Monograph:
Leontaritis, I.J. and Billings, S.A. (1984) Input-Output Parametric Models for Nonlinear Systems. Part ii Stochastic Nonlinear Systems. Research Report. Acse Report 252 . Dept of Automatic Control and System Engineering. University of Sheffield

[redacted]

INPUT-OUTPUT PARAMETRIC MODELS FOR NONLINEAR SYSTEMS

PART II STOCHASTIC NONLINEAR SYSTEMS

by

I.J.Leontaritis B.Sc., M.Eng., Ph.D.

and

S.A.Billings B.Eng., Ph.D., C.Eng., M.I.E.E., A.F.I.M.A., M.Inst.M.C.,

Department of Control Engineering
University of Sheffield
Mappin Street
SHEFFIELD S1 3JD

Research Report No. 252

April 1984

Abstract

In the first part of this paper [Leontaritis and Billings (1984)] recursive input-output models for deterministic nonlinear multivariable discrete-time systems were derived and sufficient conditions for their existence were defined. In this the second part, the nonlinear model is compared with other system representations, several examples are introduced and the results are extended to create prediction error input-output models for multivariable nonlinear stochastic systems. These latter models are a generalisation of the ARMAX models for linear systems and are referred to as NARMAX or Nonlinear Auto Regressive Moving Average models with eXogenous inputs.

1. Introduction

In the first part of this paper [Leontaritis and Billings (1984)] recursive input-output models for multivariable nonlinear discrete systems were derived. To recapitulate, let the input set U of a system S be an r-dimensional vector space and the output set Y on m-dimensional vector space. For some specified ordered bases of U and Y the inputs and outputs can be represented by the column vectors

$$u(t) = [u_1(t), u_2(t), \dots u_r(t)]^T \qquad (1.1)$$

$$y(t) = [y_1(t), y_2(t), \dots y_m(t)]^T \qquad (1.2)$$

Let the vector of all inputs from time 1 to time t be

$$u^t = [(u(t))^T, (u(t-1))^T, \dots (u(1))^T]^T \qquad (1.3)$$

and the zero state response function f of the system be

$$y(t) = f(u^t) \qquad (1.4)$$

If the response f satisfies two mild conditions the following recursive model descibes the system (1.4) in a region around the zero equilibrium point

$$\begin{aligned} y_i(t+p) = q_i [\, & y_1(t+n_1-1), \dots y_1(t), \\ & y_2(t+n_2-1), \dots y_2(t), \\ & \vdots \\ & y_m(t+n_m-1), \dots y_m(t), \\ & u_1(t+p), \dots\dots u_1(t) \\ & u_2(t+p), \dots\dots u_2(t) \\ & \vdots \\ & u_r(t+p), \dots\dots u_r(t)] \end{aligned} \qquad (1.5)$$

where $i = 1,2,...m$ and $p = \max(n_1,n_2,...n_m)$

The integers $n_1,n_2 ...n_m$ are the observability indices defined in part I of the paper. The order of the model is $n=n_1+n_2...+n_m$. Every index n_i corresponds to the specific output y_i and the model (1.5) can be regarded as m interconnected single-output models each of order n_i. A multivariable system can have more than one set of observability indices and each set corresponds to a different input-output model (1.5). The sum of all the observability indices is however an invariant of the system equal to the order of the system n.

If the system is single-input single-output the model becomes

$$y(t+n) = q[y(t+n-1);...y(t),u(t+n),...u(t)] \qquad (1.6)$$

which is nothing more than the generalisation to the nonlinear case of the difference equation which is satisfied by a linear system of order n where $q[\cdot]$ is some nonlinear function.

In this second part of the paper a comparison between the globally valid input-output models developed by Sontag [1976,1979a,b] and the model in equ.(1.6) is made. Several examples which illustrate the two models are introduced and their usefulness in identification studies is considered. A rigorous definition of stochastic nonlinear discrete-time systems is then presented. The innovation form of the stochastic nonlinear system is also derived and necessary and sufficient conditions which must be satisfied by the innovations are found. Considering a different interpretation of the meaning of input and output for stochastic systems, the deterministic model in eqn. (1.5) is transformed into a stochastic input-output model. Such models are essentially the generalization to the nonlinear case of the widely used ARMAX models of linear systems. [Goodwin and Payne (1977)] The stochastic nonlinear models are given the name multistructural innovation input-output models, since their special structure is inherited from the multistructural forms of linear multivariable systems. [Wertz, Grevers and Hannon (1982)] For single-input single-output systems the nonlinear innovation models are called NARMAX or Nonlinear AutoRegressive Moving Average models with eXogenous inputs

2. Comparison with other nonlinear models

The realization of the special case of discrete-time polynomial response function has been studied in great detail in [Sontag (1976),(1979a)] using mainly algebraic methods. In [Sontag (1979b)] a similar type of response function is introduced where an arbitrary set of linearly independent non-linear functions is used to generate the response function in the same way the monomials $x, x^2, x^3, \ldots,$ are used to generate a polynomial response function. The theory and the results for such response functions correspond exactly to the theory and the results for the polynomial response funcitons and thus they will not be reviewed here. A summary of the basic results in [Sontag (1976), (1979)] now follows so that a comparison with the present work can be made.

A polynomial input-output map or response function is a function $f: U^* \to Y$ such that every individual function $f_t: U^t \to Y$ where $y(t)=f_t(u(t),u(t-1),\ldots,u(1))$ is a polynomial of finite degree on all variables. However, the power of the variables of the polynomial f_t are raised to can tend to ∞ as $t \to \infty$. The assumption that the input-output map is polynomial is not a severe restriction since any system restricted to operate in any region around the zero state has a response function that can be approximated as closely as desired by a polynomial input-output map. However, if the accuracy of the required approximation increases, the degree of the polynomials f_t must also increase.

If the functions f_t are analytic functions then the response function can be represented exactly by an infinite series, the Volterra series [Billings (1980)]. Then

$$
\begin{aligned}
y(t) = & \sum_{T_1=0}^{t-1} h_1(T_1)u(t-T_1) + \\
& \sum_{T_1=0}^{t-1} \sum_{T_2=0}^{T_1} h_2(T_1,T_2)u(t-T_1)u(t-T_2) + \\
& \cdots \cdots \\
& \sum_{T_1=0}^{t-1} \cdots \sum_{T_s=0}^{T_{s-1}} h_s(T_1,T_2,\ldots,T_s)u(t-T_1)\ldots u(t-T_s) + \\
& \cdots \cdots
\end{aligned}
\qquad (2.1)
$$

The functions $h_1(T_1), h_2(T_1,T_2), \ldots$ are called Volterra kernels. An infinite Volterra series is not a polynomial response function since the degree every varaible in (2.1) is raised to is not finite. If the series (2.1) terminates with the term involving the s^{th} kernel, it is called finite Volterra series of length s. A finite Volterra series is a polynomial response function since the maximum degree any input is raised to is s.

A polynomial response function f_t is called bounded if the power any input is raised to is always less than some constant integer independent of t. The maximum power any input is raised to is called the degree of the polynomial function. A finite Volterra series of length s is obviously a bounded polynomial response function of degree s but the inverse is not true. A bounded polynomial response function is a much more general response function. For instance the response function

$$y(t)=u(t-1)+u(t-1)u(t-2)+u(t-1)u(t-2)u(t-3)+\dots \qquad (2.2)$$

is bounded of degree 1 since no input is raised to a power higher than 1, but it is obviously not a finite Volterra series.

A polynomial response map is called finitely realizable if Assumption 1 of section 4.1 Part I [Leontaritis and Billings (1984)] is satisfied. The finite rank condition of Assumption 1 is thus as vital for the realization of any differentiable response function.

The realization of polynomial response functions is done in great detail in [Sontag (1976),(1979a)]. In that realization a system is considered minimal (it is called canonical in references [Sontag(1976),(1979a)]), when it is quasi-reachable and algebraically observable, in contrast to the Nerode realization where minimality is equivalent to reachability and observability. The state-space is an abstract topological construction that has finite dimension if the response function is finite realizable. The one step ahead state-transition function and the output functions are polynomial functions. The more striking result is that a finitely realizable response function also satisfies a rational difference equation

$$a[y(t+p-1),\dots,y(t),u(t+p),\dots,u(t)]y(t+p)=$$

$$b[y(t+p-1),\dots y(t),u(t+p),\dots\ u(t)] \qquad (2.3)$$

where a and b are polynomials of finite degree. The reverse is also true, if a response function satisfies a rational difference equation (2.3), it is finite realizable. The rational difference equation (2.3) is actually an input-output model valid everywhere in contrast to the recursive input-output model (1.5). The difference between the two models and their potential usefulness in the identification of non-linear systems will be examined later.

Some more results for bounded polynomial response functions and finite Volterra series are given first.

If the response function is bounded as well as finitely realizable, the input-output difference equation (2.3) takes the special form of an output-affine difference equation

$$\sum_{i=0}^{p} a_i\left[u(t+p),\ldots,u(t)\right]y(t+i)=b\left[u(t+p),\ldots,u(t)\right] \qquad (2.4)$$

where a_i for $i = 1,2,\ldots,p$ are polynomials of finite degree.

The bounded finitely realizable polynomial response functions can be realized by state-affine state-space mdoels. In such realizations a system is considered minimal if it is span-reachable and observable. A state-affine state-space model is given by the equations

$$x(t+1)=A(u(t))x(t)+B(u(t))$$

$$y(t)=C(u(t))x(t)+D(u(t)) \qquad (2.5)$$

where $A(u(t))$ and $C(u(t))$ are linear maps for every $u(t)\varepsilon U$ and $B(u(t))$ and $D(u(t))$ are vectors for every $u(t)\varepsilon U$. The state-space X is a finite dimensional vector space. Furthermore, fixing an ordered basis for the state-space, the linear maps $A(u(t))$ and $C(u(t))$ are represented by matrices and every element of these matrices is a polynomial of the variable $u(t)$. The vectors $B(u(t))$ and $D(u(t))$ are also column vectors with every element a polynomial of the variable $u(t)$. The state-affine realization of a bounded polynomial response function can be done in a manner analogus to the realization of linear systems by finding a row basis of a generalization of the Hankel matrix called behaviour matrix $B(f)$ [Sontag(1979b)]. The state-affine systems are very easy to use because the state-space is a vector space. They suffer however, the disadvantage that they are only span-reachable. The state-space is the span of all the reachable states and thus the reachable states may be a very thin set in the state-space. Also the dimension of the vector space can change if the non-linearities of the system change slightly. These problems will be clarified with examples later.

The finite Volterra series is a bounded polynomial response function and thus, if it is finitely realizable, it satisfies an output-affine difference equation (2.4). It can also be realized by a state-affine state-space model (2.5). The state-space model, in this case, takes a special form that actually corresponds to the structure of a cascade of linear multivariable systems with polynomial interconnections.

The realization of a finite Volterra series can be done alternatively by a state-space model that is state-affine in the one step ahead state-transition function while the output function is a general polynomial function. The state-transition function can still keep the special structure of the cascade of linear systems with polynomial interconnections. The big advantage of such a realization is that the dimension of the state-space does not have to be artificially large as in the state-affine realizations. Such realizations have been carried out in detail only in the continuous-time case in [Crouch (1981)] . An example will try to clarify the concepts and the results presented so far.

Example 2.1

Consider a system described by the equations

$$x(t+1)=x(t)+u(t)$$
$$y(t)=x(t)+x^2(t) \qquad (2.6)$$

where u(t) is the scalar input and y(t) the scalar output. The zero state response function is

$$\begin{aligned} y(t) &= f(u(t),u(t-1), \dots ,u(1)) \\ &= u(t-1)+u(t-2)+ \dots +u(1)+(u(t-1)+u(t-2)+ \dots ,u(1))^2 \\ &= u(t-1)+u(t-2)+ \dots +u(1)+u^2(t-1)+u^2(t-2)+ \dots +u^2(1)+ \\ &\quad 2u(t-1)u(t-2)+2u(t-1)u(t-3)+ \dots +2u(2)u(1) \end{aligned} \qquad (2.7)$$

This is obviously a polynomial response function. It is also bounded of degree 2 since the highest power any input is raised to is 2. It is also a finite Volterra series of length 2 since it can be put in the form (2.1) with only the first two terms non-zero. The response function is finitely realizable since it is derived from a state-space description (2.6) that has a state-space of dimension 1 and thus Assumption 1 in Part I is satisfied with maximum rank less or equal to 1. If the response function (2.7) is not known to be derived from the state description (2.6), the functions $F_{k,t}(z,x)$ must be constructed first. Let for example k=2 and t=2. Then

$$F_{2,2}(z,x)= \begin{bmatrix} u(t-1)+u(t-2)+[u(t-1)+u(t-2)]^2 \\ u(t)+u(t-1)+u(t-2)+[u(t)+u(t-1)+u(t-2)]^2 \end{bmatrix} \qquad (2.8)$$

where z= u(t) and x = $[u(t-1),u(t-2)]^T$

$$D_x F_{2,2}(z,x)= \begin{bmatrix} 1+2[u(t-1)+u(t-2)] & 1+2[u(t-1)+u(t-2)] \\ 1+2[u(t)+u(t-1)+u(t-2)] & 1+2[u(t)+u(t-1)+u(t-2)] \end{bmatrix} \qquad (2.9)$$

and thus

$$\text{rank } D_x\ F_{2,2}(z,x)=1 \qquad (2.10)$$

when $u(t-1)+u(t-2)\neq -0.5$ or $u(t)+u(t-1)+u(t-2)\neq -0.5$

and

$$\text{rank } D_x\ F_{2,2}(z,x)=0 \qquad (2.11)$$

when $u(t-1)+u(t-2)=-0.5$ and $u(t)=0$

Similarly $D_x\ F_{k,t}(z,x)$ for any $k>0$ and $t>0$ has a maximum rank equal to 1. The response function is bounded and finite realizable and thus it must satisfy an output-affine finite difference equation (2.4). In fact

$$y(t)=x(t)+x^2(t) \qquad (2.12)$$

$$\begin{aligned} y(t+1)&=x(t)+u(t)+[x(t)+u(t)]^2 \\ &=x(t)+x^2(t)+u(t)+u^2(t)+2x(t)u(t) \\ &=y(t)+u(t)+u^2(t)+2x(t)u(t) \end{aligned} \qquad (2.13)$$

$$\begin{aligned} y(t+2)&=x(t)+u(t)+u(t+1)+[x(t)+u(t)+u(t+1)]^2 \\ &=x(t)+u(t)+[x(t)+u(t)]^2+ \\ &\quad u(t+1)+u^2(t+1)+2[x(t)+u(t)]u(t+1) \\ &=y(t+1)+u(t+1)+u^2(t+1)+2x(t)u(t+1)+2u(t)u(t+1) \end{aligned} \qquad (2.14)$$

Solving $x(t)$ from (2.13) and substituting in (2.14)

$$\begin{aligned} [u(t)]y(t+2)=[u(t)+u(t+1)]y(t+1)-[u(t+1)]y(t) \ + \\ u^2(t+1)u(t)+u(t+1)u^2(t) \end{aligned} \qquad (2.15)$$

The difference equation (2.15) is affine in the output and it is an input-output model valid everywhere.

The response function is bounded and finite realizable and thus it must be realizable by a span-reachable and observable state-affine state-space model. Let

$$\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x^2(t) \end{aligned} \qquad (2.16)$$

then from (2.6)

$$x_1(t+1)=x_1(t)+u(t) \qquad (2.17)$$

$$\begin{aligned} x_2(t+1)=[x_1(t)+u(t)]^2&=x_1^2(t)+2u(t)x_1(t)+u^2(t) \\ &=x_2(t)+2u(t)x_1(t)+u^2(t) \end{aligned} \qquad (2.18)$$

The state-affine model then is

$$x_1(t+1)=x_1(t)+u(t)$$
$$x_2(t+1)=x_2(t)+2u(t)x_1(t)+u^2(t) \qquad (2.19)$$
$$y(t)=x_1(t)+x_2(t)$$

where the state-space is the vector space R^2. The reachable states of this model are the points (x_1,x_2) of the vector space R^2 such that

$$x_2=x_1^2 \qquad (2.20)$$

i.e. the points of the plane given in figure 2.1

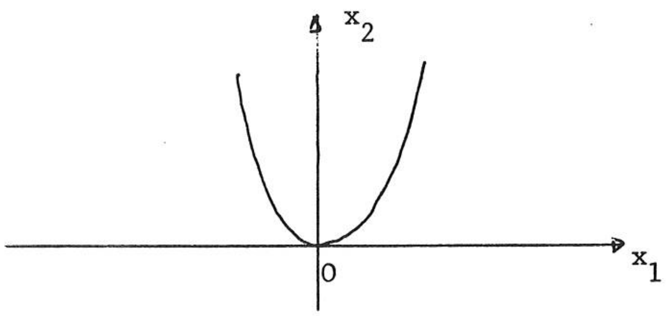

Figure 2.1

The reachable states span the whole state-space and thus the system is span-reachable. If the output function is not restricted to be state-affine the realization of the response function (2.7) is the model (2.6) with state-space the vector space R^1. These realizations are the ones considered in [Crouch (1981)] for continuous-time systems. The input-output model (1.5) developed in Part I can also be derived for this system. The linearised function around the origin has response function

$$y(t)=u(t-1)+u(t-2)+\ \dots\ +u(1) \qquad (2.21)$$

which has Hankel matrix of rank equal to 1, the maximum possible rank of the derivatives $D_x F_{x,t}(z,x)$. According to Theorem 2 Part I, a recursive input-output model can be found valid in a region around the origin. In fact if $x(t)>-0.5$, from (2.6)

$$2x(t)=-1+\sqrt{1+4y(t)} \qquad \text{for } x(t)>-0.5 \qquad (2.22)$$

but

$$y(t+1)=x(t)+u(t)+[x(t)+u(t)]^2$$
$$=x(t)+x^2(t)+u(t)+u^2(t)+2x(t)u(t)$$
$$=y(t)+u(t)+u^2(t)+2x(t)u(t)$$

$$=y(t)+u(t)\left[-1+\sqrt{1+4y(t)}\right]+u(t)+u^2(t)$$
$$=y(t)+u(t)\sqrt{1+4y(t)}+u^2(t) \qquad (2.23)$$

Thus the recursive input-output model is

$$y(t+1)=y(t)+u(t)\sqrt{1+4y(t)}+u^2(t)=q\left[y(t),u(t)\right] \qquad (2.24)$$

for

$$x(t)=(u(t-1)+u(t-2)+\ \dots\ +u(1))>-0.5 \qquad (2.25)$$

Now a comparison between the two input-output models, Sontag's model (2.15) and the recursive model (2.24), can be made. Sontag's input-output model is valid globally. However, given the values of the inputs and outputs before the time t+2, u(t+1),u(t)... and y(t+1),y(t),... , the output at time t+2,y(t+2), can only be calculated if $u(t)\neq 0$. Even if u(t) is different from zero but of small magnitude, Sontag's model requires very high accuracy in the execution of the algebraic operations. The biggest disadvantage is that when the system is operating around zero, where the linear approximation should be valid, Sontag's model fails completely to degenerate to the linear difference equation that is satisfied by the linearized system. The recursive input-output model (2.24) may be valid in a restricted region around the equilibrium point but for small u(t) and y(t) it becomes

$$y(t+1)=y(t)+u(t) \qquad (2.26)$$

the input-output model of the linearized system. The approximation

$$\sqrt{1+z}=1+\frac{1}{2}z \quad \text{for } |z|<<1 \qquad (2.27)$$

was used to derive (2.26).

Sontag's input-output model has another disadvantage. If the non-linearity of the system changes even slightly, the non-linear difference equation (2.15) changes completely. For instance if the output equation of the system (2.6) becomes

$$y(t)=x(t)+x^2(t)+\alpha x^3(t) \qquad (2.28)$$

Sontag's input-output model becomes

$$a_1\left[u(t+2),\ u(t+1),\ u(t)\right]y(t+3)+$$
$$a_2\left[u(t+2),\ u(t+1),\ u(t)\right]y(t+2)+$$
$$a_3\left[u(t+2),\ u(t+1),\ u(t)\right]y(t+1)+$$
$$a_4\left[u(t+2),\ u(t+1),\ u(t)\right]y(t)=b\left[u(t+2),\ u(t+1),\ u(t)\right] \qquad (2.29)$$

where a_1, a_2, a_3, a_4 and b are polynomials with coefficients that depend on the parameter α. If α is put equal to zero in (2.29), the model does not reduce the model in (2.15). The recursive input-output model in this case is still the model

$$y(t+1)=q'(y(t),u(t)) \tag{2.30}$$

where q' is a non-linear function that depends on α but for $\alpha=0$ it reduces to the function q of (2.24). The region of validity also depends on α but again for $\alpha=0$ it becomes the region (2.25).

The two types of input-output models compared here are sometimes identical. For instance the system

$$x(t+1)=x(t)+u^2(t) \tag{2.31}$$
$$y(t)=x(t)$$

has input-output model

$$y(t+1)=y(t)+u^2(t) \tag{2.32}$$

which is both a model (1.5) and (2.3). In such a case the recursive input-output model is obviously valid globally. A globally valid recursive input-output model is not necessarily identical to Sontag's model. For instance the system

$$x(t+1)=x(t)+u(t) \tag{2.33}$$
$$y(t)=x(t)+x^3(t)$$

has recursive input-output model (1.5)

$$y(t+1)=y(t)+3x^2u(t)+3xu^2(t)+u^3(t) \tag{2.34}$$

where

$$x=\left[y(t)/2+(y^2(t)/4+1/27)^{1/2}\right]^{1/3}+\left[y(t)/2-(y^2(t)/4+1/27)^{1/2}\right]^{1/3} \tag{2.35}$$

and it is globally valid. Sontag's model is a model like (2.29) which is a completely different model.

3. Some Special Model Structures

There are some models of very special form which have been used, because of their simplicity, in the identification of non-linear systems. One such group of models is the one that consists of a cascade of single-input single-output linear systems and static non-linearities [Billings and Fakhouri (1982)]. If the static non-linearities are polynomial functions these models can be represented by a finite Volterra series and all the results mentioned before are valid for such models. Two very well known models belong to this group, the Weiner and the Hammerstein models.

The Weiner model consists of a cascade of a linear system followed by a static non-linearity as in figure (3.1).

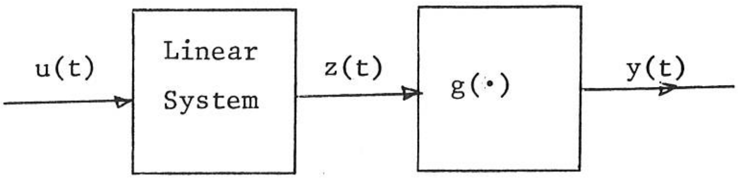

Figure 3.1

The static non-linearity is given by the function g, thus

$$y(t)=g(z(t)) \quad (3.1)$$

The linear system has response function

$$z(t)=h_0u(t)+h_1u(t-1)+ \ldots +h_{t-1}u(1) \quad (3.2)$$

and it is supposed to be finitely realizable so that it satisfies the input-output model

$$z(t+n)=a_1z(t+n-1)+\ldots+a_nz(t)+b_0u(t+n)+\ldots+b_nu(t) \quad (3.3)$$

where n is the order of the linear system. The overall response function of the Wiener model is

$$y(t)=g(h_0u(t)+h_1u(t-1)+\ldots+h_{t-1}u(1)) \quad (3.4)$$

The input-output model of such a response function can be found using (3.3) without the use of Theorem2 Part I. Suppose that the function g is an invertible function, then

$$z(t)=g^{-1}(y(t)) \quad (3.5)$$

substituting in (3.3)

$$g^{-1}(y(t+n))=a_1g^{-1}(y(t+n-1))+\ldots+a_ng^{-1}(y(t))+$$
$$b_0u(t+n)+\ldots+b_nu(t) \quad (3.6)$$

or

$$y(t+n)=g\left[a_1g^{-1}(y(t+n-1))+\ldots+a_ng^{-1}(y(t))+\right.$$
$$\left.b_0u(t+n)+\ldots+b_nu(t)\right] \quad (3.7)$$

Suppose that the function g is invertible only in the interval (a,b) where zero belongs to that interval. Then the input-output model (3.7) is valid in the restricted region of operation

$$a<h_0u(t)+\ldots+h_{t-1}u(1)<b$$

$$a<h_0u(t+1)+\ldots+h_tu(1)<b$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$a<h_0u(t+n-1)+\ldots+h_{t+n-2}u(1)<b \tag{3.8}$$

The function g is invertible in the interval (a,b) if it has non-zero derivative in that interval. It is interesting to see how Theorem 2 Part I could be applied in this particular case. Assumption 1 Part I is satisfied with n the order of the linear sub-system. This can be proved easily by writing the Wiener model in the state-space form. Assumption 2 Part I is satisfied if the function g has non-zero derivative at the origin. Let

$$Dg(0) = k \tag{3.9}$$

The response function of the linearized system then is

$$y(t)=kh_0u(t)+kh_1u(t-1)+\ldots+kh_{t-1}u(1) \tag{3.10}$$

which has order n if $k \neq 0$ and order 0 if $k = 0$. Consequently Theorem 2 Part I applies if g has a non-zero derivative at the origin. The region of validity derived by requiring the level submanifolds to be connected is given by (3.8). Thus the validity region provided by the condition of connected level submanifolds is the largest possible in this situation.

The Hammerstein model consists of a cascade of a static non-linearity followed by a linear system as in figure 3.2.

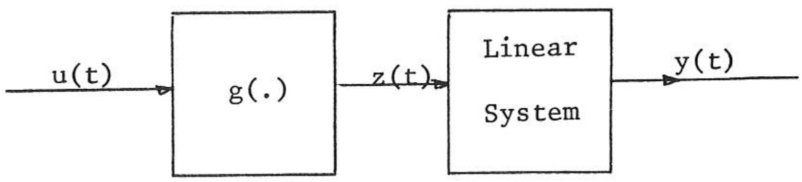

Figure 3.2

The static non-linearity is given by the function g, thus

$$z(t) = g(u(t)) \tag{3.11}$$

The linear system has response function

$$y(t)=h_0z(t)+h_1z(t-1)+\ldots+h_{t-1}z(1) \tag{3.12}$$

and it is supposed to be finitely realizable so that it satisfies the input-output model

$$y(t+n)+a_1y(t+n-1)+\ldots+a_ny(t)+b_0u(t+n)+\ldots+b_nu(t) \quad (3.13)$$

where n is the order of the linear sub-system. The overall response function of the Hammerstein model is

$$y(t)=h_0g(u(t))+h_1g(u(t-1))+\ldots+h_{t-1}g(u(1)) \quad (3.14)$$

The input-output model of such a response function can be found trivially using (3.13) without the use of Theorem 2 Part I. It is given by

$$y(t+n)=a_1y(t+n-1)+\ldots+a_ny(t)+b_0g(u(t+n))+\ldots+b_ng(u(t)) \quad (3.15)$$

and it is valid globally. It is very interesting however to investigate what Theorem 2 would require in this particular case. First of all Assumption 1 is satisfied with n the order of the linear sub-system. This is easy to see by writing the Hammerstein model in the state-space form. Assumption 2 is satisfied only if the function g has a non-zero derivative at the point zero. If the function g has

$$Dg(0) = k \quad (3.16)$$

then the response function of the linearized system is

$$y(t)=kh_0u(t)+kh_1u(t-1)+\ldots+kh_{t-1}u(1) \quad (3.17)$$

which has order n if $k \neq 0$ and order 0 if $k = 0$. Consequently Theorem 2 applies only if g has a non-zero derivative at the origin. The region of validity of the recursive input-output model given by the connected level submanifold requirement can be seen to be

$$a<u(t-1)<b \quad a<u(t-2)<b \quad \ldots \quad a<u(1)<b \quad (3.18)$$

where the function g is invertible in the interval (a,b). Consequently Theorem 2 and the validity region given by the condition of connected level submanifolds is quite restrictive since the input-output model (3.15) is actually valid globally. The reason for this is that the requirements imposed basically insure that the response function behaves similarly to a linear function so that function dependency can be checked using rank conditions and connected level submanifolds. More general conditions for function dependency would result in more relaxed conditions but these conditions would be very much more difficult to check.

Another model which is the combination of the Wiener and the Hammerstein models is the one with a static non-linearity followed by a linear system which is followed by another static non-linearity. Such a model has an input-output model globablly valid only if the output non-linearity is an invertible function similar to the Wiener model. The Wiener model, the Hammerstein model and the model with both input and output static non-linearities have input-output models globally valid in the case of invertible non-linearities. It is reasonable to hope that this may be true for any system which is a cascade of invertible static non-linearities and single-input single-output linear systems. Unfortunately this is not true. The following example demonstrates this fact.

Let the system be the one represented in figure 3.3.

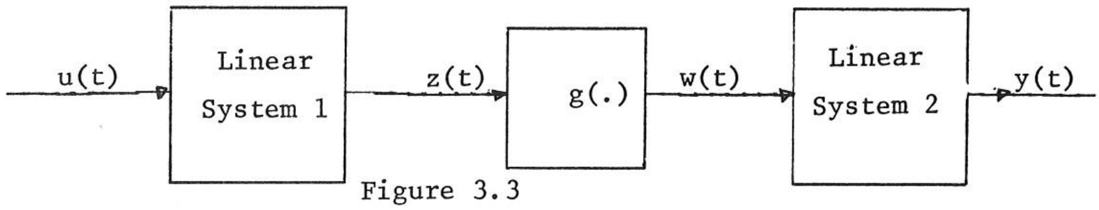

Figure 3.3

The linear system 1 is given by

$$z(t+1) = z(t) + u(t) \tag{3.19}$$

The static non-linearity g is the hyperbolic tangent function

$$w(t) = g(z(t)) = \tanh(z(t)) = \frac{e^{z(t)} - e^{-z(t)}}{e^{z(t)} + e^{-z(t)}} \tag{3.20}$$

The hyperbolic tangent function is an invertible function which is shown in figure 3.4

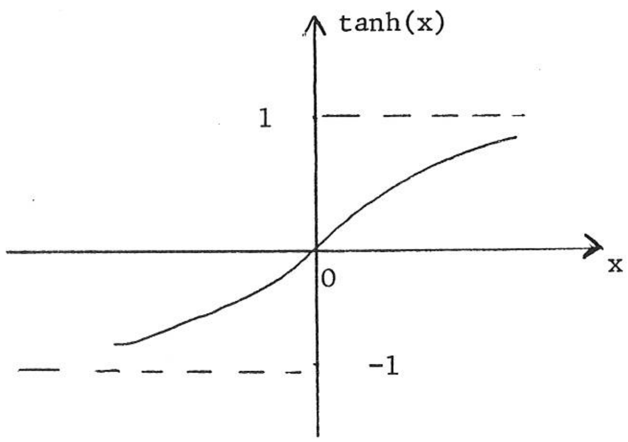

Figure 3.4

The linear system 2 is given by

$$y(t) = w(t) - 0.25w(t-1) \tag{3.21}$$

The response function of this system is

$$y(t) = g[u(t-1)+u(t+2)+ \dots +u(1)] -0.25g[u(t-2)+u(t-3)+ \dots + u(1)] \tag{3.22}$$

If the function $F_{k,t}$ is constructed it can be readily seen that Assumption 1 is satisfied with n=1. The linearized system is the cascade of the two linear sub-systems since the derivative of tanh at zero is equal to 1. The linearized system has order equal to 1. An input-output model

$$y(t+1) = q[y(t),u(t)] \tag{3.23}$$

is thus valid in some region. The exact region of validity of the input-output model (3.23) will now be found. The output at time t and t+1 are given by

$$y(t) = g[u(t-1)+z(t-1)] -0.25g[z(t-1)] \tag{3.24}$$

$$y(t+1) = g[u(t)+u(t-1)+z(t-1)]-0.25g[u(t-1)+z(t-1)] \tag{3.25}$$

If the model (3.23) is to be valid, equation (3.24) should be solved for z(t-1) so that it can be substituted in equation (3.25). The region of validity of the model (3.23) is thus the region in which equation (3.24) can be solved for z(t-1). Let for notational simplicity

$$u = u(t-1)$$

$$x = z(t-1) \tag{3.26}$$

so that from (3.24)

$$y(t) = \tanh(u+x)-0.25\tanh(x) = f(u,x) \tag{3.27}$$

The problem then is to find the region of (u,x) in which the function f(u,x) can be solved for x. In this simple case, this region is any region in which the derivative of f(u,x) with respect to x is different from zero. Thus the points (u,x) where the derivative of f(u,x) with respect to x is equal to zero must be found first

$$\frac{\partial f(u,x)}{\partial x} = \frac{1}{\cosh^2(u+x)} - \frac{0.25}{\cosh^2(x)} = 0$$

$$\Rightarrow 4\cosh^2(x) = \cosh^2(u+x) \Rightarrow 2\cosh(x)=\cosh(u+x)$$

$$\Rightarrow e^x+e^{-x} = \tfrac{1}{2}(e^{u+x}+e^{-u-x})$$

$$\Rightarrow 2e^x+2e^{-x} = e^x e^u+e^{-x}e^{-u}$$

$$\Rightarrow e^{2x}(2-e^u) = e^{-u}-2$$

$$\Rightarrow e^{2x} = \frac{e^{-u}-2}{2-e^u} \tag{3.28}$$

The points (u,x) that satisfy equation (3.28) are given in figure 3.4

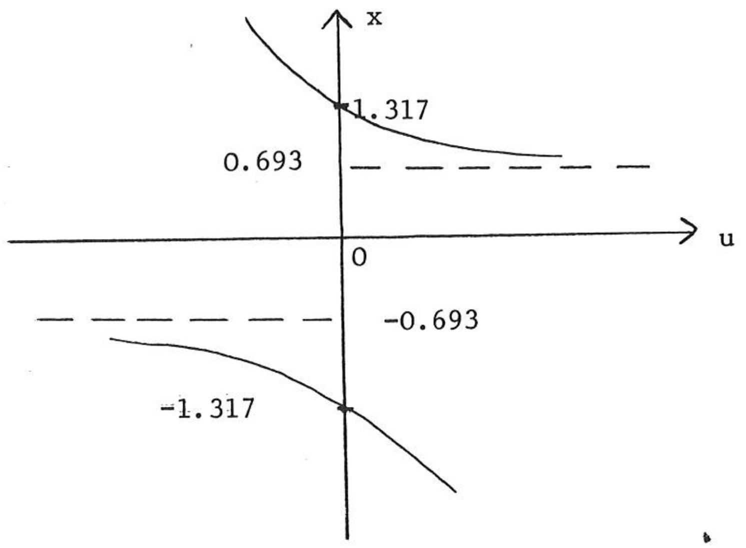

Figure 3.4

These points separate the plane into three regions. Let the region that contains the origin be called W. The input-output model is valid only if $u=u(t-1)$ and $x=z(t-1)=u(t-2)+u(t-3)+ \ldots +u(1)$ belong to W_{t-1}, i.e. when

$$(u(t-1),(u(t-2)+u(t-3)+ \ldots + u(1)))\epsilon W_{t-1} \qquad (3.29)$$

4. Stochastic systems and input-output models

The input set U of a general stochastic system is, similar to that of a general deterministic system, an r-dimensional vector space and the output set Y is an m-dimensional vector space. The output set Y is however a set of vector random variables. The response function of a stochastic system is inevitably a probability distribution function or the corresponding probability density function. Only probability density functions will be used, but in the case where the density functions do not exist, the corresponding distribution functions can be used instead.

Assuming that ordered bases for the vector spaces U and Y are specified, the input $u(t)$ is an r-dimensional column vector and the output $y(t)$ is an m-dimensional column vector. Let the vector of all outputs from time 1 to time t be

$$y^t = \left[(y(t))^T,(y(t-1))^T,\ldots,(y(1))^T\right]^T \qquad (4.1)$$

The response function of a stochastic system is the conditional probability density function of the vector y^t for a given input vector u^t, this is

$$p(y^t \mid u^t) \qquad (4.2)$$

This conditional density function for $t = 1,2,...$ specifies completely a causal stochastic system. This happens because, for a causal stochastic system, the outputs from time 1 to time t, i.e. y^t, can be influenced only by inputs in that stretch of time, i.e. by u^t, and not by any inputs after time t. Thus all the information about the vector y^t, given the inputs at all times, is provided by the conditional density function (4.2).

There is an alternative way of describing a causal stochastic system. The conditional density function (4.2) is equal to

$$\begin{aligned} p(y^t|u^t) &= p(y(t)|y^{t-1},u^t)p(y^{t-1}|u^{t-1}) \\ &= \;.\;.\;.\;.\;.\;.\;.\;.\;. \\ &= p(y(t)|y^{t-1},u^{t-1})p(y(t-1)|y^{t-2},u^{t-2})...p(y(1)|u(1)) \end{aligned} \qquad (4.3)$$

The conditional probability density function

$$p(y(t)|y^{t-1},u^t) \qquad (4.4)$$

for $t = 1,2,...$ can thus equally well describe a causal stochastic system since the density function (4.2) decomposes as a product of density functions (4.4).

The conditional probability density function of y(t), given all the past inputs and outputs, is usually only needed to provide an estimate for the output y(t) when all the past inputs and outputs are known. An estimate of a random variable can be defines in several ways. The simplest and most important estimate is the mean value of the random variable. The mean value is actually the estimate that minimizes the mean square error and thus it is also known as mean square error estimate. The estimate $\hat{y}(t)$ of y(t), given all the past values of inputs and outputs, is called the prediction of the output at time t. The prediction $\hat{y}(t)$ then is

$$\hat{y}(t)=E[y(t)|y^{t-1},u^t]=f(y^{t-1},u^t) \qquad (4.5)$$

The difference $y(t)-\hat{y}(t)$ is another vector random variable called the prediction error vector or innovation vector e(t). Thus

$$e(t) = y(t)-\hat{y}(t) \qquad (4.6)$$

The vector e(t) is the vector random variable that shows how much the actual output at time t differs from the predicted one based on all the values of inputs and outputs prior to time t.

The prediction error e(t) can be calculared when the vectors y^{t-1},u^t and y(t) are known as

$$e(t)=y(t)-f(y^{t-1},u^t) \qquad (4.7)$$

and also

$$y(t)=f(y^{t-1},u^{t})+e(t) \quad (4.8)$$

This is the prediction error or innovation from of a stochastic system, Let the vector e^{t} be

$$e^{t}=(e(t)^{T},e(t-1)^{T},\ldots,e(1)^{T})^{T} \quad (4.9)$$

The elements of the vector e^{t-1} can be calculated from the vectors y^{t-1} and u^{t-1} using (4.7). Similarly the elements of the vector y^{t-1} can be calculated from e^{t-1} and u^{t-1} using (4.8) recursively. Thus the pair (y^{t-1},u^{t-1}) and (e^{t-1},u^{t-1}) can each be derived from the other and consequently the same is true for the pair (y^{t-1},u^{t}) and (e^{t-1},u^{t}) by adding the vector u(t). The conditional probability density function (4.4) is then equal to

$$p(y(t)|y^{t-1},u^{t})=p(y(t)|e^{t-1},u^{t}) \quad (4.10)$$

The prediction $\hat{y}(t)$ can thus alternatively be given by

$$\hat{y}(t)=E[y(t)|e^{t-1},u^{t}]=f^*(e^{t-1},u^{t}) \quad (4.11)$$

The function f* in (4.11) can be considered as the response function of a deterministic system where the input in this case is the vector $[(e(t))^{T},(u(t))^{T}]^{T}$ and the output is the vector $\hat{y}(t)$. Thus under the conditions given in Part I of this paper, the following input-putput model describes the system

$$\begin{aligned}\hat{y}(t+p)=q_i^*[&\hat{y}_1(t+n_1-1),\ldots,\hat{y}_1(t),\\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ &\hat{y}_m(t+n_m-1),\ldots,\hat{y}_m(t),\\ &u_1(t+p),\ldots u_1(t),\\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ &u_r(t+p),\ldots,u_r(t),\\ &e_1(t+p-1),\ldots,e_1(t),\\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ &e_m(t+p-1),\ldots,e_m(t)]\end{aligned} \quad (4.12)$$

where $i = 1,2,\ldots,m$ and $p = \max(n_1,n_2,\ldots,n_m)$

The integers $n_1,n_2,\ldots n_m$ are the observability indices of the stochastic non-linear system. Substituting $\hat{y}(t)=y(t)-e(t)$ in the model (4.12), it becomes

$$
\begin{aligned}
y_i(t+p)=q_i[&y_1(t+n-1),\ldots,y_1(t),\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&y_m(t+n_m-1),\ldots,y_m(t),\\
&u_1(t+p),\ldots,u_1(t),\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&u_r(t+p),\ldots,u_r(t),\\
&e_1(t+p-1),\ldots,e_1(t),\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&e_m(t+p-1),\ldots,e_m(t)]+e_i(t+p)
\end{aligned}
\qquad (4.13)
$$

where $i = 1,2,\ldots,m$ and $p = \max(n_1,n_2,\ldots,n_m)$

In this particular case of single-input, single-output systems, the model (4.13) becomes

$$y(t+n)=q[y(t+n-1),\ldots,y(t),u(t+n),\ldots,u(t),e(t+n-1),\ldots,e(t)]+e(t+n) \qquad (4.14)$$

This is the generalization to the non-linear case of the model

$$
\begin{aligned}
y(t+n)=&a_1y(t+n-1)+\ldots+a_ny(t)+\\
&b_0u(t+n)+\ldots+b_nu(t)+\\
&c_1e(t+n-1)+\ldots+c_ne(t)+e(t+n)
\end{aligned}
\qquad (4.15)
$$

This linear model usally called ARMAX model (AutoRegressive Moving Average model with eXogenous input) or controlled ARMA model is the standard model used in the identification of single-input, single-output linear systems (Goodwin and Payne 1977) The model (4.14) can by extension be called the NARMAX model, for Non-linear ARMAX model. The general multivariable model (4.13) can be called multivariable NARMAX model but such a name does not describe the special structure of the model (4.13) inherited from the state-space multi-structural forms. It should thus more appropriately be called multi-structural input-output prediction error or innovation model. It is a model that can describe a wide variety of non-linear systems that satisfy the weak requirements discussed in Part I of the paper. The innovation vector e(t) is, from (4.5) and (4.6)

$$e(t)=y(t)-E[y(t)\mid y^{t-1},u^t] \qquad (4.16)$$

or equivalently from (4.6) and (4.11)

$$e(t)=y(t)-E[y(t)\mid e^{t-1},u^t] \qquad (4.17)$$

Thus the conditional expected value of the innovations is zero

$$E\left[e(t) \mid y^{t-1}, u^{t}\right] = 0 \qquad (4.18)$$

or equivalently

$$E\left[e(t) \mid e^{t-1}, u^{t}\right] = 0 \qquad (4.19)$$

Condition (4.18), or equivalently (4.19), is the condition which the innovation sequence e(t) must satisfy if the innovation form (4.8) is to be valid.

Conclusions

In this the second part of the paper the deterministic multivariable input-output model for nonlinear discrete systems has been compared with Sontag's models, and several examples have been given. A stochastic model representation for multivariable nonlinear discrete time systems has also been derived. These latter models have been given the name multistructural innovation input-output models since their special structure is inherited from the multi-structural forms of linear multivariable systems. If the system is single-input, single-output the stochastic models are called NARMAX models.

The overall aim of the present study was to extend the parameter identification methods which are available for linear systems to nonlinear systems. In order to accomplish this objective, two relatively different types of problems had to be solved. The first was the construction of nonlinear parametric models which can be used in nonlinear identification and the second was the extension to the nonlinear case of the parameter estimation and model validity tests of linear identification. One particulat solution to the first problem has been discussed in the present study. Some solutions to the second problem are available in the literature [Billings and Leontaritis (1981)(1982), Billings and Voon (1983)(1984)] and hopefully these can be extended and augmented to provide simple to implement methods of identifying nonlinear systems.

Acknowledgements

The author gratefully acknowledges that this work is supported by the SERC under grant GR/B/31163

References

BILLINGS S.A., (1980) Identification of nonlinear systems - a survey. Proc. IEE, Part D, 127, pp 272-285.

BILLINGS S.A. and LEONTARITIS I.J., (1981) Identification of non-linear systems using parameter estimation techniques, Proc. IEE Conference, Control and Its Applications, Warwick Univ., pp 183-187.

BILLINGS S.A. and LEONTARITIS I.J., (1982) Parameter estimation techniques for non-linear systems, 6th IFAC Symp. Identification and System Parameter Estimation, WAshington DC. pp 427-433.

BILLINGS S.A. and VOON W.S.F., (1983) Structure detection and model validity tests in the identification of nonlinear systems, Proc. IEE Part D, 130, pp 193-199.

BILLINGS S.A. and VOON W.S.P., (1984) Least squares parameter estimation algorithms for nonlinear systems. Int. J.Syst.Sci. (to appear).

BILLINGS S.A. and FAKHOURI S.Y., (1982) Identification of systems containing linear dynamic and static nonlinear elements, Automatica, 18, pp 15-26.

CROUCH P.E. (1981) Dynamical realization of finite volterra series, Siam J.Control and Optimization, vol. 19, pp 177-202.

GOODWIN G.C. and PAYNE R.L., (1977) Dynamic System Identification Experiment Design and Data Analysis. Academic Press, New York.

LEONTARITIS I.J. and BILLINGS S.A., (1984) Input-output parameter models for nonlinear systems, Part I Deterministic nonlinear systems. Res. Report No. 251 University of Sheffield.

PADULO L. and ARBIB M.A., (1974) System theory; a state-space approach to continuous and discrete-time systems. W.B. Sonders, Philadelphia.

PAPOULIS A., (1965) Probability, random variables and stochastic processes. McGraw-Hill, Tokyo.

SONTAG E.D., (1976) On the internal realization of polynomial response maps. Ph.D. dissertation, University of Florida.

SONTAG E.D., (1979a) Polynomial response maps. Lecture notes in Control and Information Sciences No 13, Springer-Verlag.

SONTAG E.D., (1979b) Realization theory of discrete-time nonlinear systems - the bounded case, IEEE Transactions on circuits and systems, vol. CAS-26, pp 342-356.

WERTZ V. GEVERS M. and HANNAN E.J., (1982) The discrete determination of optimum structures for the state space representation of multivariable stochastic processes, IEEE Trans. on Automat. Control. vol. AC-27, pp 1200-1211