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GAIN-PHASE STRUCTURES FOR LINEAR

MULTIVARIABLE SYSTEMS

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Research Report No. 256

May 1984

Indexing Terms: Gains and phases; multivariable systems; feedback; positive-real systems.

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Abstract

A general gain phase structure for linear multivariable systems is proposed and related to the ideas of quasi-Nyquist loci and singular value decompositions, the analysis of absolute stability using positive real conditions and eigenvalue estimation using Gershgorin's theorem and the numerical range.

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## 1. Introduction

A number of gain and phase characterizations of a linear multivariable system described by the mxm invertible transfer function matrix G(s) have been proposed in the form of

(i) the gain and phase characteristics of the characteristic loci (Postlethwaite and MacFarlane, 1979) of G,

(ii) a use of the polar decomposition of G (Postlethwaite, Edmunds and MacFarlane, 1981) and the identification of the gains (resp. phases) of G with the eigenvalues (resp. arguments of the eigenvalues) of the Hermitian (resp. Unitary) component of the decomposition, and

(iii) the use of approximation methods (Hung and MacFarlane, 1982) and the idea of systems gains and phases as the gains and phases of the quasi-Nyquist loci of G.

In all cases we can associate 2m input/output directions that describe the way that the gain-phase structure propagates through the system. In general, the input and output directions are distinct unless G is normal.

It is the purpose of this paper to explore the possibilities of systems theoretical concepts of gain and phase from a physical definition, to underline the infinity of possibilities that can be generated and to relate the existence of particular gain-phase structures to system properties such as positive-realness (Anderson, 1967; Kalman, 1963). The use of gain-phase structures is also considered in eigenvalue estimation with results similar in structure to those using diagonal dominance concepts (Rosenbrock, 1974) and singular value regions (Postlethwaite, Edmunds and MacFarlane, 1981).

## 2. A Gain-phase Structure for G

The input/output relations of G can be represented as

$$y(s) \quad = \quad G(s)u(s) \qquad \ldots(1)$$

where u and y are the input and output vector transforms respectively. The physical notion of a gain-phase structure on G(s) can be formulated as follows:

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Definition 1: Let $\Omega$ be a subset of the complex plane on which G is bounded and nonsingular. A gain-phase structure on G(s) relative to $\Omega$ consists of a pair of bijections $(T_u^{\ s}, T_y^{\ s})$ defined on $C^m$ and parameterized by $s \in \Omega$ and a collection $g_j(s)$, $1 \le j \le m$, of mappings of $\Omega$ into C such that the transformed inputs $\hat{u}(s) = T_u^{\ s} u(s)$ and transformed outputs $\hat{y}(s) = T_y^{\ s} y(s)$ satisfy the scalar relations

$$\hat{y}_j(s) \quad = \quad g_j(s)\hat{u}_j(s) \quad , \quad 1 \le j \le m \ , \quad s \in \Omega \qquad \ldots(2)$$

If $g_j(s) = b_j(s)e^{i\phi_j(s)}$ with $b_j$ and $\phi_j$ real and $b_j \ge 0$ then the sets $\{b_j(s)\}_{1 \le j \le m}$ and $\{\phi_j(s)\}_{1 \le j \le m}$ will be termed gains and phases of G(s) in

$\Omega$. The gain-phase structure will be termed continuous if $T_u^s$, $T_y^s$ and $g_j(s)$, $1 \leq j \leq m$, vary continuously with s on $\Omega$.

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The definition says, in effect, that the system can be transformed into m independent scalar systems by redefinition of inputs and outputs. In principle, the transformations could be nonlinear but, to remain consistent with the linearity of the system, we will only consider linear transformations. The obvious example of a gain-phase structure is obtained for systems with a spectral decomposition in $\Omega$ of the form

$$G(s) = W(s) \operatorname{diag} \{\tilde{g}_j(s)\}_{1 \leq j \leq m} V(s) \qquad \ldots(3)$$

with eigenvalues $\tilde{g}_j(s)$, $1 \leq j \leq m$, eigenvector matrix W(s) and inverse eigenvector matrix $V(s) = \hat{W}(s)$ (where $\hat{M}$ is used to denote the inverse of a matrix M). The transformations $T_u^s$ and $T_y^s$ are defined by $\hat{y}(s) = V(s)y(s)$ and $\hat{u}(s) = V(s)u(s)$ respectively and $g_j(s) \equiv \tilde{g}_j(s)$, $1 \leq j \leq m$. Note that the structure can always be taken to be continuous.

The form of gain phase structure considered in this paper will be of a 'projective' form. Let Q(s) be a Hermitian, positive-definite matrix that is continuously dependent on s in $\Omega$. For each point $s \in \Omega$, let $C^m$ be given a Hilbert space structure defined by the inner product

$$\langle x,y \rangle = x^* Q(s) y \qquad \ldots(4)$$

(where * denotes complex conjugate transpose). Let $x_j(s)$, $1 \leq j \leq m$, be m linearly independent vectors in $C^m$ depending continuously on s in $\Omega$ and normalized by the natural normalization

$$\langle x_j(s), x_j(s) \rangle = x_j^*(s) Q(s) x_j(s) = 1 \quad , \quad 1 \leq j \leq m \ , \ s \in \Omega \qquad \ldots(5)$$

but not necessarily orthogonal with respect to $\langle \cdot , \cdot \rangle$. Define the matrix

$$X(s) = [x_1(s), \ldots, x_m(s)] \qquad \ldots(6)$$

and $T_u^s$ and $T_y^s$ by

$$\hat{u}(s) = X^*(s)Q(s)u(s) \quad , \qquad y(s) = X(s)\hat{y}(s) \qquad \ldots(7)$$

In effect $\hat{u}_j(s) = x_j^* Q(s)u(s)$ is the orthogonal projection of the input onto $x_j(s)$ and $\hat{y}_j(s)$ is the jth coordinate of $y(s)$ expressed in terms of the basis $\{x_j(s)\}_{1\leq j\leq m}$ in $C^m$. The existence of a gain-phase structure depends upon the relation between $\hat{y}$ and $\hat{u}$. More precisely,

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<u>Definition 2</u>: With $\Omega$ and $Q(s)$ given, the basis $\{x_j(s)\}_{1\leq j\leq m}$ induces a gain-phase structure on $G(s)$ in $\Omega$ if there exists mappings $g_j(s)$ of $\Omega$ into $C$, $1\leq j\leq m$, such that relation (2) holds with $\hat{u}$ and $\hat{y}$ defined by (6) and (7). The structure is continuous on $\Omega$ if, and only if, the mappings $g_j(s)$, $1\leq j\leq m$, are continuous.

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<u>Definition 3</u>: If the conditions of definition 2 are satisfied for some choice of (continuous) $\{g_j(s)\}_{1\leq j\leq m}$, the system $G(s)$ is said to permit a (continuous) gain-phase structure on $\Omega$ with respect to $Q(s)$.

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The following matrix definition follows trivially from the above:

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<u>Theorem 1</u>: The basis $\{x_j(s)\}_{1\leq j\leq m}$ induces a (continuous) gain-phase structure on $G(s)$ in $\Omega$ if, and only if,

(a) $G(s) = X(s)D(s)X^*(s)Q(s) \quad , \quad s\in\Omega \qquad \ldots(8)$

(b) the diagonal terms of $N(s) \triangleq X^*(s)Q(s)X(s)$ are all unity,

(c) $D(s)$ is diagonal of the form

$$D(s) \;=\; \text{diag}\,\{g_j(s)\}_{1\leq j\leq m} \qquad \ldots(9)$$

with (continuous) diagonal terms $g_j(s)$, $1\leq j\leq m$.

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To illustrate the generality of the decomposition, write the spectral decomposition (3) in the form

$$G(s) = W(s)D(s)W^*(s)(V^*(s)V(s)) \quad , \quad s\in\Omega \qquad ...(10)$$

and choose $Q(s) = V^*(s)V(s)$, $X(s) = W(s)$ and $g_j(s) = \tilde{g}_j(s)$, $1\le j\le m$. Conditions (a) and (c) of theorem 1 are satisfied and also condition (b) as $N = X^*QX = W^*V^*VW = I_m$. We conclude that the spectral decomposition of G is a special case of a gain-phase structure described in Definition 2. If G(s) does not have a diagonal canonical form at every point $s\in\Omega$ then the above analysis fails. The following result demonstrates, however, that suitable choice of Q(s) guarantees the existence of gain-phase structures.

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Theorem 2: Let G(s) have a polar decomposition of the form

$$G(s) = U(s)H(s) \qquad ...(11)$$

with U(s) unitary with eigenvalues $e^{i\theta_j(s)}$, $1\le j\le m$, and H(s) Hermitian, positive-definite with real eigenvalues $\underline{\sigma}(G(s)) \triangleq \sigma_1(s) \le \sigma_2(s) \le \ldots \le \sigma_m(s) \triangleq \bar{\sigma}(G(s))$. If U(s) has the spectral decomposition

$$U(s) = X_o(s)\ \text{diag}\ \{e^{i\theta_j(s)}\}_{1\le j\le m}\ X_o^*(s) \quad , \quad s\in\Omega \qquad ...(12)$$

with $X_o(s)$ unitary, then G(s) permits a gain-phase structure on $\Omega$ with respect to $Q(s) \triangleq H(s)$ with

$$X(s) \triangleq X_o(s)\ \text{diag}\ \{a_j^{-\frac{1}{2}}(s)\}_{1\le j\le m} \qquad ...(13)$$

$$g_j(s) \triangleq a_j(s)e^{i\theta_j(s)} \quad , \quad 1\le j\le m \qquad ...(14)$$

where $\quad a_j(s) \triangleq (X_o^*(s)H(s)X_o(s))_{jj} = \overline{a_j(s)} > 0,\ 1\le j\le m.$

Proof: We verify conditions (a), (b) and (c) of theorem 1 by noting that $G = X_o\text{diag}\{e^{i\theta_j}\}X_o^*H \equiv X\ \text{diag}\{g_j\}X^*Q$ from the definitions. This verifies

(a) and (c) whilst (b) follows from the relation $N = \mathrm{diag}\{a_j^{-\frac{1}{2}}\}X_o^* H X_o$ $\mathrm{diag}\{a_j^{-\frac{1}{2}}\}$ and examination of the diagonal terms.

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The result can be interpreted as stating that G permits a gain-phase structure with phases $\phi_j(s) = \theta_j(s)$ equal to the principal phases (Postlethwaite, Edmunds and MacFarlane, 1981) and gains $b_j(s) = a_j(s)$ bounded by the principal gains $\sigma_1(s),\ldots,\sigma_m(s)$ by the relations

$$\underline{\sigma}(G(s)) \leq b_j(s) \leq \bar{\sigma}(G(s)),\ 1\leq j\leq m,\ s\in\Omega \qquad \ldots(15)$$

$$\sum_{j=1}^{m} b_j(s) \equiv \sum_{j=1}^{m} \sigma_j(s) \qquad \ldots(16)$$

as is easily proved by the definition of $a_j(s)$, $1\leq j\leq m$, and noting that $\mathrm{tr}\, H \equiv \mathrm{tr}(X_o^* H X_o)$. The gain-phase structure defined is very similar to the quasi-Nyquist decomposition (Hung and MacFarlane, 1982) of G but differs in that it associates gain with the principal phases rather than phase with the principle gains.

It is clear that gain-phase structures do exist by suitable choice of Q. In fact, an infinite number appear to exist and can be characterized as follows:

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Theorem 3: Given $Q(s)$ and $\Omega$, $G(s)$ permits a gain-phase structure on $\Omega$ with respect to $Q(s)$ if, and only if, the matrix

$$M(s) \triangleq Q(s)\hat{G}(s)\hat{Q}(s)G^*(s) \qquad \ldots(17)$$

possesses m linearly independent eigenvectors $z_j(s)$, $1\leq j\leq m$, at each point $s\in\Omega$ with the property that

$$z_j^*(s)G(s)\hat{Q}(s)z_k(s) = 0 \quad , \quad j \neq k,\ s\in\Omega \qquad \ldots(18)$$

Moreover, under these conditions, the structure is induced by $x_j(s)$, $1 \le j \le m$, defined by

$$x_j(s) \triangleq \hat{z}_j(s) \Big/ (\hat{z}_j^{*} Q(s) \hat{z}_j(s))^{\frac{1}{2}} \quad , \quad 1 \le j \le m, \quad s \in \Omega \quad \ldots(19)$$

where $\hat{z}_j(s)$ is the jth column of $\hat{Z}^{*}(s)$ where $Z(s) \triangleq [z_1(s), \ldots, z_m(s)]$. The corresponding scalars $g_j(s)$, $1 \le j \le m$, are given by

$$g_j(s) \triangleq (\hat{z}_j^{*}(s) Q(s) \hat{z}_j(s))(z_j^{*}(s) G(s) \hat{Q}(s) z_j(s)) \quad \ldots(20)$$

for $1 \le j \le m$, $s \in \Omega$.

<u>Proof</u>: If G possesses the gain-phase structure (8) on $\Omega$, then

$$M(s) \equiv \hat{X}^{*}(s) \hat{D}(s) D^{*}(s) X^{*}(s) \quad , \quad s \in \Omega \quad \ldots(21)$$

so that $\hat{X}^{*}(s)$ is an eigenvector matrix of M(s) in $\Omega$. We can therefore identify $\hat{z}_j(s)$ with $x_j(s)$, equation (18) following from the consequent identity $\hat{X}(s) G(s) \hat{Q}(s) \hat{X}^{*}(s) \equiv D(s)$ by equating off-diagonal terms. Conversely, if M(s) has a complete set of eigenvectors $z_j(s)$, $1 \le j \le m$, satisfying (18), it is clear that

$$Z^{*}(s) G(s) \hat{Q}(s) Z(s) \triangleq D_o(s) = \text{diag}\{z_j^{*}(s) G(s) \hat{Q}(s) z_j(s)\}_{1 \le j \le m} \quad \ldots(22)$$

or $G \equiv \hat{Z}^{*} D_o \hat{Z} Q$ which has the required form if the normalization (19) of the columns of $X_o \triangleq \hat{Z}^{*}$ is performed and leads directly to the form (20) for $g_j(s)$, $1 \le j \le m$.

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If the conditions on M(s) are satisfied, the result provides a computation scheme for the factorization. In fact, the existence of linearly independent eigenvectors is generically satisfied but not always as evidenced by the example

$$G(s)\hat{Q}(s) \equiv \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow M(s) \equiv \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \qquad \ldots(23)$$

If however, $M(s)$ has a complete set of eigenvectors in $\Omega$, necessary and sufficient conditions for the existence of the gain-phase structure can be stated as follows:

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Theorem 4: $M(s)$ satisfies the conditions of theorem 3 if, and only if, it has a complete set of eigenvectors and eigenvalues $\lambda_j(s)$, $1 \le j \le m$, of unit modulus only at each point $s \in \Omega$.

Proof: Writing $M(s)z_j(s) = \lambda_j(s)z_j(s)$ leads to

$$\hat{Q}G^* z_j = \lambda_j G\hat{Q}z_j \quad , \quad 1 \le j \le m \qquad \ldots(24)$$

and hence

$$z_k^* \hat{Q}G^* z_j = \lambda_j z_k^* G\hat{Q}z_j \quad , \quad 1 \le j,k \le m \qquad \ldots(25)$$

By symmetry we obtain

$$z_j^* \hat{Q}G^* z_k = \lambda_k z_j^* G\hat{Q}z_k \quad , \quad 1 \le j,k \le m \qquad \ldots(26)$$

and hence

$$(\lambda_j^{-1} - \bar{\lambda}_k) z_k^* G\hat{Q}z_j = 0 \quad , \quad 1 \le j,k \le m \qquad \ldots(27)$$

If all eigenvalues $\{\lambda_j\}$ are distinct and of unit modulus, it follows that

$$z_k^*(s)G(s)\hat{Q}(s)z_j(s) = 0 \quad , \quad k \ne j,\ s \in \Omega \qquad \ldots(28)$$

as required. If all eigenvalues are not distinct then (27) indicates that (28) holds if $\lambda_j \ne \lambda_k$. Let M have distinct eigenvalues $\mu_j$, $1 \le j \le q$, of multiplicity $q_j$, $1 \le j \le q$, and let $V_j$ be the eigensubspace of $C^m$ corresponding to $\mu_j$. Let $V_j^b$ be a $m \times q_j$ basis matrix for $V_j$ and set the eigenvector matrix $Z = [V_1^b, V_2^b, \ldots, V_q^b]$. Relation (28) indicates that

$$Z^* G\hat{Q}Z \;=\; \text{block diag } \{G_j\}_{1\leq j\leq q} \qquad \ldots(29)$$

where $G_j$ is nonsingular of dimension $q_j x q_j$, $1\leq j\leq q$. A simple calculation yields

$$M \;=\; Z \text{ block diag } \{\hat{G}_j G_j^*\}_{1\leq j\leq q} \hat{Z} \qquad \ldots(30)$$

and hence $\hat{G}_j G_j^* = \mu_j I_{q_j}$, $1\leq j\leq q$. Each $G_j$ is hence normal and we can choose unitary $q_j x q_j$ matrices $U_j$, $1\leq j\leq q$, such that $U_j^* G_j U_j = \text{diag}\{g_{jk}\}_{1\leq k\leq q_j}$, $1\leq j\leq q$. Setting $U = \text{block diag}\{U_j\}_{1\leq j\leq q}$ and applying to (29) yields $(ZU)^* G\hat{Q}(ZU)$ to be diagonal and hence the columns of ZU are eigenvectors of M satisfying the required condition (18). This proves sufficiency. To prove necessity, note that (18) requires that $z_k^* G\hat{Q} z_k \neq 0$, $1\leq k\leq m$, $s\in\Omega$ and hence, using (25) with $j = k$, $\lambda_k = z_k^* \hat{Q} G^* z_k / z_k^* G\hat{Q} z_k$ has unit modulus, $1\leq k\leq m$, $s\in\Omega$.

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Corollary 4.1: Under the conditions of theorem 4,

$$\arg g_j(s) = n_j\pi - \tfrac{1}{2}\arg\lambda_j(s) \quad , \quad 1\leq j\leq m,\ s\in\Omega \qquad \ldots(31)$$

by suitable ordering and choice of integers $n_j$.

Proof: Use (20) to show that $\arg g_j = \arg(z_j^* G\hat{Q} z_j) = n_j\pi - \tfrac{1}{2}\arg\lambda_j$ as (25) indicates that $\bar{\lambda}_j = z_j^* G\hat{Q} z_j / (z_j^* G\hat{Q} z_j)$.

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Although the unit modulus requirement of the result appears to be restrictive, the paper will illustrate that this is not so. For example,

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Theorem 5: Sufficient conditions for M(s) to have linearly independent eigenvectors and eigenvalues of unit modulus are that either

(a) $G(s)\hat{Q}(s)$ is a normal matrix in $\Omega$ or

(b) $0 \notin V_0(M(s))$ , $s\in\Omega$

where

$$V_o(M(s)) \triangleq \{\mu : \mu = x^*\hat{Q}(s)G^*(s)x \quad , \quad x^*\hat{Q}(s)x = 1\} \qquad \ldots(32)$$

is the numerical range (Bonsall and Duncan, 1971) of $G^*$ with respect to the inner product $<x,y>' \triangleq x^*\hat{Q}(s)y$ in $C^m$.

Proof: For case (a), let $G\hat{Q} = VD_oV^*$ with $V = [v_1, v_2, \ldots, v_m]$ unitary and $D_o$ diagonal. Clearly $M \equiv V\hat{D}_oD_o^*V^*$ so that $|\lambda_j| = 1$, $1 \le j \le m$, $s \in \Omega$ and we can choose $z_j = v_j$, $1 \le j \le m$, $s \in \Omega$. For case (b), note that $z_k^*G\hat{Q}z_k \ne 0$, $1 \le k \le m$, $s \in \Omega$ and hence all eigenvalues of M have unit modulus from (25). In fact M must have a complete set of eigenvectors otherwise we can choose an eigenvalue $\lambda$ and non-zero vectors x,y such that $Mx = \lambda x$ and $My = \lambda y + x$. It follows that

$$\lambda x^*G\hat{Q}y = x^*\hat{Q}G^*y \qquad \ldots(33)$$

$$x^*\hat{Q}G^*y = \lambda x^*G\hat{Q}y + x^*G\hat{Q}x \qquad \ldots(34)$$

from which $x^*\hat{Q}G^*x = 0$ which is impossible.

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There is hence a rich class of systems possessing gain-phase structures for a given choice of Q, whilst theorem 2 indicates that a given system permits a gain phase structure on $\Omega$ with respect to a non-empty set of Q.

Previous studies (Postlethwaite, Edmunds and MacFarlane, 1981; Hung and MacFarlane, 1982) have noted that important design implications follow from normality of G (when the input/output directions are orthogonal). A similar situation can be identified for the gain-phase structures presented above:

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Theorem 6: If G(s) permits a gain-phase structure (8) on $\Omega$ with respect to Q(s), then we can choose $X^*(s)Q(s)X(s) = I_m$ on $\Omega$ if, and only if, $G(s)\hat{Q}(s)G^*(s)Q(s) \equiv \hat{Q}(s)G^*(s)Q(s)G(s)$.

(Remarks: (1) $X^*QX \equiv I_m$ simply means that the vectors $x_j(s)$, $1 \leq j \leq m$, are orthonormal with respect to the inner product $\langle x,y \rangle = x^*Q(s)y$ in $C^m$ which, if $Q(s) \equiv I_m$, reduces to the more well-known orthonormality condition $x_j^*(s)x_k(s) = \delta_{jk}$, $s \in \Omega$ used in Postlethwaite, Edmunds and MacFarlane (1981) and Hung and MacFarlane (1982).

(2) The identity $G\hat{Q}G^*Q \equiv \hat{Q}G^*QG$ simply states that G is normal in the sense that it commutes with its adjoint $\hat{Q}G^*Q$ with respect to the inner product $\langle \cdot,\cdot \rangle$ in $C^m$. This can be viewed as stating that $Q^{\frac{1}{2}}GQ^{-\frac{1}{2}}$ is normal. In both interpretations, we obtain the classical normality condition $GG^* = G^*G$ if $Q = I$).

Proof: If $G = XDX^*Q$ and $X^*QX = I_m$ we have $GX = XD$ or $G^*\hat{X}^* = \hat{X}^*D^*$. As (theorem 3) $\hat{X}^*$ can be identified with an eigenvector matrix Z of M, we conclude that $G^*$ commutes with M ie $G^*\hat{Q}\hat{G}QG^* = \hat{Q}\hat{G}QG^*G^*$ or $G\hat{Q}G^*Q = \hat{Q}G^*QG$ as required. Conversely, if $G\hat{Q}G^*Q \equiv \hat{Q}G^*QG$, $s \in \Omega$, we can use the implied normality of $Q^{\frac{1}{2}}GQ^{-\frac{1}{2}}$ (remark (2) above) to write $Q^{\frac{1}{2}}GQ^{-\frac{1}{2}} = VD_1V^*$ where V is unitary and $D_1$ is diagonal. After rearrangement this takes the form $G = (Q^{-\frac{1}{2}}V)D_1(Q^{-\frac{1}{2}}V)^*Q$. We note now that (theorem 1) this is a gain-phase decomposition of G in $\Omega$ with $D_1$ identified with D and X with $Q^{-\frac{1}{2}}V$ and $X^*QX = V^*Q^{-\frac{1}{2}}QQ^{-\frac{1}{2}}V = I_m$ as required.

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The normality conditions have a strong impact on the form of the $g_j(s)$, $1 \leq j \leq m$:

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Corollary 6.1: Under the conditions of theorem 6, X(s) is an eigenvector matrix of G(s) in $\Omega$, $g_j(s)$, $1 \leq j \leq m$, are its eigenvalues and $\hat{X}(s) \equiv X^*(s)Q(s)$, $s \in \Omega$.

(Remark: under the stated conditions, the gain-phase structure is essentially that of (10) induced by the spectral decomposition).

Proof: If $G \equiv XDX^*Q$ and $X^*QX = I$, then $GX = XD$ or $G = X\hat{D}X$.

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This 'spectral' result indicates that the $g_j(s)$, $1 \leq j \leq m$, can be strongly related to the eigenvalues (and hence characteristic loci) of G, equality holding under the 'normality' conditions of theorem 6. Intuitively, it can be expected that the $g_j(s)$, $1 \leq j \leq m$, can be used as approximations to the eigenvalues $\tilde{g}_j(s)$, $1 \leq j \leq m$, even if normality is not present, the deviation from normality (represented by the elements of $X^*QX-I_m$) being a measure of the quality of the approximation. We will return to this problem in section 4 and content ourselves with the observation that the gain-phase structures permitted by G have a strong physical interpretation and links with well-established methods. This theme is continued in the next section by relating the gain-phase structure to positive-real properties of G.

## 3. Linear Absolute Stability and Positive-realness

The system G is considered in the presence of unity-negative feedback with forward path controller $K = pI_m$ where p is a positive scalar.

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Definition 4: G is absolutely stable in the linear sense (ASILS) if it is closed-loop stable in the presence of all scalar gains $p \geq 0$.

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Lemma 1: G is ASILS if, and only if, it is stable and no eigenvalue $\tilde{g}_j(s)$ of G lies on the negative real axis $R^- \triangleq \{z : z = \bar{z} < 0\}$ for $s = i\omega$, $-\infty < \omega < \infty$.

Proof: A trivial application of characteristic locus methods (Postlethwaite and MacFarlane, 1979).

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The following development indicates a strong relationship between ASILS and the structure of gain-phase decompositions by taking $\Omega = \{z : z = i\omega, -\infty<\omega<\infty\}$. The analysis is initiated by the following eigenvalue phase location result:

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Theorem 7: Let $G(s)$ permit a gain-phase structure on $\Omega$ with respect to $Q(s)$. Then, for each point $s \in \Omega$, all eigenvalues of $G(s)$ lie in the smallest closed cone $P(g_j(s);\ 1\leq j\leq m)$ of vertex the origin of the complex plane containing all $g_j(s)$, $1\leq j\leq m$.

Proof: If $|\lambda I_m - G(s)| = 0$, then $|\lambda I_m - X^* QXD| = 0$. Defining the self-adjoint 'frame-matrix'

$$N(s) \triangleq X^*(s)Q(s)X(s) \qquad \ldots(35)$$

this indicates the existence of a non-zero vector $x \in C^m$ such that $\lambda \hat{N}x = Dx$ and hence

$$\lambda = \frac{1}{x^* \hat{N} x} \sum_{j=1}^{m} g_j(s)|x_j|^2 \in P(g_j(s);\ 1\leq j\leq m) \qquad \ldots(36)$$

as $x^*\hat{N}x$ is real and strictly positive.

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In many cases (namely when the phases of the $g_j(s)$ have a spread of more than $\pi$) $P(g_j(s);1\leq j\leq m)$ will consist of the whole complex plane. We will concentrate on the case when P is proper however. This situation is formally characterized as follows:

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Lemma 2: $P(g_j(s);1\leq j\leq m)$ is a proper subset of the complex plane if, and only if, there exists a phase rotation $\theta(s)$ such that

$$\mathrm{Re}\{e^{i\theta(s)} g_j(s)\} \geq 0 \quad , \quad 1\leq j\leq m \qquad \ldots(37)$$

(Remark: that is, all $g_j(s)$ can be notated into the closed-right-half plane by a common rotation $\theta(s)$).

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Turning now to ASILS we obtain the following sufficient condition:

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Theorem 8: Let $G(s)$ be stable and permit a gain-phase structure on $\Omega = \{z = i\omega, \omega \geq 0\}$ with respect to $Q(s)$ with the property that either

$$\text{(a)} \quad Re\{e^{i\theta(s)} g_j(s)\} > 0 \quad , \quad 1 \leq j \leq m, \; s \in \Omega \qquad \ldots(38)$$

or

$$\text{(b)} \quad Re\{e^{i\theta(s)} g_j(s)\} \geq 0 \quad , \quad 1 \leq j \leq m, \; s \in \Omega \qquad \ldots(39)$$

where $\theta(s)$ is an angular rotation in the complex plane in the ranges (a) $[-\frac{\pi}{2}, \frac{\pi}{2}]$ or (b) $]-\frac{\pi}{2}, \frac{\pi}{2}[$ respectively. Then $G(s)$ is ASILS.

Proof: If $G$ is not ASILS, there will exist $\omega \geq 0$, $p > 0$ such that $|I_m + pG(i\omega)| = 0$ and hence $|e^{i\theta(i\omega)}\hat{N}(i\omega) + pe^{i\theta(i\omega)}D(i\omega)| = 0$. Let $x$ be any non-zero vector in the kernel of $e^{i\theta}\hat{N} + pe^{i\theta}D$ and note that

$$e^{i\theta(i\omega)} x^*\hat{N}(i\omega)x + px^*(e^{i\theta(i\omega)}D(i\omega))x = 0 \qquad \ldots(40)$$

Taking real parts yields

$$0 = \cos\theta(i\omega)x^*\hat{N}(i\omega)x + p\sum_{j=1}^{m} |x_j|^2 Re\{e^{i\theta(i\omega)} g_j(i\omega)\} \qquad \ldots(41)$$

which is impossible under the stated conditions. $G(s)$ is hence ASILS.

---

The geometrical interpretation of the conditions is simply that

$$(-1,0) \notin P(g_j(s);\; 1 \leq j \leq m) \quad , \quad s \in \Omega \qquad \ldots(42)$$

which (lemma 2) precludes the possibility of any characteristic locus crossing the negative real axis of the complex plane. A systems theoretical connection is obtained by writing

$$e^{i\theta(s)}G(s)\hat{Q}(s) + e^{-i\theta(s)}\hat{Q}(s)G^*(s) = X(s)\,\text{diag}\{2Re\{e^{i\theta(s)}g_j(s)\}\}_{1 \leq j \leq m} X^*(s) \qquad \ldots(43)$$

which indicates that the conditions (38) and (39) of theorem 8 are satisfied if, and only if, $e^{i\theta}G\hat{Q} + e^{-i\theta}\hat{Q}G^*$ is positive definite and positive semidefinite respectively. These connections with positive realness can be strengthened as follows:

---

<u>Theorem 9</u>: G(s) is strictly positive real on $\Omega$ in the sense that

$$G(s) + G^*(s) > 0 \qquad , \qquad s \in \Omega \qquad \ldots(44)$$

if, and only if, there exists an mxm nonsingular matrix X(s) and functions $g_j(s)$, $1 \leq j \leq m$, defined on $\Omega$ such that

(a) $G(s) \equiv X(s)\ \mathrm{diag}\ \{g_j(s)\} X^*(s) \quad , \quad s \in \Omega$

and (b) $g_j(\Omega)$ lies in the <u>open</u> right-half complex plane, $1 \leq j \leq m$. (Remark: The conditions correspond to the requirement that G permits a gain-phase structure on $\Omega$ with respect to $Q(s) = I_m$ with the property that the phases of the generated $g_j(s)$, $1 \leq j \leq m$, lie in the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$ in $\Omega$).

<u>Proof</u>: Sufficiency follows from (43) with $\theta(s) \equiv 0$ and $Q(s) \equiv I_m$. Necessity follows from theorem 5(b) by noting that (44) implies that $0 \notin V_0(M(s))$ and hence that the factorization (a) above exists. Condition (b) follows from (43) again as $G+G^*>0$ implies $\mathrm{Re}\{g_j(s)\}>0$, $1 \leq j \leq m$, $s \in \Omega$.

---

In essence, the result states that G(s) is strictly positive real in $\Omega$ if, and only if, the m scalar functions $g_j(s)$ are strictly positive real on $\Omega$. This is rather a nice intuitive result consistent with the idea pervading multivariable control (Postlethwaite and MacFarlane, 1979; Owens, 1978; Rosenbrock, 1974) that mxm systems are best characterized for design purposes by m (independent) scalar systems. A similar (but slightly weaker) statement can be made for positive-real systems, but requires the process G to possess a defined (generic) structural property:

---

<u>Theorem 10</u>: There exists an mxm nonsingular matrix X(s) and functions $g_j(s)$, $1\leq j\leq m$, defined on $\Omega$ such that

(a) $G(s) = X(s)\ \text{diag}\ \{g_j(s)\}_{1\leq j\leq m} X^*(s)$ , $s\in\Omega$

(b) $g_j(\Omega)$ lies in the <u>closed</u> right-half complex plane, $1\leq j\leq m$,

if, and only if,

(i) G is positive real on $\Omega$ in the sense that

$$G(s) + G^*(s) \geq 0 \quad , \quad s\in\Omega \qquad \ldots(45)$$

and (ii) $R(G(s)+G^*(s)) \cap G(s)\ker(G(s)+G^*(s)) = \{0\}$ , $s\in\Omega$ ...(46)

(Note: Here R(K) and ker(K) denote the range and kernel of K respectively).

<u>Proof</u>: To prove necessity note that (i) follows from (43) with $\theta(s) \equiv 0$ and $Q(s) \equiv I_m$ whilst (b) follows by noting that, for $s\in\Omega$,

$$R(G(s)+G^*(s)) = \text{span}\{x_j(s);\ g_j(s) + \overline{g_j(s)} \neq 0\} \qquad \ldots(47)$$

whilst

$$G(s)\ker(G(s)+G^*(s)) = \text{span}\{x_j(s);\ g_j(s) + \overline{g_j(s)} = 0\} \qquad \ldots(48)$$

where $X(s) = [x_1(s),\ldots,x_m(s)]$. To prove sufficiency, suppose that z is an eigenvector of $M = G^{-1}G^*$ with eigenvalue $\lambda$ ie $G^*z = \lambda Gz$. If $z^*Gz = 0$, then $z^*(G+G^*)z = 0$ and hence $(G+G^*)z = 0$ ie $\lambda = -1$. If $z^*Gz \neq 0$ we have $\lambda = z^*G^*z/z^*Gz$ which has unit modulus. We conclude that all eigenvalues of M have unit modulus on $\Omega$. To prove the existence of a complete set of eigenvectors, let $\lambda$ be an eigenvalue of M. Suppose that there exists non-zero vectors x,y such that $Mx = \lambda x$ and $My = \lambda y+x$. Equations (33) and (34) then hold with $Q = I_m$ and hence $x^*Gx = 0$. From the above this requires $\lambda = -1$ ie all eigenvalues $\lambda \neq -1$ of M have geometric multiplicity one as required. If $\lambda = -1$ then $(G+G^*)x = 0$ and $(G+G^*)y = Gx$. Condition (ii) then implies that $(G+G^*)y = 0$ ie $My = -y$ and $x = 0$ contrary to assumption. M(s) hence has a complete set of eigenvectors and eigenvalues of unit modulus only on $\Omega$. Theorem 4 then indicates the existence of the gain-phase structure (a) for G(s) on $\Omega$ whilst (b) follows from (43) and (45). This completes the proof of the theorem.

---

The interpretation of the result is identical to that of theorem 9 with positive-realness of G being equivalent to positive realness of the m scalar functions $g_j(s)$, $1\leq j\leq m$, provided that G(s) possesses the structural characteristic (46). This constraint is important as can be seen by the example of equation (23) with $Q(s) \equiv I_2$ which satisfies $G+G^* \geq 0$ in any set $\Omega$ but does not permit a gain-phase factorization as M does not have a complete set of eigenvectors.

To conclude this section, we note that the gain-phase structures described above are capable of describing many dynamic phenomena in linear open-loop and closed-loop systems. The examples included demonstrate that both spectral and polar decompositions can be set in this framework including an intuitive relationship between the spectrum of G on $\Omega$ and the functions $g_j(s)$, $1\leq j\leq m$, expressed in terms of the deviations of G from normality. Also the existence of positive-realness properties of G(s) is equivalent to the existence of certain types of gain-phase structures. This connections are pursued further in the next section by examination of the applications in eigenvalue location.

## 4. Eigenvalue-estimation using Gain-phase Structures

Eigenvalue estimation plays an important role in design theory (Postlethwaite, Edmunds and MacFarlane, 1981; Hung and MacFarlane, 1982; Rosenbrock, 1974; Owens, 1978), and, in the authors opinion, can be fruitfully developed further. The use of gain-phase structures of the general type discussed in this paper is considered in this section based on the use of Gershgorins theorem (Rosenbrock, 1974; Owens, 1978) and singular values and the numerical range (Bonsall and Duncan, 1971). Throughout the section it will be assumed that G(s) permits a gain-phase structure on $\Omega$ with respect to Q(s). The 'frame-matrix' N(s) defined by (35) will play an important role in the development.

4.1. Eigenvalue Estimation using Gershgorins Theorem

A complex number $\lambda$ is an eigenvalue of G(s) if, and only if, $|\lambda I_m - G(s)| = 0$.

---

Theorem 11: If $\lambda$ is an eigenvalue of G(s), then $\lambda \in B_j(s)$ for some j in the range $1 \leq j \leq m$ where $B_j(s)$ is the circle

$$B_j(s) \triangleq \{\mu \in C : |\mu - g_j(s)| \leq |g_j(s)| \sum_{k \neq j} |N_{jk}(s)|\} \quad \ldots(49)$$

of centre $g_j(s)$ and radius

$$r_j(s) \triangleq |g_j(s)| \sum_{k \neq j} |N_{jk}(s)| \quad \ldots(50)$$

Proof: The eigenvalues of G(s) are identical to those of D(s)N(s) and the result follows by application of Gershgorins theorem (Rosenbrock, 1974; Owens, 1978), noting that $N_{jj}(s) = 1$, $1 \leq j \leq m$.

---

Corollary 11.1: If $G(s)\hat{Q}(s)G^*(s)Q(s) = \hat{Q}(s)G^*(s)Q(s)G(s)$ then the eigenvalues of G(s) are identical to $g_j(s)$, $1 \leq j \leq m$, and $r_j(s) = 0$, $1 \leq j \leq m$. (Note: this result duplicates corollary 6.1 and is stated for interest only as it relates the normality condition to the Gershgorin circles defined above).

Proof: From theorem 6, $N(s) = I_m$ and hence $r_j(s) = 0$, $1 \leq j \leq m$, the result being completed by application of corollary 6.1, or the following result:

---

Corollary 11.2: If A is a subset of $\{1,2,\ldots,m\}$ with $q(A)$ distinct entries with the property that

$$\{\bigcup_{j\in A} B_j(s)\} \cap \{\bigcup_{j\notin A} B_j(s)\} = \emptyset \qquad \ldots(51)$$

then $\bigcup_{j\in A} B_j(s)$ contains exactly $q(A)$ eigenvalues of $G(s)$.

Proof: Replace $N(s)$ by $N_\varepsilon(s)$ with entries $(N_\varepsilon(s))_{jk} = (\delta_{jk}+\varepsilon(1-\delta_{jk}))N_{jk}(s)$ and $0 \leq \varepsilon \leq 1$ and apply an eigenvalue continuity argument using theorem 11.

---

Note that the radii of the circles $r_j(s)$ depend critically on the off-diagonal structure of the 'frame matrix' $N(s)$. Corollary 6.1 and 11.1 indicate that, if the off-diagonal terms are zero the $g_j(s)$, $1 \leq j \leq m$, are equal to the eigenvalues of $G(s)$ whilst theorem 11 expresses the magnitude of the approximation errors involved in approximating the eigenvalues by $g_j(s)$, $1 \leq j \leq m$, in terms of the off-diagonal terms of $N(s)$. The closer G is to normality (in the sense of theorem 6), the better the approximation involved! These results are an exact analogue of those to be found in Hung and MacFarlane (1982) but valid for the more general gain-phase structures considered here.

The structure of $N(s)$ has other implications:

---

Proposition 1: $0 \notin B_j(s)$, $1 \leq j \leq m$, if, and only if, $N(s)$ is diagonally dominant.

Proof: $0 \notin B_j(s)$ iff $r_j(s)/|g_j(s)| < 1$ which is equivalent to row dominance of $N(s)$. As $N(s) = N^*(s)$, an identical conclusion follows if N is column dominant.

---

In particular, it is possible to prove a simple stability result similar in structure to the direct Nyquist array (Rosenbrock, 1974; Owens, 1978) but likely to be less conservative as the Gershgorin circles used will have small radii if G is near normal in the sense of theorem 6.

---

<u>Theorem 12</u>: Let $\Omega$ be identified with the Nyquist contour in the complex plane. Then G(s) is stable in the presence of unity-negative feedback if

$$(1)\quad n_o + \sum_{j=1}^{m} n_j = 0 \qquad \ldots(52)$$

where $n_j$ is the number of clockwise encirclements of the (-1,0) point by the image $\Gamma_j$ of $\Omega$ under the map $s \mapsto g_j(s)$ and $n_o$ is the number of poles of G in the interior of $\Omega$, and

$$(2)\quad (-1,0) \notin B_j(s) \quad , \quad 1 \le j \le m \ , \ s \in \Omega \qquad \ldots(53)$$

<u>Proof</u>: The proof is based on theorem 11 but is identical to standard proofs to be found in Rosenbrock (1974) and Owens (1978). It is hence omitted for brevity.

---

An inverse Nyquist form of the result is stated as follows,

---

<u>Theorem 13</u>: If $\Omega$ is the Nyquist contour, then G(s) is closed-loop stable if

$$(a)\quad n_o + \sum_{j=1}^{m} (\hat{n}_j - \tilde{n}_j) = 0 \qquad \ldots(54)$$

(b) $(-1,0) \notin \hat{B}_j(s)$ , $1 \le j \le m$, $s \in \Omega$ where $\hat{B}_j(s)$ is the circle

$$\hat{B}_j(s) \triangleq \{\mu \in C : |\mu - g_j^{-1}(s)| \le \sum_{k \ne j} |N_{jk}(s)|\} \qquad \ldots(55)$$

where $\hat{n}_j$ (resp. $\tilde{n}_j$) is the number of clockwise encirclements of the (-1,0) (resp. (0,0)) point of the complex plane by the image $\hat{\Gamma}_j$ of $\Omega$ under the map $s \mapsto g_j^{-1}(s)$.

Proof: Write $|I+G(s)| = |I+DN| = |\hat{D}+N|/|\hat{D}|$ and apply standard INA arguments remembering that $N_{jj}(s) \equiv 1$, $1 \le j \le m$.

---

A number of other results can be derived based on gain-phase structures on $\hat{G}(s)$, but they are omitted for brevity. Note however that the possibility of either approach exists simultaneously:

---

Proposition 2: If G(s) permits a gain-phase structure on $\Omega$ with respect to Q(s) that so does $\hat{G}(s)$.

Proof: If $G \equiv XDX^*Q$, then $\hat{G} = \hat{Q}\hat{X}^*\hat{D}\hat{X} = (\hat{Q}\hat{X}^*)D(\hat{Q}\hat{X}^*)^*Q$ which is a gain-phase structure on $\hat{G}$ after normalization of the columns of $\hat{Q}\hat{X}^*$.

---

4.2. Eigenvalue Estimation and Singular Values

In this section we continue with the use of $g_j(s)$, $1 \le j \le m$, and the 'frame matrix' N(s) in eigenvalue estimation to obtain results similar in structure to those of Postlethwaite, Edmunds and MacFarlane (1981). The results essentially add gain information to the phase spread information provided by theorem 7. We begin with a statement of Postlethwaite's result (1981) using the cone notation of theorem 7 and using

$$A(a,b) \triangleq \{z \in C : a \le |z| \le b\} \qquad \ldots(56)$$

to denote an annulus in the complex plane of internal and external radii a and b respectively.

---

Theorem 14: With the notation of theorem 2, all eigenvalues of G(s) lie in the set

$$S_{sV}(s) \triangleq A(\underline{\sigma}(G(s)),\bar{\sigma}(G(s))) \cap P(e^{i\theta_j(s)}\ ;\ 1\le j\le m) \qquad \ldots(57)$$

---

The following results have a structural similarity to the above but are expressed in terms of general gain-phase structure permitted by G.

---

Theorem 15: Let the Hermitean, positive-definite matrix $N_o(s) \triangleq N(s)D^*(s)D(s)N(s)$ have eigenvalues $0 < \mu_1^2(s) \le \mu_2^2(s) \le .. \le \mu_m^2(s)$, then all eigenvalues of G(s) lie in the set

$$S_{GP}(s) \triangleq A(\mu_1(s),\mu_m(s)) \cap P(g_j(s);\ 1\le j\le m) \qquad \ldots(58)$$

Proof: If $\lambda$ is an eigenvalue of G(s), then theorem 7 indicates that $\lambda \in P(g_j(s);\ 1\le j\le m)$. Note that $\lambda$ is also an eigenvalue of D(s)N(s) and write $DNy = \lambda y$ in the form $|\lambda|^2 y^* y = y^* N_o y$ ie $\mu_1^2 \le |\lambda|^2 \le \mu_m^2$ and $\lambda \in A(\mu_1,\mu_m)$ as required.

---

The similarity in the structure of theorems 14 and 15 is best made apparent by noting that $\mu_1(s) = \underline{\sigma}(D(s)N(s))$ and $\mu_m(s) = \bar{\sigma}(D(s)N(s))$ and that (using the notation of definition 1) $P(g_j(s);\ 1\le j\le m) = P(e^{i\phi_j(s)};\ 1\le j\le m)$. The results are therefore identical in structure with DN replacing G for the calculation of the annulus and the phases $\phi_j$ replacing the principal phases in the calculation of the cone P. Note the following observations:

(1) As the sufficient conditions for closed-loop stability described by Postlethwaite, Edmunds and MacFarlane (1981) depend only on the fact that all eigenvalues of G(s) lie in the region $S_{SV}(s)$ for all $s\in\Omega$, the results still hold true if $S_{SV}(s)$ is replaced by $S_{GP}(s)$ for all $s\in\Omega$. The details are omitted for brevity but we note that the flexibility available in the choice of Q(s) can make possible a refinement of their results. More precisely,

(2) the flexibility in the choice of $Q(s)$ can locate the eigenvalues to the interaction of an annulus and a cone of smallest area. This is easily proved by noting that, if $Q(s)$ is chosen to satisfy the normality conditions of theorem 6 we have $N(s) = I_m$, $g_j(s) = \tilde{g}_j(s)$, $1 \le j \le m$, and $S_{GP}(s) = A(\min_j |\tilde{g}_j(s)|, \max_j |\tilde{g}_j(s)|) \cap P(\tilde{g}_j(s);\ 1 \le j \le m)$ which is clearly the smallest set (of the type considered) containing the eigenvalues of $G(s)$, and is contained by $S_{SV}(s)$ unless $G(s)$ is normal when equality holds.

## 4.3. Eigenvalue Estimation and the Numerical Range

The numerical range $V(R)$ of a complex mxm matrix $R$ on $C^m$ regarded as a Hilbert space with inner product $\langle x,y\rangle = x^* y$ is defined (Bonsall and Duncan, 1971) to be the subset of the complex plane defined by

$$V(R) \triangleq \{\mu \in C \ :\ \mu = x^* R x \ ,\ x^* x = 1\} \qquad \ldots(59)$$

$V(R)$ is known to be compact and convex but, for our purposes, we will need only the following structural and spectral properties (Bonsall and Duncan, 1971).

---

Proposition 3: If $R = \mathrm{diag}\{r_j\}_{1 \le j \le m}$ is diagonal, then $V(R)$ is the closed convex hull $\overline{\mathrm{co}}\{r_j\}_{1 \le j \le m}$ of the point set $\{r_j\}_{1 \le j \le m}$.

---

Proposition 4: If $R_1$ and $R_2$ are complex mxm matrices with $0 \notin V(R_1)$, then $R_1$ is nonsingular and all eigenvalues of $R_1^{-1} R_2$ lie in the compact set

$$V(R_1, R_2) \triangleq \{\mu \in C \ :\ \mu = \lambda^{-1}\xi \ ,\ \ \lambda \in V(R_1) \ ,\ \ \xi \in V(R_2)\} \qquad \ldots(60)$$

---

Proposition 5: If R is Hermitian with (real) eigenvalues $r_1 \leq r_2 \leq \dots \leq r_m$, then V(R) is the real interval $[r_1, r_m]$ regarded as a subset of the complex plane.

---

Our main concern here is the combination of the gain-phase structure with the eigenvalue location possibilities inherent in the numerical range as indicated by Proposition 4. These possibilities are formalized in the following result.

---

Theorem 16: All eigenvalues of G(s) lie in the compact, convex set

$$S_{NR}(s) \triangleq \{\mu \in C : \mu = \lambda\xi , \quad \lambda \in [n_1(s), n_m(s)] \quad , \xi \in \overline{co}\{g_j(s)\}_{1\leq j\leq m}\} \qquad \dots(61)$$

where $0 < n_1(s) \leq n_2(s) \leq \dots \leq n_m(s)$ are the (ordered) real eigenvalues of $N(s) \triangleq X^*(s)Q(s)X(s)$.

Proof: All eigenvalues of G(s) are identical to those of ND. Apply Proposition 4 with $R_2 = D$ and $R_1 = \hat{N}$ noting that (Proposition 5) $V(R_1) = [n_m^{-1}(s), n_1^{-1}(s)] \not\ni 0$ and (Proposition 3) $V(R_2) = \overline{co}\{g_j(s)\}_{1\leq j\leq m}$.

---

Note that $S_{NR}(s)$ is easily computed graphically as the area contained between all lines joining points in $\{g_j(s)\}_{1\leq j\leq m}$ and scaling using points in $[n_1(s), n_m(s)]$. The result has a very similar structure to theorems 14 and 15 as can be seen by writing

$$S_{SV}(s) = \{\mu \in C : \mu = \lambda\xi , \quad \lambda \in [\underline{\sigma}(G(s)), \bar{\sigma}(G(s))] \quad \text{and}$$
$$\xi = \alpha/|\alpha| \quad \text{with} \quad 0 \neq \alpha \in P(e^{i\theta_j(s)} ; 1\leq j\leq m)\} \qquad \dots(62)$$

and

$$S_{GP}(s) = \{\mu \in C : \mu = \lambda\xi , \quad \lambda \in [\mu_1(s), \mu_m(s)] \quad \text{and}$$
$$\xi = \alpha/|\alpha| \text{ with } 0 \neq \alpha \in P(g_j(s); 1\leq j\leq m)\} \qquad \dots(63)$$

The set $S_{NR}(s)$ has a distinctly different shape however. This fact is underlined by the example of section 4.3. We conclude this section however with the following observations paralleling those following theorem 15,

(1) The results of Postlethwaite, Edmunds and MacFarlane remain valid with $S_{SV}(s)$ replaced by $S_{NR}(s)$. In fact any combination of $S_{SV}$, $S_{GP}$ and $S_{NR}$ will do as, for any eigenvalue $\lambda$ of $G(s)$,

$$\lambda \in S_{SV}(s) \cap S_{GP}(s) \cap S_{NR}(s) \qquad \ldots(64)$$

which could represent a considerable refinement in eigenvalue location.

(2) Theorem 16 potentially provides the best choice of eigenvalue estimates as, under the normality conditions of theorem 6, $N(s) \equiv I_m$, $g_j(s) = \tilde{g}_j(s)$, $1 \leq j \leq m$ and $S_{NR}(s)$ is just the convex hull of the spectrum of $G(s)$. Under these conditions, it is trivially verified that

$$S_{SV}(s) \supset S_{GP}(s) \supset S_{NR}(s) \qquad \ldots(65)$$

whenever $\inf\{z : z \quad S_{NR}(s)\}$ is achieved by $z = g_j(s)$ for some index j.

### 4.3. Illustrative Example

To illustrate the form of the results described above consider the constant transfer function matrix

$$G(s) = \begin{pmatrix} 2i & i \\ 1 & 1 \end{pmatrix} \qquad \ldots(66)$$

with polar decomposition $G = UH$ of the form

$$G(s) = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \ldots(67)$$

The corresponding gain-phase structure obtained from theorem 2 by choosing $Q(s) = H$ is simply

$$G(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \ldots(68)$$

with the natural identification of

$$X(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad g_1(s) = 2i \quad , \quad g_2(s) = 1 \qquad \ldots(69)$$

Considering first the eigenvalue estimation using singular values using theorem 14, the principal phases are simply $\theta_1 = \frac{\pi}{2}$, $\theta_2 = 0$ and the principal gains are $\underline{\sigma} = 0.382$ $\bar{\sigma} = 2.618$. That is

$$S_{SV}(s) = \{z : z = ce^{i\psi} \;,\; 0.382 \leq c \leq 2.618 \;,\; 0 \leq \psi \leq \frac{\pi}{2}\} \qquad \ldots(70)$$

Turning now to the gain-phase structure (68), a simple calculation yields

$$N_O(s) = \begin{pmatrix} \frac{9}{2} & \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & 3 \end{pmatrix} \qquad \ldots(71)$$

and hence $\mu_1 = 0.3685$, $\mu_2 = 2.7137$. Invoking theorem 15 leads to

$$S_{GP}(s) = \{z : z = ce^{i\psi} \;,\; 0.3685 \leq c \leq 2.7137 \;,\; 0 \leq \psi \leq \frac{\pi}{2}\} \qquad \ldots(72)$$

which is slightly more conservative than (70). In contrast theorem 16 leads to

$$S_{NR}(s) = \{z = \lambda\xi \ , \ 0.2929 \leq \lambda \leq 1.7071 \ , \xi = 2i\alpha+(1-\alpha) \ , \ 0\leq\alpha\leq 1\} \quad \ldots(73)$$

as

$$N(s) = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \quad \ldots(74)$$

and hence $n_1(s) = 0.2929$, $n_2(s) = 1.7071$. The relative form of the results is shown in Fig.1 where both $S_{SV}(s)$ and $S_{NR}(s)$ are given together with the eigenvalues

$$\lambda_1 = \tfrac{1}{2} + i(1 + \frac{\sqrt{3}}{2}) \quad , \quad \lambda_2 = \tfrac{1}{2} + i(1 - \frac{\sqrt{3}}{2}) \quad \ldots(75)$$

of G(s). Note the distinctly different shapes of the regions, the singular value estimate being a better estimate of maximum eigenvalue gains corresponding to phases in the vicinity of $\frac{\pi}{2}$ and the numerical range estimate being better in all other areas. Clearly the intersection of $S_{SV}$ and $S_{NR}$ yields an improved estimate and further improvements are possible by incorporation of the Gershgorin circles (theorem 11) as illustrated in Fig.1.

Finally, the improvements possible by choosing Q(s) can be illustrated by choosing

$$Q(s) = \frac{1}{3}\begin{pmatrix} 2 & 1-2i \\ 1+2i & 4 \end{pmatrix} \quad \ldots(76)$$

when $G = XDX^{*}Q$ with

$$g_1(s) = \tfrac{1}{2} + i(1 + \frac{\sqrt{3}}{2}) \quad , \quad g_2(s) = \tfrac{1}{2} + i(1 - \frac{\sqrt{3}}{2}) \quad \ldots(77)$$

and

$$X(s) = \begin{pmatrix} -\frac{1}{2}+i(1+\frac{\sqrt{3}}{2}) & -\frac{1}{2}+i(1-\frac{\sqrt{3}}{2}) \\ 1 & 1 \end{pmatrix} \quad \ldots(78)$$

A simple calculation yields $N = X^*QX = I_2$ when theorem 15 leads to

$$S_{GP}(s) = \{z : z = ce^{i\psi} , 0.517 \leq c \leq 1.93 , 0.26 \leq \psi \leq 1.30\} \quad \ldots(79)$$

whilst $S_{NR}(s) = \overline{co}\ (g_1(s), g_2(s))$ is just the line segment joining $g_1(s)$ to $g_2(s)$. In fact, as $N = I_2$, Corollary 6.1 indicates that $g_1(s)$ and $g_2(s)$ are the eigenvalues of $G(s)$.

## 5. Interchange Phenomena

Although theorems 3 and 4 provide both a computational and spectral characterization of the existence of gain-phase structures, they do leave open the structure of the system if the conditions are not satisfied. In particular, it leaves open the question of whether an alternative structural decomposition exists that permits a physical interpretation. The following treatment is not exhaustive but indicates that the answer can be positive.

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Lemma 3: If $\lambda$ is an eigenvalue of $M$, then so is $\bar{\lambda}^{-1}$.

Proof: If $Mz = \lambda z$ then $G^*z = \lambda Gz$ or $z^*G = \bar{\lambda}z^*G^*$ ie $\bar{\lambda}^{-1}$ is an eigenvalue of $G^*G^{-1}$ and hence an eigenvalue of $M = G^{-1}G^*$.

---

For simplicity, we concentrate on the case when $M(s)$ has a complete set of eigenvectors $z_j(s)$, $1 \leq j \leq m$, corresponding to distinct eigenvalues $\lambda_j$, $1 \leq j \leq m$, but allow the possibility that the eigenvalues do not have unit modulus. Using lemma 3, there exists a permutation $\ell_s$ of $\{1,2,\ldots,m\}$ with the property that $\ell_s(\ell_s(j)) = j$, $1 \leq j \leq m$, and such that $\bar{\lambda}_j^{-1} = \lambda_{\ell_s(j)}$. Using this fact in (27) yields the 'orthogonality' relation, $s \in \Omega$

$$z_k^*(s)G(s)\hat{Q}(s)z_j(s) = 0 \quad , \quad k \neq \ell_s(j) \qquad \ldots(80)$$

Let $X(s) \triangleq [x_1(s), x_2(s), \ldots, x_m(s)]$ be defined by (19) and let $P_{\ell_s}$ be the mxm permutation matrix defined by

$$P_{\ell_s}^2 = I_m \quad , \quad P_{\ell_s} e_j = e_{\ell_s(j)} \quad , \quad 1 \leq j \leq m \qquad \ldots(81)$$

where $e_1 = (1,0,0\ldots,0)^T$, $e_2 = (0,1,0,\ldots,0)$,... are the natural basis elements in $C^m$. We obtain the result:

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Theorem 17: If M(s) has a complete set of eigenvectors on $\Omega$ with distinct eigenvalues, then there exists a nonsingular matrix X(s) on $\Omega$ and permutation $P_{\ell_s}$ on $\Omega$ such that

(a) $G(s) = X(s)P_{\ell_s}D(s)X^*(s)Q(s)$ ...(82)

(b) the diagonal terms of $X^*(s)Q(s)X(s)$ are all unity, and

(c) D(s) is diagonal of the form of equation (9).

Proof: Using the above discussion, equation (80) indicates that the matrix

$$D(s) \triangleq P_{\ell_s}\hat{X}(s)G(s)\hat{Q}(s)\hat{X}^*(s) \qquad \ldots(83)$$

has zero off-diagonal terms. Requirements (a) and (c) follow immediately from the fact that $P_{\ell_s}^{-1} = P_{\ell_s}$, (b) following from the normalization (19).

---

In the case when M has eigenvalues of unit modulus only, theorem 4 indicates that $\ell_s$ is the identity and $P_{\ell_s} = I_m$ which reproduces the previous results. If, however, M has eigenvalues of non-unit modulus, $\ell_s$ is not an identity permutation and the result differs from previous work. To obtain a physical interpretation of the gain-phase structure

(82), note that the input-output transformations (7) are still defined and possess the same physical significance but that $y = Gu$ reduces to

$$\hat{y}_{\ell_s(j)}(s) = g_j(s)\ \hat{u}_j(s) \quad , \quad 1\le j\le m,\ s\in\Omega \qquad \ldots(84)$$

Comparing with (2) we see that a physical gain-phase structure still exists but that the projection $\hat{u}_j(s)$ of the input onto $x_j(s)$ is modified in gain and phase by $g_j(s)$ and injected into the output coordinate $\hat{y}_{\ell_s(j)}(s)$ rather than $\hat{y}_j(s)$. As $\ell_s(\ell_s(j)) = j$, $1\le j\le m$, we see that $\hat{u}_{\ell_s(j)}$ is injected into $\hat{y}_j(s)$ and hence the input-output relations have suffered a loop-interchange.

In principle, the modified gain-phase structure (82) could be used in eigenvalue estimation in a similar manner to that described in section 3. In particular, this will change the form of the Gershgorin circle based results (theorem 11) as N is replaced by $X^*QXP_\ell$, producing substantial off-diagonal terms in DN even if $X^*QX$ is diagonal. This problem is not pursued further here except to state the following special case motivated by theorem 6 and corollary 6.1.

---

<u>Proposition 6</u>: If G(s) permits a gain-phase structure on $\Omega$ with respect to Q(s) of the form of (82) with loop interchange and $X^*(s)Q(s)X(s) \equiv I_m$, then G(s) has eigenvalues $\lambda = g_j(s)$ if $\ell_s(j) = j$ and $\pm\sqrt{g_j(s)g_{\ell_s(j)}(s)}$ if $\ell_s(j) \neq j$.

<u>Proof</u>: Note that the eigenvalues of G are those of $DX^*QXP_\ell = DP_\ell$.

---

Note, in particular, that the presence of loop interchange ensures that at least two eigenvalues of G(s) differ in phase by exactly $\pi$ radians.

Theorem 15 fails in the presence of loop interchange as theorem 7 is no longer valid. It is possible to state however that all eigenvalues of G(s) lie in $A(\mu_1(s),\mu_m(s))$ if $N_o$ is replaced by $N(s)D^*(s)P^*_{\ell_s}P_{\ell_s}D(s)N(s)$, but this result carries no phase information whatsoever. In the case of theorem 16, a similar result still holds with $S_{NR}(s)$ replaced by

$$S_{NR}(s) \triangleq \{\mu \in C : \mu = \lambda\xi \;,\; \lambda \in [n_1(s), n_m(s)] \;,\; \xi \in V(P_{\ell_s}D(s))\} \quad \text{...(85)}$$

but the form of the numerical range $V(P_\ell D)$ is relatively complex. These problems are not pursued further in this paper.

## 6. A Note on Normalization

In mathematical terms, the normalization (5) is not necessary for the existence of factorizations of G(s) of the form of (8), although, physically, it is preferred to use the normalization to provide a physical interpretation of the structure. If the normalization requirement is removed then the existence of one factorization implies the existence of an infinity of factorizations with the gains of $g_j(s)$, $1\le j\le m$, arbitrary. In this situation the polar decomposition of G(s), written in the form (11) with U(s) expressed in spectral form (12) is an unnormalized gain-phase structure with $Q(s) \equiv H(s)$, $X(s) \equiv X_o(s)$ and $g_j(s) = e^{i\theta_j(s)}$, $1\le j\le m$, consisting of a pure phase change.

The effects of the removal of the normalization condition is to require substantial modifications to the material of section 4.1 (which relies heavily on the fact that the diagonal termsof N(s) are unity). All of the results of section 4.2 and 4.3 still hold however and illustrate the fact that an infinity of eigenvalue location sets can be identified. For example, if G(s) is expressed in polar form (as described above) theorem 15 still holds with $\mu_j(s) = \sigma_j(s)$, $1\le j\le m$, $s\in\Omega$, so that

$S_{GP}(s) = A(\sigma_1(s),\sigma_m(s)) \cap P(e^{i\theta_j(s)}\ ;\ 1\leq j\leq m) \equiv S_{SV}(s)$. That is the location of eigenvalues using the polar decomposition (theorem 14) is simply a special case of the eigenvalue-location possibilities using unnormalized gain-phase structures. The possibilities inherent in the numerical range methodology (theorem 16) are more interesting however. Again, we will consider G(s) in polar form as described above and note that theorem 16 still holds with

$$S_{NR}(s) \triangleq \{\mu \in C : \mu = \lambda\xi\ ,\ \lambda \in [\underline{\sigma}(G(s)),\bar{\sigma}(G(s))],\ \xi \in \overline{co}\{e^{i\theta_j(s)}\}_{1\leq j\leq m}\} \qquad \ldots(86)$$

which represents a previously unnoted eigenvalue location theorem expressed in terms of principal gains and phases of G(s). For the example of section 4.3, it is easily verified that $\theta_1 = \frac{\pi}{2}$, $\theta_2 = 0$ and the unnormalized gain-phase structure leads to

$$S_{NR}(s) = \{z : z = \lambda\xi\ ,\ 0.382 \leq \lambda \leq 2.618\ ,\xi = \alpha + i(1-\alpha),\ 0\leq\alpha\leq 1\} \qquad \ldots(87)$$

as illustrated in Fig.2. A comparison with Fig.1 indicates that the change of normalization leads to a change in the eigenvalue estimate.

Finally, to underline the potential benefits of combining eigenvalue estimates, we note the following special case:

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Proposition 7: If G(s) is unitary on $\Omega$ and $S_{SV}(s)$ and $S_{NR}(s)$ are defined by (57) and (86) respectively, then $S_{SV}(s) \cap S_{NR}(s)$ is identical (with appropriate multiplicities) to the set of eigenvalues of G(s).

Proof: If G is unitary then $H(s) \equiv I_m$ so that $\underline{\sigma}(G) = \bar{\sigma}(G) = 1$ and the eigenvalues of G are simply $e^{i\theta_j}$, $1\leq j\leq m$. Clearly $S_{SV}(s)$ is a subset of the unit circle in C whilst $S_{NR}(s)$ (as given by (86)) is a polygon contained in the closed unit circle and touching that circle at the points $e^{i\theta_j}$, $1 \leq j \leq m$.

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## 7. Conclusions

Starting from a physically motivated definition, the paper has demonstrated that linear dynamic systems represented by an mxm transfer function matrix G(s) can be viewed as permitting an infinite number of gain-phase interpretations characterized by positive-definite, self-adjoint matrices Q(s), bases $\{x_j(s)\}_{1 \leq j \leq m}$ for $C^m$ and m scalar 'transfer functions' $g_j(s)$, $1 \leq j \leq m$, describing the input-output relationships between 'projections' of the input onto the basis $\{x_j\}$ and coordinates of the output in the basis $\{x_j\}$. The generality of the framework has been demonstrated by the fact that the spectral decomposition and polar decomposition of G(s) can be regarded as special forms of gain-phase structures and yet an infinity of other decompositions can exist by suitable choice of Q(s).

The basic physical motivation for the gain-phase structures indicates that they can have a real physical interpretation and systems theoretical implications. This fact has been underlined by a proof of the equivalence of positive-realness to the existence of a specific form of gain-phase structure characterized by m positive-real scalar systems. The possibilities of their application in systems analysis and control design seems to be promising as illustrated by applications to eigenvalue (characteristic loci) estimation using Gershgorins theorem, the polar decomposition and the numerical range. In particular, the approach yields general results identical in structure to previous work on the polar decomposition and reproduces those results exactly as a special case if unnormalized gain-phase structures are permitted.

Overall, the results suggest that there is a larger degree of flexibility available for system description than previously assumed. Further work will attempt to identify the physical and computational possibilities that this flexibility could allow.

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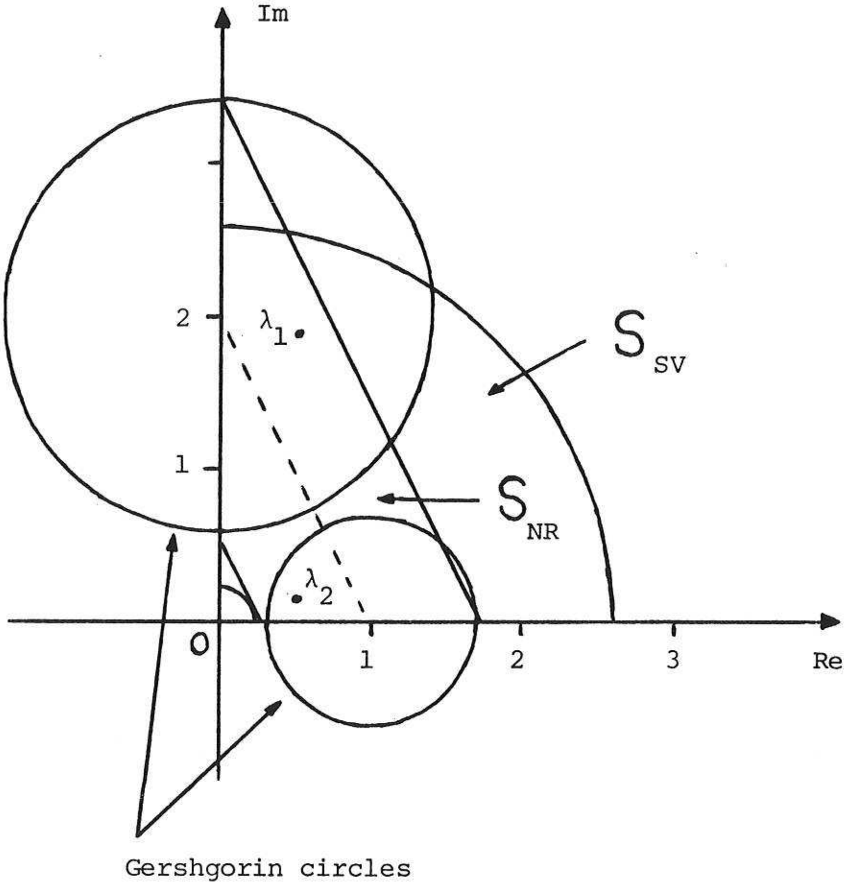


Fig. 1. Comparison of $S_{SV}$, $S_{NR}$ and Gershgorin Circles eigenvalue location regions

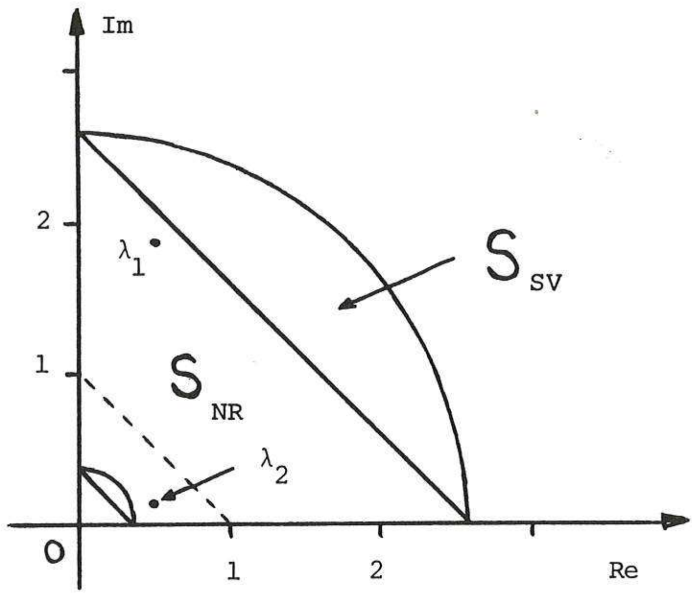




Fig.2. Unnormalized Estimates of Eigenvalue Locations