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ON SYSTEMS WITH SPECTRAL CLUSTER POINTS AND THEIR RELATION TO H^p SPACES

by

S. P. Banks & F. Abbasi-Ghelmansarai

University of Shefffield,
Department of Control Engineering,
Mappin Street,
Sheffield. S1 3JD.

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1. Introduction

In this paper we continue our investigation of the general theory of root locus for distributed parameter systems which we began in Banks and Abbasi-Ghelmansarai (1983a,1983b). In the latter paper we considered the root locus of systems whose defining operator had a spectrum which did not cluster at any point of the plane, and we obtained a generalisation of the classical theory by considering 'generalised poles and zeros'. The motivation for the present work is to remove the restriction on the clustering of singulatities of the transfer function. In doing this we shall see that the general theory of root locus is intimately connected with the theory of H^p spaces.

The basic theory of H^p spaces is therefore presented in section 2, together with two important approximation theorems for H^∞. In section 3 we shall consider systems whose transfer function is a Blaschke product and show that, if D denotes the unit disk in the s plane and ∂D its boundary, then the root locus of the system clusters around ∂D. Then in section 4 we shall consider systems whose transfer function G is singular (that is, G is defined via the integral of an expontential function with respect to a singular measure on ∂D). This will lead to a rather surprising example of a simple system with a bounded defining operator whose transfer function behaves (at s=1) like a pole and a zero of infinite order at the same time. When the output of the system is fed back with a gain k, this singularity splits up into an infinite number of root locus branches which all converge back at the point s=1 as $k\to\infty$. These branches leave the singularity from outside D and return to s=1 from inside D.

The case of a general H^∞ function is dealt with in section 5 using the fact that Blaschke products generate H^∞ as an algebra. We shall again give a simple example of a system with bounded defining operator such that the transfer function has an infinity of separated poles which cluster at $s=\pm i$.

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However the points $s=\pm i$ again behave like infinite order zeros and attract all the root locus branches.

We shall finally show that systems whose spectrum is not bounded in the finite plane may be included in this theory by interpreting the root locus on a Riemann surface and regarding the point at infinity as just another point on the surface. Many systems whose singularities are not bounded may therefore be regarded as systems defined on a surface in which the singularities cluster at some point; the fact that the point is labelled ∞ is irrelevant.

2. H^p Spaces

In this section we shall review the theory of H^p spaces which we shall need in this paper. Proofs of the results can be found, for example, in Garnett (1981). Let us first recall definition of an H^p space.

Definition 2.1 If $D=\{z:|z|<1\}$ denotes the open unit disc in $\mathbb{C}$, then H^p is the set of functions f analytic on D such that

$$\sup_r \frac{1}{2\pi} \int |f(re^{i\theta})|^p \, d\theta = \|f\|^p_{H^p} < \infty \quad , \qquad \text{if } 0<p<\infty$$

and for which

$$\| f \|_\infty = \sup_{z\in D} |f(z)| < \infty \quad , \qquad \text{if } p=\infty$$

In this paper we shall be particularly interested in H^∞ functions which are the transfer functions of some systems. The importance of H^p spaces in realisation theory has already been pointed out by Baras and Brockett (1975). We are mainly concerned here with the root locus of systems which have cluster points in their spectra, and later we shall see some particularly surprising behaviour in the case of H^∞ transfer functions.

Definition 2.2 Let $\{z_n\}$ be a sequence in D such that

$$\sum (1-|z_n|) < \infty \, . \tag{2.1}$$

Then the function

$$B(z) = z^m \cdot \prod_{|z_n|\neq 0} \left(\frac{-\bar{z}_n}{|z_n|} \right) \left(\frac{z-z_n}{1-\bar{z}_n z} \right) \tag{2.2}$$

is called a Blaschke product.

It is easy to see that B(z) has m zeros at the origin and also zeros at $z_n (\neq 0)$. The condition (2.1) implies that if an infinite number of the z_n's are distinct, they must accumulate at the boundary, ∂D, of D. If E denotes the subset of ∂D consisting of the accumulation points of $\{z_n\}$, then B(z) extends analytically to $\mathbb{C} \setminus (E \cup \{1/\bar{z}_n : n=1,2,\ldots\})$.

Definition 2.3 A singular function on D is a function of the form

$$S(z) = \exp\left[-\int\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) d\mu(\theta)\right] , \tag{2.3}$$

where $d\mu$ is a positive measure which is singular with respect to the usual Lebesgue measure $d\theta$ on ∂D. If $E\subseteq\partial D$ is the closed support of μ, then $S(z)$ extends analytically to $\mathbb{C}\setminus E$.

Definition 2.4 An inner function is a function $f\varepsilon H^{\infty}$ such that $|f(e^{i\theta})|=1$ almost everywhere.

It can be shown (Duren, 1970) that every inner function f may be written in the form

$$f(z) = e^{i\alpha} B(z)S(z) ,$$

where α is real, B is a Blaschke product and S is a singular function. Finally, in order to characterise H^p spaces, we must introduce the notion of outer function.

Definition 2.5 An outer function for the space H^p is a function of the form

$$F(z) = e^{i\beta} \exp\left\{\frac{1}{2\pi} \int_o^{2\pi} \left(\frac{e^{it}+z}{e^{it}-z}\right) \log \psi(t)dt\right\}$$

where β is real, $\psi(t)\geq o$, $\log \psi \ \varepsilon L^1[o,2\pi]$ and $\psi \ \varepsilon L^p[o,2\pi]$.

Then we have,

Theorem 2.6 (Duren, 1970; Garnett, 1981). Every function $f\neq o$ in H^p $(o<p\leq\infty)$ has a unique factorisation of the form $f(z)=B(z)S(z)F(z)$, where B is a Blaschke product, $S(z)$ is singular and $F(z)$ is outer. □

This is a fundamental factorisation theorem for H^p functions. Since it can be shown that S and F have no zeros in D it follows that the zeros of any H^p function are contained in the Blaschke product factor. However, the behaviour of a function $f\varepsilon H^p$ on ∂D can be very complex. Consider, for example, the singular function

$$S(z) = \exp\left(\frac{z+1}{z-1}\right), \quad z \varepsilon D.$$

obtained from (2.3) by taking μ to be the Dirac measure at the point 1. Then S and all its derivatives have the nontangential limit 0 at z=1. (By a nontangential limit we mean that the limit is approached from inside a cone with vertex at the limit and contained in D). However, if we extend S analytically to $\mathbb{C}\backslash\{1\}$ then, if x is real, $\lim_{x\to 1^+} \exp\left(\frac{x+1}{x-1}\right) = \infty$. Hence the point z=1 is an isolated essential singularity of S, and moreover, we shall see that in system theory terms, S behaves like an infinite order zero inside D and like an infinite order pole outside D.

The following two results are important in the particular case of H^∞ functions, in view of the simplicity of the structure of Blaschke products compared with singular and outer functions.

Theorem 2.7 (Garnett, 1981). The set of Blaschke products is dense (in the topology of H^∞) in the set of inner functions.

Theorem 2.8 (Marshall, 1976). H^∞ is generated by Blaschke products. (since H^∞ is an algebra, by generated we mean that any H^∞ function can be uniformly approximated by a finite sum of products of finitely many Blaschke functions).

The last two theorems make the space H^∞ particularly attractive for the study of the distributed root locus as we shall see later.

3. Root Locus of Blaschke Products

In this section we shall consider the root locus of a system whose transfer function is a Blaschke product. The definitions of transfer function and root locus for distributed systems have been given in a previous paper (Banks and Abbasi-Ghelmansarai, 1983b) and, as in the finite-dimensional case, we must solve the equation

$$1 + kB(s) = 0 \quad , \quad k \varepsilon [0, \infty] \tag{3.1}$$

where $B(s) = s^m \prod_{|s_n| \neq 0} \frac{-\bar{s}_n}{|s_n|} \frac{s-s_n}{1-\bar{s}_n s}$ is a Blaschke product.

Let E denote the accumulation points of B on ∂D and let X be the set

$$X = \mathbb{C} \setminus (E \cup \{1/\bar{s}_n : n=1,2,\ldots\}). \tag{3.2}$$

and denote by $A(\varepsilon)$ the annulus of width 2ε about ∂D, i.e.

$$A(\varepsilon) = \{s \in \mathbb{C} : 1-\varepsilon < |s| < 1+\varepsilon\}$$

Then we have

<u>Lemma 3.1</u> If a system has transfer function B(s) (a Blaschke product), then the root locus of the system begins in the set $P=\overline{\{1/\bar{s}_n : n=1,2,\ldots\}}$, ends in the set $Z=\overline{\{s_n : n=1,2,\ldots\}}$ and is such that, for any $\varepsilon>0$, has only a finite number of branches in $\mathbb{C}\setminus A(\varepsilon)$.

<u>Proof</u> Consider the function

$$\frac{1}{k} + B(s).$$

Then, if $s \in \mathbb{C}\setminus Z$, it follows that $d(s,Z)>0$ (since Z is closed; d denotes the distance from s to Z). Since $B(s) \neq 0$ it follows that $B(s) \neq -1/k$ for sufficiently large k. Hence the root locus ends on Z. Similarly, the root locus begins on P.

To prove the final part note that, on the set X defined in (3.2), the function $f(s)=1 + kB(s)$, for any $k \in [0,\infty)$, is analytic and is not identically zero on any open subset of X. Hence f cannot have a limit of zeros in X. (The root locus is compact with any limit points included). Since f clearly does not have any zeros in $\{1/\bar{s}_n : n=1,2,\ldots\}$ the result follows. □

<u>Remark 3.2</u> In particular, it follows from the lemma that all but a finite number of branches of the root locus lie arbitrarily close to the unit circle ∂D. In fact, if any $\varepsilon>0$ is given, then outside the annulus $A(\varepsilon)$ the root locus approximates that of a rational function corresponding to the first few terms of the Blaschke product, since the last terms of the product must be arbitrarily close to 1 in $\mathbb{C}\setminus A(\varepsilon)$. (cf. fig.3.1)

<u>Remark 3.3</u> We have taken P and Z as the closures of the singular and zero sets of the Blaschke product since the limit points in ∂D are also singularities of B(s).

4. Systems with Singular Transfer Functions

Having described the root locus of systems whose transfer function is a Blaschke product, we shall consider now those systems with any inner transfer function; i.e. products of singular functions and Blaschke products. In particular we shall be interested in the behaviour of singular systems.

Lemma 4.1 Let $S(s)$ be a singular function in $H^{\infty}(D)$ and suppose that $B_n(s)$ is a sequence of Blaschke products such that

$$B_n \to S \quad \text{in } H^{\infty}(D).$$

Then the root locus of the system with transfer function $S(s)$ is the limit in the Hausdorff metric of the root loci of the systems with transfer functions $B_n(S)$.

Remark 4.2 The Hausdorff metric is defined on the set $\mathcal{C}$ of closed subsets of a normed space $(N, \|\cdot\|)$ as follows:

If $A,B \in \mathcal{C}$ then define

$$\delta(A,B) = \max\left\{ \sup_{b \in B} \left(\inf_{a \in A} \|a-b\|\right), \sup_{a \in A} \left(\inf_{b \in B} \|a-b\|\right) \right\}.$$

Then $(\mathcal{C},\delta)$ is a complete metric space.

Proof of lemma 4.1 Note first that it is easy to see that the root locus of $S(s)$ does not contain ∞. For, if

$$1 + kS(\infty) = 0 \quad , \quad \text{some finite } k$$

then $S(\infty) = -1/k$ and since $B_n \to S$ (by analytic continuation) it follows that there must exist a sequence s_n such that

$$1 + kB_n(s_n) = 0$$

and $s_n \to \infty$. However, $B_n(\infty)$ is either ∞ or

$$|s_{mn}| \neq 0 \left\{ \frac{1}{|s_{mn}|} \right\}$$

where (s_{mn}) are the zeros of B_n. Hence $B_n(\infty)$ must be bounded away from $-1/k$.

The root loci of the Blaschke systems are therefore compact and it suffices to prove that the zeros of $1+kB_n(s)$ converge uniformly for k in compact subsets of $[o,\infty)$ to those of $1+kS(s)$ and similarly the zeros of $1/k+B_n(s)$ converge uniformly for k in compact subsets of $(o,\infty]$ to those of $1/k+S(s)$. However, from lemma 3.1 it is clear that the zeros and poles of the Blaschke products $B_n(s)$ converge to the support of the singular measure of $S(s)$ and so we need to show on that the zeros of $1+kB_n(s)$ converge to those of $1+kS(s)$ uniformly for $k\varepsilon[\varepsilon,1/\varepsilon]$ for any $\varepsilon>o$. However, it is easy to see that, by the definition of H^∞, the zero sets of $1+kB_n(s)$, for $k\varepsilon[\varepsilon,1/\varepsilon]$ form a Cauchy sequence in $(\mathcal{C},\delta)$ (which is complete) and the result follows. □

Example 4.3

Consider the system governed by the equation

$$\frac{\partial^2}{\partial x \partial t} - \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t} + \phi , \qquad x\varepsilon[o,\infty) \tag{4.1}$$

with boundary condition $\phi(o,t) = 1$, and suppose the observation is

$$y = \phi(1,t)$$

Then we may also write the system in the form

$$\frac{\partial \phi}{\partial t} = \left(\frac{\partial}{\partial x} -1\right)^{-1}\left(\frac{\partial \phi}{\partial x} + \phi\right) = A\phi$$

where

$$A = \left(\frac{\partial}{\partial x} -1\right)^{-1}\left(\frac{\partial}{\partial x} +1\right)$$

which can be extended to a bounded operator on $L^2[o,\infty]$. The transfer function is easily seen to be

$$G(s) = \exp\left(\frac{s+1}{s-1}\right)$$

(i.e. a singular inner function with a single singularity at s=1).

The root locus is given by

$$1 + kG(s) = 0$$

and so, if $s = x + iy$, we have

$$\frac{-1}{k} = \exp\left(\frac{x^2-1+y^2}{(x-1)^2+y^2}\right)\left(\cos\left\{\frac{2y}{(x-1)^2+y^2}\right\} - i\sin\left\{\frac{2y}{(x-1)^2+y^2}\right\}\right)$$

and since the imaginary part of the left hand side is zero we must have

$$\frac{2y}{(x-1)^2+y^2} = n\pi . \qquad (4.2)$$

However, for $k>o$, the left hand side is negative and exp is always positive so n must be odd to make cos negative. Hence,

$$\exp\left(\frac{x^2-1+y^2}{(x-1)^2+y^2}\right) = \frac{1}{k}$$

and so

$$x^2 + \frac{2\alpha}{1-\alpha}x + y^2 = \frac{1+\alpha}{1-\alpha}, \qquad \alpha \neq 1. \qquad (4.3)$$

where $\alpha = \log\frac{1}{k}$.

From (4.2) we have

$$x^2 - 2x + 1 + y^2 - \frac{2y}{n\pi} = 0 \qquad (4.4)$$

From (4.3) and (4.4) it now follows easily that

$$x = 1 - \left(\frac{1-\alpha}{n\pi}\right)y$$

and so

$$x^2\left(1+\frac{n^2\pi^2}{(1-\alpha)^2}\right) + 2x\left(\frac{\alpha}{1-\alpha} - \frac{n^2\pi^2}{(1-\alpha)^2}\right) + \frac{n^2\pi^2}{(1-\alpha)^2} - \frac{1+\alpha}{1-\alpha} = 0$$

Therefore,

$$x = 1 \text{ or } x = \left(\frac{n^2\pi^2}{(1-\alpha)^2} - \frac{1+\alpha}{1-\alpha}\right) \Big/ \left(1 + \frac{n^2\pi^2}{(1-\alpha)^2}\right) .$$

Alternatively, (4.3) and (4.4) may be written respectively as

$$\left(x + \frac{\alpha}{1-\alpha} \right)^2 + y^2 = \frac{1}{(1-\alpha)^2} \tag{4.5}$$

and

$$(x-1)^2 + \left(y - \frac{1}{n\pi} \right)^2 = \frac{1}{n^2\pi^2} \tag{4.6}$$

and so the root locus of the system consists of the loci of the intersection of these two circles as α changes from ∞ to $-\infty$ (i.e. as k increases from 0 to ∞). These circles and their intersections are shown in fig 4.1 and the root locus of the system (4.1) is shown in fig. 4.2. From fig. 4.2 we see that the root locus is the union of a set of circles with centre at $1/n\pi$ and touching the point s=1. Hence for the system (4.1) the singularity at s=1 splits into an infinity of zeros when k>o which are all outside the disk $|s| \leq 1$ until α=o (i.e. k=1) at which point the zeros move inside the unit disk, and are eventually attracted back to s=1. Hence the point s=1 behaves like a infinite order open loop pole for low k and as an infinite order open loop zero for large k, which is to be expected, since the nontangential limit of an inner function inside the unit disk is zero as shown above. However, the nontangential limit of a singular function outside the unit disk towards a support point of the measure is $\pm\infty$.

5. Systems with Transfer Functions in H^∞

We shall now consider systems which have transfer functions which are elements of H^∞ and which are analytic on some arc Γ of $\partial D^\dagger$. From theorem 2.8 we know that, as a function algebra, H^∞ is generated by Blaschke products. Hence if a system has the transfer function $G(s) \varepsilon H^\infty$, then we may write

$$G(s) = \sum_{i=1}^{\infty} P_i(s) \quad , \quad |s|<1 \tag{5.1}$$

† This condition is required to ensure that the outer factor of G(s) has an analytic continuation outside D; see Garnett, 1981, p.78.

where

$$P_i(s) = \prod_{j=1}^{n_j} \left(B_{ij}(s) \right)$$

and each B_{ij} is a Blaschke product. The sum in (5.1) converges in the topology of H^∞. Let $S_1 \subseteq \mathbb{C}$ be the set on which G(s) has an analytic continuation, and let S_2 be the set on which all the Blaschke products B_{ij} have an analytic continuation. Then we can extend (5.1) to $S_1 \cup S_2$.

Lemma 5.1 The root locus of the system with transfer function $G(s) \in H^\infty$, which is analytic across some arc Γ of ∂D, and which when extended analytically outside D has only finitely many singularities in $\mathbb{C} \setminus \{D \cup A(\varepsilon)\}$ for any $\varepsilon > 0$, has only a finite number of branches outside any neighbourhood of $\partial D \cup S_1$ (and so behaves, in such a neighbourhood, essentially like a finite dimensional system).

Proof The proof is very similar to that of lemma 3.1, and therefore only an outline will be given. Clearly the poles of the Blaschke products B_{ij} must cluster around $\mathbb{C} \setminus S_1$ and so outside any neighbourhood N of $\partial D \cup S_1$ we may write (5.1) in the approximate form

$$G(s) \simeq \sum_{i=1}^{m} \left\{ \prod_{j=1}^{n_j} \left(B_{ij}(s) \right) \right\}$$

and we may write each Blaschke product $B_{ij}(s)$ in the form

$$B_{ij}(s) = \alpha_{ij}(s) B_{ij}^f(s)$$

where $B_{ij}^f(s)$ is a Blaschke product with only finitely many zeros and $\alpha_{ij} \simeq 1$. Hence, in $\mathbb{C} \setminus N$, $1+kG(s)$ has the form

$$1+kG(s) \simeq 1+k \cdot \sum_{i=1}^{m} \left\{ \prod_{j=1}^{n_j} \left(B_{ij}^f(s) \right) \right\}$$

which is rational. □

<u>Example 5.2</u> Consider the system

$$\frac{\partial^4}{\partial t^2 \partial x^2}\phi = -\frac{\partial^2 \phi}{\partial x^2} + \phi \qquad (5.2)$$

defined in the interval $[0,1]$ with boundary conditions

$$\phi(0,t) = 0 , \quad \phi(1,t) = u(t) \quad \text{(control)}$$

and with the observation

$$y = \phi(\tfrac{1}{2},t)$$

Then (5.2) can be written

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2}\right)^{-1}\left(-\frac{\partial^2 \phi}{\partial x^2} + \phi\right)$$

and the operator

$$A = \left(\frac{\partial^2}{\partial x^2}\right)^{-1}\left(-\frac{\partial^2}{\partial x^2} + 1\right), \quad \mathcal{D}(A) = \{\phi \varepsilon H^2[o,1] : \phi(0)=0, \phi(1)=u\}$$

can be extended to a bounded operator on $L^2[o,1]$. The stability of the system is therefore determined by $\sigma(A)$.

Now the transfer function of (5.2) is easily seen to be

$$G(s) = \frac{1}{2\cosh\left(\frac{1}{2}\sqrt{\frac{1}{s^2+1}}\right)} ,$$

and the root locus is given by

$$\cosh\left(\frac{1}{2}\sqrt{\frac{1}{s^2+1}}\right) + \frac{k}{2} = 0 \qquad (5.3)$$

Now it is well known (Abramowitz and Stegun, 1968) that

$$\cosh \frac{1}{2\sqrt{(s^2+1)}} = \prod_{k=1}^{\infty}\left[\frac{(s^2+1)(2k-1)^2\Pi^2+1}{(s^2+1)(2k-1)^2\Pi^2}\right] \qquad (5.4)$$

and so G(s) has open loop poles at

$$s = \pm i \left(1 + \frac{1}{(2k-1)^2 \Pi^2} \right)^{\frac{1}{2}}$$

which cluster at ±i and intuitively it appears from (5.4) that G(s) should have an infinity of zeros at ±i. Note that $G(s) \varepsilon H^\infty$ but is not an inner function although we know that on D it must be the H^∞- limit of a sum of products of Blaschke functions, and by lemma 5.1 we expect 'most' of the root locus to cluster near ∂D. That this is so can now be seen by an elementary calculation. In fact, from (5.3), if we put $\zeta = x+iy = \frac{1}{2\sqrt{s^2+1}}$ we have

$$\cos y \cosh x = -\frac{k}{2} \quad , \quad \sin y \sinh x = 0$$

and so x = 0 or y = nπ. For $k \varepsilon [0, \frac{1}{2})$ we must have x = 0 and so

$$\cos y = -\frac{k}{2} .$$

For $k \varepsilon [\frac{1}{2}, \infty)$ we must have y = nπ for odd n and then

$$\cosh x = \frac{k}{2}$$

The solutions of these equations can easily be seen graphically as in fig 5.1. If we transfer this information from the ζ-plane back into the s-plane, it is easy to see that the root locus of (5.2) is as shown in fig. 5.2. (we have shown only the roots clustering about +i; there is a similar locus around -i.) As we saw above, the points s=±i behave like infinite order zeros which attract all the open loop poles. Note also that, because of the square root in the transfer function, this root locus should be interpreted on a Riemann surface obtained by cutting the s-plane between ±i.

6. Relation to systems with compact resolvent

In the previous sections we have been concerned with systems whose spectrum is bounded in the finite complex plane. We shall now consider a simple system whose spectrum is separated on the negative real axis and tends to infinity. Such a system is given by the heat equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

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$$\phi(o,t) = 0$$

$$\phi(1,t) = u(t), \ t>o$$

$$\phi(x,o) = 0 \ , \quad x\varepsilon[o,1].$$

together with the observation

$$y = \phi(\xi,t) \ , \quad \xi\varepsilon[o,1]$$

(cf. Pohjolainen, 1981). Then the transfer function is

$$G(s) = \frac{\sinh\sqrt{s}\,\xi}{\sinh\sqrt{s}} \tag{6.1}$$

The spectrum of the system consists of the eigenvalues

$$s_i = -n^2\pi^2 .$$

If we put

$$s' = 1 - \frac{1}{s} \tag{6.2}$$

then we obtain the transfer function

$$G(s_1) = \frac{\sinh\left(\sqrt{\frac{1}{1-s'}}\,\xi\right)}{\sinh\left(\sqrt{\frac{1}{1-s'}}\right)} \tag{6.3}$$

in the new variable s'. It is easy to see that $G(s')\varepsilon H^\infty(s')$ and the previous theory applies. Hence it is natural to consider the root locus of (6.1) as clustering around $s=\infty$. (It is easy to see that the same argument applies to any system whose defining operator has compact resolvent.)

This example and the preceding theory essentially confirm the suggestion made in a previous paper (Banks and Abbasi-Ghelmansarai, 1983b) that it is reasonable to consider the root locis of general distributed systems on a Riemann surface. In the simplest case of a surface of genus 0 we obtain the Riemann sphere S shown in fig. 6.1, which is 'symmetric' with respect to each point $s \in S$. Hence if D is a disk on S (or even a topological disk D_T) as shown in fig. 6.1, we can define the space $H^\infty(D)$ just as before. The distinction between the systems with spectrum in the finite plane and those whose spectrum clusters at ∞ now essentially vanishes, and hence regarding $s=\infty$ as a well-defined point on the sphere S we can see that the change of variables (6.2) is unnecessary. It merely corresponds to shifting equivalent points on S.

7. Conclusions

In this paper we have seen that the root locus theory for distributed systems may be extended to include systems whose transfer function is in H^∞. The examples which we presented show that the root locus of such systems differ radically from the finite- dimensional analogue although the classical result that the number of zeros must match the number of poles carries over provided we 'count' the zeros and poles in the correct way. By saying that the zeros and poles match we mean, of course, that we include any open loop zeros at ∞, so that the root locus of $1/s$ goes from $s=o$ to $s=\infty$, and the system $1/s$ has a pole at $s=o$ and zero at $s=\infty$. This simple system provides a clue that even the root locus of a finite dimensional system is most naturally conceived of on the Riemann sphere, since then the zeros or poles at 'infinity' are counted in just the same way as at any other point. The classical result that a finite-dimensional system with m zeros and n poles (in the finite plane) has n-m branches of the root locus tending to ∞ becomes simply that on the Riemann sphere the system has precisely n zeros and n poles and n branches (almost everywhere) which tend to the zeros as $k \to \infty$.

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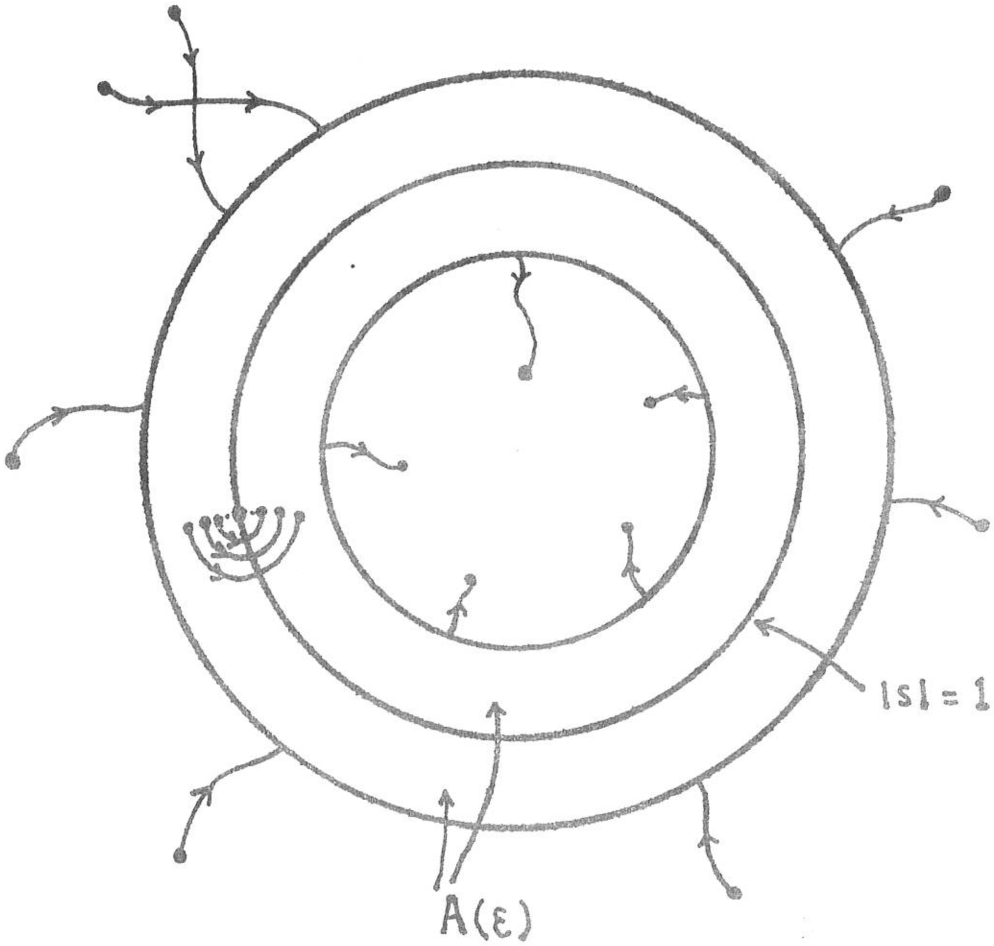


fig. 3.1.

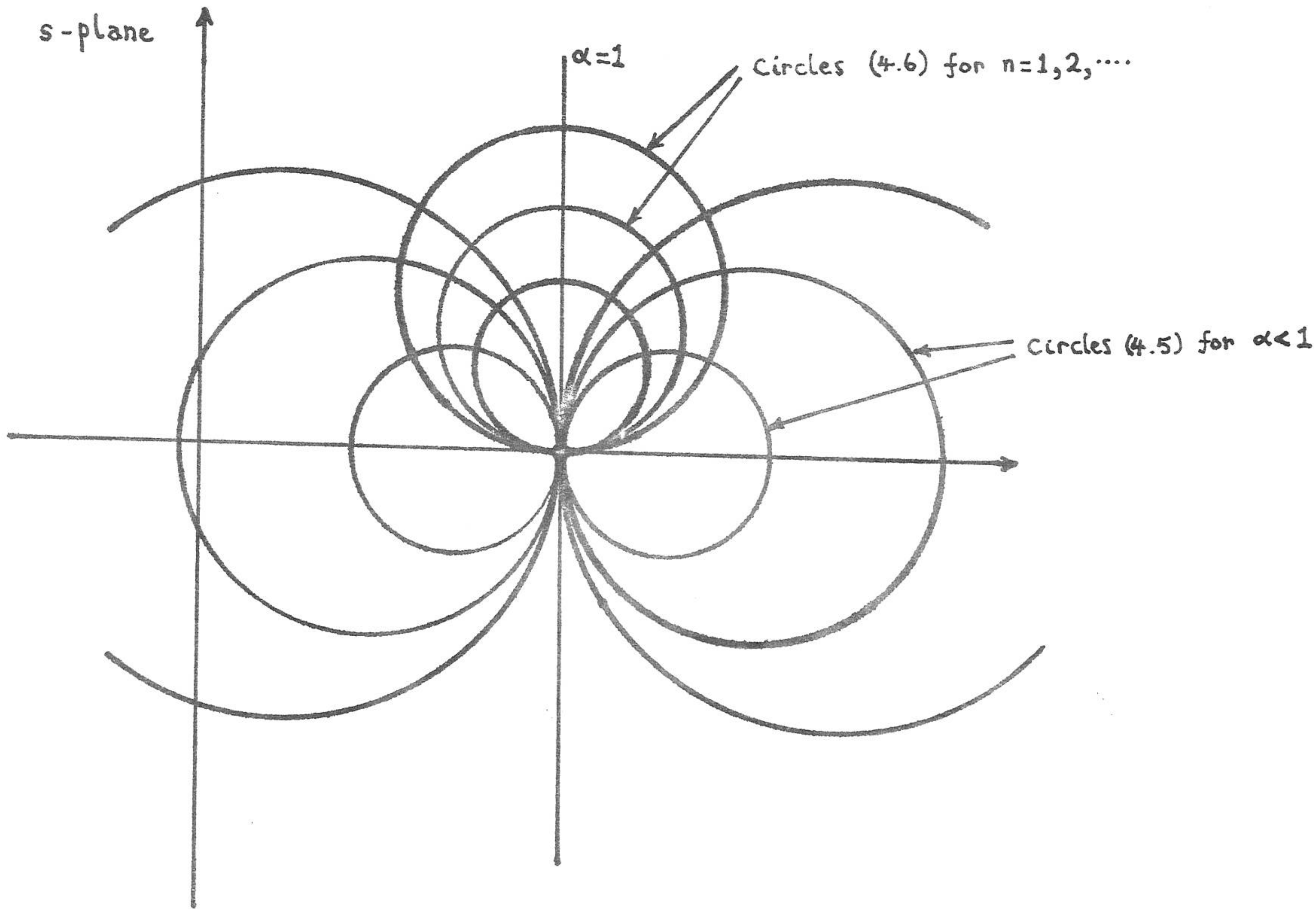


fig.4.1.

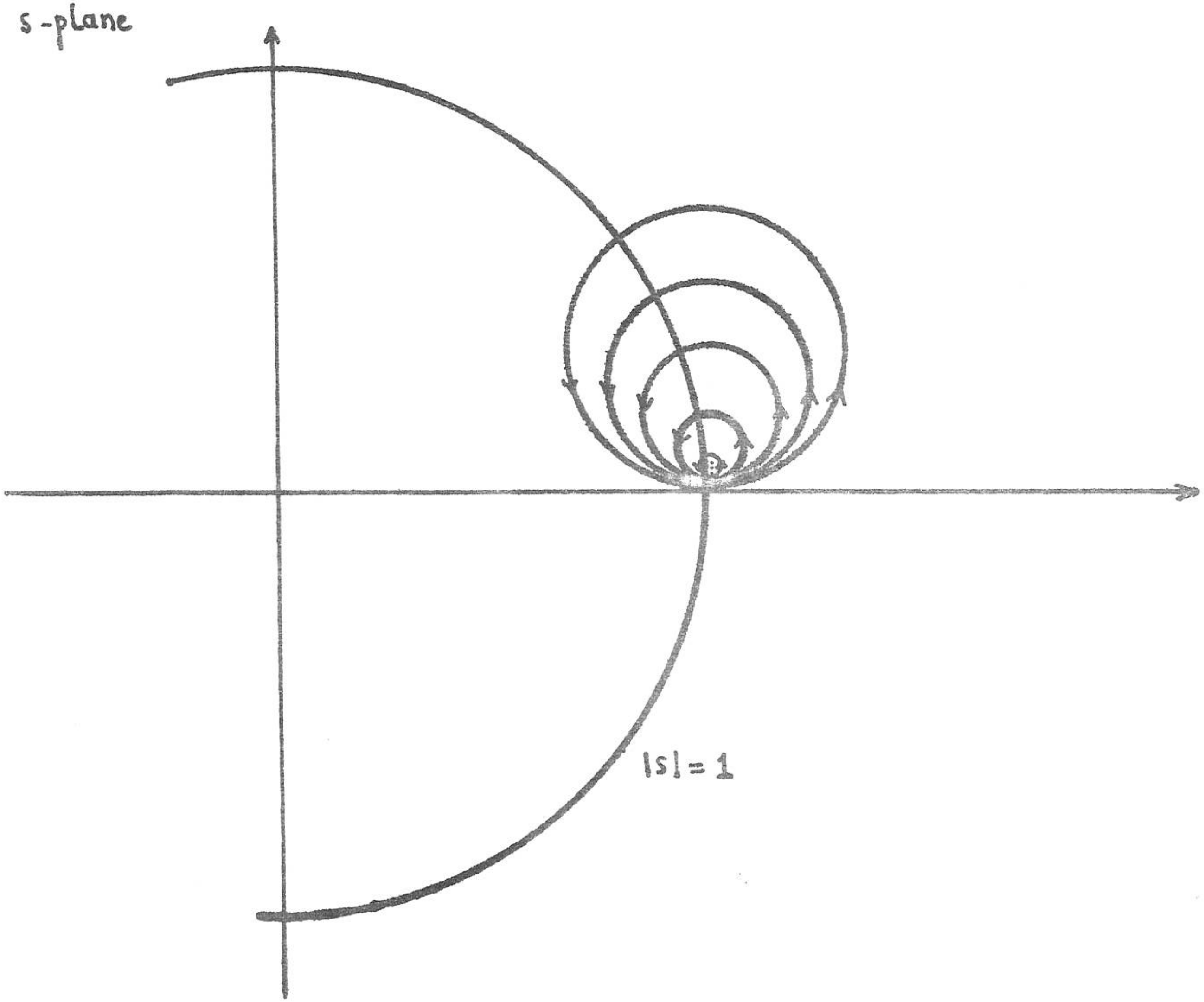
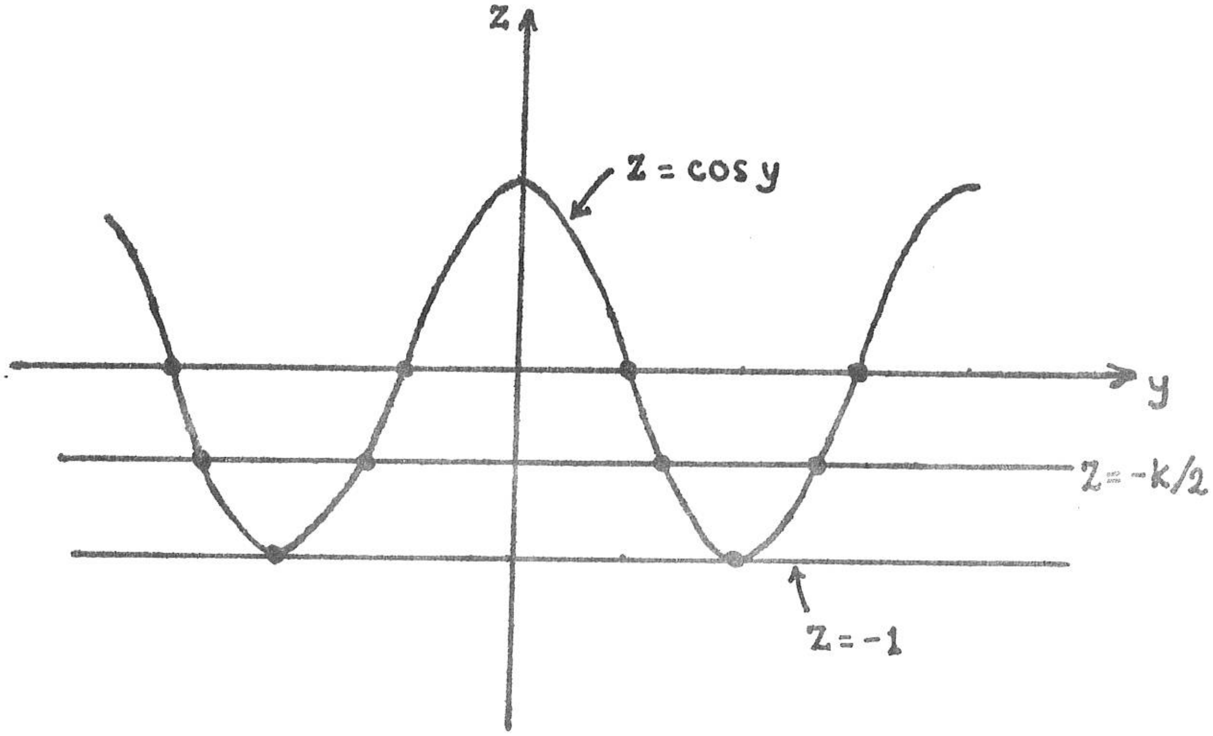
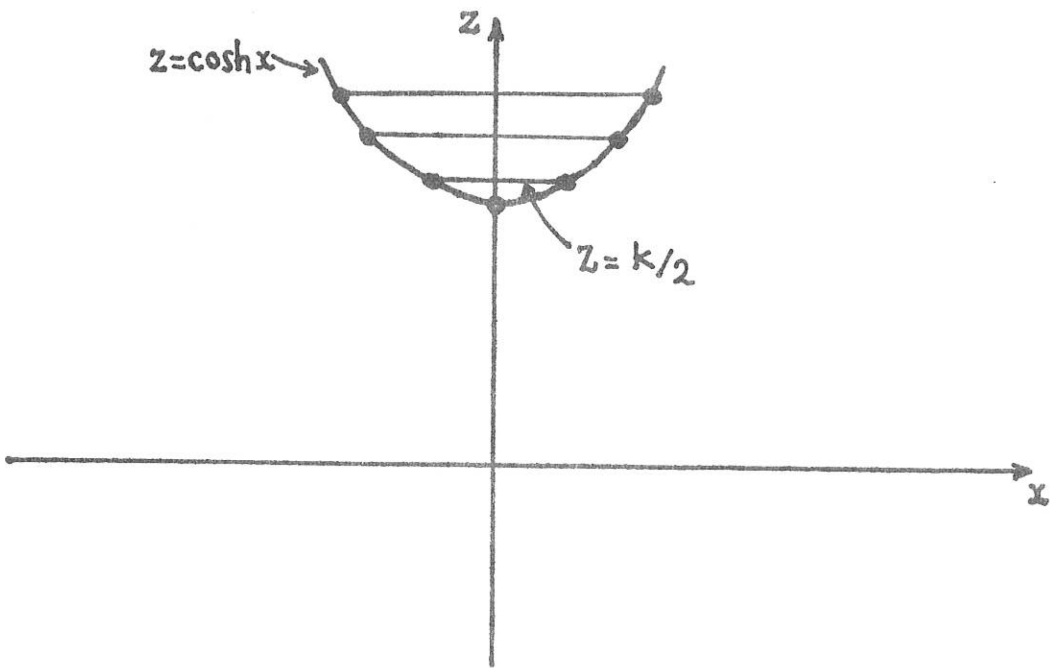


fig.4.2.



(a). $k \in [0, 1/2)$, $x = 0$



(b). $k \in [1/2, \infty)$, $y = n\pi$ (n odd)

fig. 5.1.

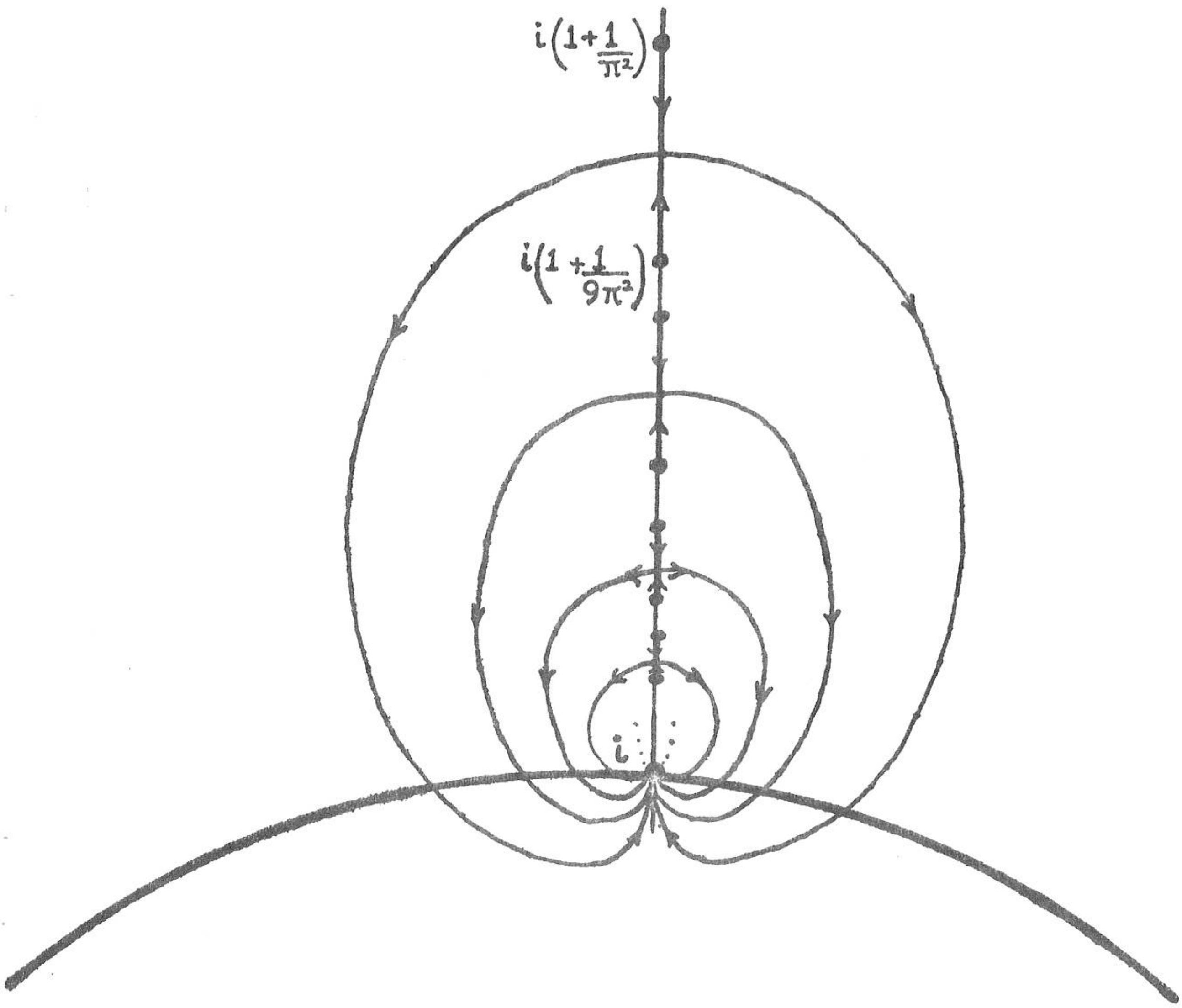


fig. 5.2.