetilde{\boldsymbol{x}}_{1\tau} = (\overline{\mathbb{A}}_0)^{-1}\widetilde{\boldsymbol{f}}_1 \equiv \widetilde{\boldsymbol{g}}, \quad \widetilde{\boldsymbol{g}} = (\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_N), \tag{3.12}$$

$$\widetilde{g}_i \equiv \widetilde{f}_{1i}/R_i, \quad (\overline{\mathbb{A}}_0)^{-1} = \mathrm{diag}\{1/R_1, 1/R_2, \ldots, 1/R_N\}. \tag{3.13}$$

One can see that (3.12), (2.12) yield $\widetilde{x}_{\alpha+1,\tau} - \widetilde{x}_{\alpha,\tau} = \widetilde{g}_{\alpha+1} - \widetilde{g}_\alpha = \widetilde{l}_{\alpha\tau}$, while (2.13) leads to $\sum_i R_i\widetilde{g}_i = 0$. Both restrictions can be written with the use of $N\times N$ constraint matrix $\mathbb{C}$:

$$\mathbb{C}\widetilde{\boldsymbol{g}} = \widetilde{\boldsymbol{l}}_\tau, \quad \mathbb{C} \equiv \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ R_1 & R_2 & R_3 & \cdots & R_{N-1} & R_N \end{pmatrix}, \quad \widetilde{\boldsymbol{l}} \equiv \begin{pmatrix} \widetilde{l}_1 \\ \widetilde{l}_2 \\ \cdots \\ \widetilde{l}_{N-1} \\ 0 \end{pmatrix}. \tag{3.14}$$

The substitution of its inverse form

$$\widetilde{\boldsymbol{x}}_{1\tau} = \mathbb{C}^{-1}\widetilde{\boldsymbol{l}}_\tau \tag{3.15}$$

into (3.11) yields

$$\overline{X}_{0s} = -\frac{\delta}{\rho}\boldsymbol{I}\cdot\left\langle\widetilde{\mathbb{A}}_0'\mathbb{C}^{-1}\widetilde{\boldsymbol{l}}_\tau\right\rangle \tag{3.16}$$

where

$$(-1)^{N+1}\rho\mathbb{C}^{-1} \equiv \begin{pmatrix} \rho_1-\rho & \rho_2-\rho & \rho_3-\rho & \cdots & \rho_{N-1}-\rho & 1 \\ \rho_1 & \rho_2-\rho & \rho_3-\rho & \cdots & \rho_{N-1}-\rho & 1 \\ \rho_1 & \rho_2 & \rho_3-\rho & \cdots & \rho_{N-1}-\rho & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{N-1}-\rho & 1 \\ \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{N-1} & 1 \end{pmatrix}, \tag{3.17}$$

$$\rho_k \equiv \sum_{i=1}^{k} R_i, \quad k \geqslant 1; \quad \rho \equiv \rho_N, \quad \boldsymbol{I}\cdot\overline{\mathbb{A}}_0\boldsymbol{I} = \rho. \tag{3.18}$$

The expression for inverse matrix (3.17) can be checked by direct calculations of product $\mathbb{C}^{-1}\mathbb{C}$; these calculations become particularly simple if the matrix on the right-hand side of (3.17) is decomposed into two matrices: one containing all the ρ, and another with identical rows $(\rho_1, \rho_2, \rho_3, \dots, \rho_{N-1}, 1)$. It is worth emphasizing that the analytic procedure used is especially simple, since in order to calculate self-propulsion velocity (3.11) we only need to know the main approximation $\widetilde{\boldsymbol{x}}_1$ for mutual oscillations of spheres; this approximation is completely described by simple equations (3.12), (3.15).

4. Self-propulsion velocity

One can see from (3.16) that $\overline{X}_{0s} = \boldsymbol{I} \cdot \overline{\boldsymbol{x}}_s/N = O(\delta)$. Hence, the order of magnitude of the dimensionless physical velocity is

$$\overline{V}_0 \equiv \boldsymbol{I} \cdot \overline{\boldsymbol{x}}_t/N = \varepsilon^2 \overline{X}_{0s} = O(\varepsilon^2\delta). \tag{4.1}$$

The substitution of $\widetilde{\mathbb{A}}'_0$ (3.2) and $\mathbb{C}^{-1}$ (3.17) into (3.16) and subsequent algebraic transformations lead to

$$\overline{V}_0 = \frac{\varepsilon^2\delta}{\rho^2} \sum_{i<k<l} \overline{G}_{ikl}, \tag{4.2}$$

$$\overline{G}_{ikl} \equiv 2R_iR_kR_l \left(\frac{1}{L_{ik}^2} + \frac{1}{L_{kl}^2} - \frac{1}{L_{il}^2} \right) \langle \widetilde{l}_{ik}\widetilde{l}_{kl\tau} \rangle, \tag{4.3}$$

where the sum (4.2) is taken over all possible triplets $(i, k, l) : 1 \leqslant i < k < l \leqslant N$. In (4.3) one can also take $2\langle \widetilde{l}_{ik}\widetilde{l}_{kl\tau} \rangle = \langle \widetilde{l}_{ik}\widetilde{l}_{kl\tau} - \widetilde{l}_{ik\tau}\widetilde{l}_{kl} \rangle$, which could be proved using integration by parts. The formulae (4.2), (4.3) have been obtained for $N = 3, 4, 5$ by explicit analytical calculations, and for any N by the method of mathematical induction. These calculations are straightforward but rather too cumbersome to be presented here. However, as soon as (4.2) and (4.3) are known, they can be verified by separate calculations of all terms proportional to $1/L_{ik}^2$ for each particular pair i, k. For example, for $i = 1, k = 2$ the relevant part of matrix $\widetilde{\mathbb{A}}'_0 \equiv \widetilde{A}'_{ik}$ (3.2) contains only $\widetilde{A}'_{12} = \widetilde{A}'_{21} = R_{12}\widetilde{l}_1/L_{12}^2$, while all other components are zero. The related part of (3.16) can be easily calculated; it leads to the same expression as the corresponding extraction from (4.2), (4.3), which in this case contains only $(N - 2)$ terms with $l = 3, 4, \dots, N$.

The number of terms/triplets in (4.2) rapidly increases with N: for a three-sphere swimmer the sum (4.2) contains the only triplet, for a four-sphere swimmer there are four triplets, for a five-sphere swimmer 10 triplets, while for a ten-sphere swimmer the number of triplets grows to 120. In general (4.2) contains $N!/[(N - 3)!3!]$ triplets. It is important to emphasize that (4.2) contains all triplets in a micro-robot, not only triplets of the neighbouring spheres. If one takes into account only the triplets of the neighbouring spheres (say, $(1, 2, 3)$ and $(2, 3, 4)$ out of $(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)$ for $N = 4$), then they give only the rough estimation of $\overline{V}_0$, which can be misleading, since different correlations $\langle \widetilde{l}_\alpha \widetilde{l}_{\beta\tau} \rangle$ can have different values (see below).

Function $\overline{G}_{123}$ (4.3) was introduced by Golestanian & Ajdari ( ) and studied by Alouges *et al.* ( ), Alexander *et al.* ( ) and Golestanian & Ajdari ( ) in the context of a three-sphere micro-robot. We call $\overline{G}_{ikl}$ *a Golestanian function.*

Formulae (3.16), (4.2), (4.3) give the main result of this paper: the self-swimming velocity of an N-sphere micro-robot represents a linear combination of Golestanian functions for all available triplets.

5. Examples

The explicit formulae (4.2), (4.3) can be used to obtain physically interesting results (optimal strokes, required power, related forces, efficiency, etc.) for various N-sphere swimmers. We briefly address some of these questions below. In all these examples, we consider only homogeneous micro-robots, consisting of equal spheres $R_i = 1$ and of equal arms $L_\alpha = 1$.

The scalar product of the main equation (2.11) and $\boldsymbol{x}_t$ leads to the average power of a micro-robot

$$\overline{\mathscr{P}} \equiv \langle \boldsymbol{f} \cdot \boldsymbol{x}_t \rangle = \varepsilon^2 \langle \widetilde{\boldsymbol{x}}_{1\tau}^2 \rangle + O(\varepsilon^2 \delta), \tag{5.1}$$

where we have taken into account that $\boldsymbol{x}_t = \varepsilon \widetilde{\boldsymbol{x}}_{1\tau} + O(\varepsilon^2)$, which follows from (3.7) and (3.5). Another expression, $\overline{\mathscr{P}}_s = \rho \overline{V}_0^2$, represents the power which is required to drag a micro-robot with velocity $\overline{V}_0$ in the absence of its oscillations (when the main approximation for the dimensionless Stokes's friction force is $-\rho \overline{V}_0$). Lighthill's swimming efficiency (see Becker *et al.*) is the ratio $\mathscr{E} \equiv \overline{\mathscr{P}}_s / \overline{\mathscr{P}}$, which in our case is

$$\mathscr{E} \simeq \frac{\varepsilon^2 \delta^2}{\rho^3} \frac{\left(\sum_{i<k<l} \overline{G}_{ikl} \right)^2}{\langle \widetilde{\boldsymbol{x}}_{1\tau}^2 \rangle}, \tag{5.2}$$

where $\widetilde{\boldsymbol{x}}_{1\tau}$ is determined by (3.15).

For a three-sphere swimmer, (4.2), (4.3) yield

$$\overline{V}_0 = \frac{\varepsilon^2 \delta}{9} \overline{G}_{123}, \quad \overline{G}_{123} = \frac{7}{2} \left\langle \widetilde{l}_1 \widetilde{l}_{2\tau} \right\rangle. \tag{5.3}$$

Further simplification can be achieved if we accept that oscillations of both arms are harmonic and have equal amplitudes:

$$l_\alpha = \cos(\tau + \varphi_\alpha) \quad \text{then } 2\langle \widetilde{l}_1 \widetilde{l}_{2\tau} \rangle = \sin(\varphi_1 - \varphi_2) \tag{5.4}$$

with constant phases $0 \leqslant \varphi_\alpha \leqslant 2\pi$. The substitution of (5.4) into (5.3) and (5.2) gives

$$\overline{V}_0 = \varepsilon^2 \delta \frac{7}{36} \sin \phi, \quad \mathscr{E} = \left(\frac{7 \varepsilon \delta}{12} \right)^2 \frac{\sin^2 \phi}{2 + \cos \phi}, \quad \phi \equiv \varphi_1 - \varphi_1 \tag{5.5}$$

which shows that their maxima take place at different ϕ:

$$\max \overline{V}_0 \simeq 0.19 \varepsilon^2 \delta \quad \text{at } \phi = \pi/2 \simeq 1.57, \tag{5.6}$$

$$\max \mathscr{E} = 0.182 \varepsilon^2 \delta^2 \quad \text{at } \phi = 1.80. \tag{5.7}$$

Similar consideration for a four-sphere swimmer yields

$$\overline{V}_0 = \frac{\varepsilon^2 \delta}{16} (\overline{G}_{123} + \overline{G}_{124} + \overline{G}_{134} + \overline{G}_{234}), \tag{5.8}$$

which (with the use of (5.4)) leads to

$$\overline{V}_0 = \varepsilon^2\delta\frac{7}{64}\mathscr{S}_4(\phi,\psi), \quad \mathscr{S}_4(\phi,\psi) \equiv (1+C)(\sin\phi + \sin\psi) + 2C\sin(\phi+\psi), \tag{5.9}$$

where $C = 41/63 \simeq 0.65$, $\phi \equiv \varphi_1 - \varphi_2$, and $\psi \equiv \varphi_2 - \varphi_3$. Then (5.2) can be expressed as

$$\mathscr{E} = \left(\frac{7\varepsilon\delta}{16}\right)^2 \frac{\mathscr{S}_4^2(\phi,\psi)}{5 + 2\cos\phi + 2\cos\psi + \cos(\phi+\psi)}. \tag{5.10}$$

The computations show that

$$\max\overline{V}_0 \simeq 0.44\varepsilon^2\delta \quad \text{at } \phi \simeq \psi \simeq 1.10, \tag{5.11}$$

$$\max\mathscr{E} \simeq 0.55\varepsilon^2\delta^2 \quad \text{at } \phi \simeq \psi \simeq 1.38. \tag{5.12}$$

For a five-sphere swimmer, one can write

$$\overline{V}_0 = \frac{\varepsilon^2\delta}{25}(\overline{G}_{123} + \overline{G}_{124} + \overline{G}_{125} + \overline{G}_{134} + \overline{G}_{135} + \overline{G}_{145} + \overline{G}_{234} + \overline{G}_{235} + \overline{G}_{245} + \overline{G}_{345}), \tag{5.13}$$

which leads to

$$\overline{V}_0 = \frac{\varepsilon^2\delta}{50}\mathscr{S}_5, \quad \mathscr{S}_5 \equiv (4a+b+c)(\sin\phi + \sin\chi) + (5a+2b)\sin\psi \\ + (a+2b+c)[\sin(\phi+\psi) + \sin(\psi+\chi)] + (a+2c)\sin(\phi+\psi+\chi); \tag{5.14}$$

$$\left.\begin{aligned} &a = 7/8, \quad b = 41/18, \quad c = 151/72; \\ &\phi \equiv \varphi_1 - \varphi_2, \quad \psi \equiv \varphi_2 - \varphi_3, \quad \chi \equiv \varphi_3 - \varphi_4. \end{aligned}\right\} \tag{5.15}$$

Then (5.2) takes the form

$$\mathscr{E} = \left(\frac{\varepsilon\delta}{10}\right)^2 \frac{\mathscr{S}_5^2}{\mathscr{P}_5}, \quad \mathscr{P}_5 \equiv 10 + 3\cos\phi + 4\cos\psi + 3\cos\chi \\ + 2\cos(\phi+\psi) + 2\cos(\psi+\chi) + \cos(\phi+\psi+\chi). \tag{5.16}$$

The computations give

$$\max\overline{V}_0 \simeq 0.77\varepsilon^2\delta \quad \text{at } \phi \simeq \chi \simeq 0.86, \ \psi \simeq 0.83, \tag{5.17}$$

$$\max\mathscr{E} \simeq 1.00\varepsilon^2\delta^2 \quad \text{at } \phi \simeq \chi \simeq 1.14, \ \psi \simeq 1.08. \tag{5.18}$$

The numerical results (5.6), (5.7), (5.11), (5.12), (5.16)–(5.12) show that both $\max\overline{V}_0$ and $\max\mathscr{E}$ grow when N increases. For reasonably small values of parameters (say, $\varepsilon \simeq 0.2$ and $\delta \simeq 0.2$) we have $\max\mathscr{E} \sim 0.1\%$, hence the efficiency of the micro-robots considered is low.

In order to compare the velocities of micro-robots and micro-organisms we use the dimensional variables, in which $\max\overline{V}_0^* \sim \omega^* L^* \varepsilon^2\delta$; this shows that (for typical stroke frequency of self-swimming micro-organisms, which is about several Hz, see Pedley & Kessler ; Vladimirov *et al.* ; Pedley ; Polin *et al.*) a micro-robot can move itself with the speed ∼10% of its own size per second. This estimation is 20–40 times lower than a similar value for natural micro-swimmers, see Vladimirov *et al.* ( ); it again shows the low efficiency of the micro-robots considered. If we suggest that function $\max\overline{V}_0(N)$ grows with a similar rate (as has been calculated for $N = 3, 4, 5$), then micro-robots with $N = 7$–10 could swim with speed similar to that of micro-organisms. This estimation depends on the type of extrapolation, which could be linear or exponential.

6. Discussion

Our approach (based on the two-timing method and a distinguished limit) is technically different from all previous methods employed in studies of micro-robots. The possibility deriving explicit formulae for an N-sphere micro-robot shows its strength and analytical simplicity. The version of the two-timing method used has been developed in Vladimirov (, ,).

One can see that $\overline{V}_0 = O(\varepsilon^2\delta)$ (4.2), which coincides with the result by Golestanian & Ajdari (,) for a three-sphere swimmer. At the same time our choice of slow time $s = \varepsilon^2 t$ (3.5) agrees with the classical studies of self-propulsion for low Reynolds numbers, see Taylor ( ), Blake ( ) and Childress ( ), as well as geometric studies of Shapere & Wilczek ( ).

The 'triplet' structure of a formula for self-propulsion velocity (4.2) can be expected (without any calculations) on the basis of the linearity of the original problem for Stokes's equations as well as the known result for $N = 3$.

In our examples, all arms move harmonically (5.4); this does not provide the maximum of $\overline{V}_0$. For example, for a three-sphere robot $\overline{V}_0 \sim \langle\widetilde{l}_1\widetilde{l}_{2\tau}\rangle$ (5.3). Since $\widetilde{l}_1$ and $\widetilde{l}_{2\tau}$ represent mutually independent functions that are 2π-periodic in τ, it is clear that the maximum of this correlation appears when these functions coincide or are proportional to each other. If $\max\langle\widetilde{l}_1\widetilde{l}_{2\tau}\rangle$ is calculated under the constraint of fixed amplitudes (which is natural for realistic experimental devices of variable arm lengths), then it can be found that the theoretical maximum of this correlation is $2/\pi$, which is higher than $1/2$ for harmonic oscillations (5.4). This improvement will increase with the growth of N. In particular, non-harmonic periodic $\widetilde{l}_\alpha(\tau)$, providing optimal strokes, had been discovered in computational studies of four-sphere micro-robots by Alexander *et al.* ( ).

In our study we construct an asymptotic procedure with two small parameters: $\varepsilon \to 0$ and $\delta \to 0$. Such a setting usually requires the consideration of different asymptotic paths on the plane (ε, δ) when, say $\delta = \delta(\varepsilon)$. We can avoid such considerations, since small parameters appear (in the main order) as a product $\varepsilon^2\delta$.

The mathematical justification of the presented results can be performed as in Vladimirov (,) by the estimation of an error in the original equation. One can also derive the higher approximations of $\overline{V}_0$, as has been done by Vladimirov (,) for different cases. The higher approximations can be useful for studies of motion with $\overline{V}_0 \equiv 0$.

In the literature quoted in the Introduction one can find interesting discussions about the physical mechanism of self-propulsion of micro-robots. A clear illustration of this mechanism is given by Avron, Kenneth & Oaknin ( ). At the same time, one can see that the self-propulsion of deformable bodies in an inviscid fluid represents a classical topic, e.g. Saffman ( ) and Kanso & Newton ( ). It is interesting to note that the qualitative explanations of self-propulsion in an inviscid fluid and in self-propulsion in a creeping flow could be seen as the same if one replaces the term *virtual mass* (for an inviscid fluid) by *viscous drag* (for creeping flows).

Studies of different micro-robots by the same method as used in this paper can be found in Vladimirov (,). In Vladimirov ( ) the same method resulted in a new asymptotic model and a new equation (*the acoustic-drift equation*) for oscillating flows in acoustics.

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