SOME NOTES ON OPTIMIZATION OF CONFIDENCE BANDS IN PROCESS CONTROL BASED ON APPROXIMATE MODELS

by

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1. Introduction

Let $E^{p}(t)$, $t \geq o$, be the mxm matrix

$$E^{p}(t) = Y(t)K_{p} - Y_{A}(t) \tag{1}$$

where K_p is mxm and constant, $Y(t)$ is mxm and $Y_A(t) = \text{diag}\{h_j(t)\}_{1 \leq j \leq m}$. The minimization of the widths of the confidence circles can be expressed in the form

$$\min_{K_p} \max_{j} \sum_{\ell=1}^{m} N_{\infty}(E^{p}_{j\ell}) \tag{2}$$

if 'row dominance' conditions are used or

$$\min_{K_p} \max_{j} \sum_{\ell=1}^{m} N_{\infty}(E^{p}_{\ell j}) \tag{3}$$

if 'column dominance' conditions are used. If Y_A is (as yet) unspecified then E^{p}_{jj} is unspecified, $1 \leq j \leq m$, and (2) and (3) can be replaced by

$$\min_{K_p} \max_{j} \sum_{\ell \neq j} N_{\infty}(E^{p}_{j\ell}) \tag{4}$$

and

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$$\min_{K_p} \max_{j} \sum_{\ell \neq j} N_{\infty}(E^{p}_{\ell j}) \tag{5}$$

provided that one extra constraint (say)

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$$\|K_p\| = 1 \tag{6}$$

is introduced to avoid the trivial solution $K_p = 0$. Here $\|K_p\|$ is any convenient matrix norm on K_p such as

$$\| K_p \| = \max_{ij} |(K_p)_{ij}| \tag{7}$$

Other norms or constraints are, of course, possible and may be desirable for numerical purposes or to avoid singularity of K_p. (Note: There is an advantage in using the 'column dominance' conditions as, if

$$K_p = [K_1, K_2, \ldots K_m] \tag{8}$$

then problems (3) and (5) can be represented as

$$\min_{K_{\cdot j}} \sum_{\ell=1}^{m} N_\infty (E^p_{\ell j}) \quad , \quad 1 \leq j \leq m \quad , \tag{9}$$

and

$$\min_{K_{\cdot j}} \sum_{\ell \neq j} N_\infty (E^p_{\ell j}) \quad , \quad \|K_{\cdot j}\| = 1 \quad , \quad 1 \leq j \leq m \tag{10}$$

respectively)

An alternative approach is to solve

$$\min \quad r\,(N^p_\infty\,(E^p)) \tag{11}$$

or, if G_A is unspecified,

$$\min_{K_p} \quad r\,(N^p_\infty\,(E^p) - \text{diag}\,\{N_\infty(E^p_{jj})\}_{1 \leq j \leq m})$$

$$\| K_p \| = 1 \tag{12}$$

All of these problems are non-quadratic and suffer from non-differentiability and hence need further work to produce computational algorithms. They can also be extended to cover minimization with respect to

(1) K_p diagonal (i.e. the simplest case of input scaling only)

(2) K_p structured to reflect the loop connections possible.

(3) K_p free

The could also be extended to design post <u>and</u> precompensiators

Another tentative approach is to solve the problems

$$\min_{K_p} \quad N^p_\infty\,(E^p) \tag{13}$$

or

$$\min \left[N^p_\infty\,(E^p) - \text{diag}\,\{N_\infty(E^p_{jj})\}_{1 \leq j \leq m}\right] \quad , \quad \| K_p \| = 1 \tag{14}$$

where the minimization is with respect to the partial ordering on $L(R^m)$ defined by $A \geq B$ iff $A_{ij} \geq B_{ij}$, $1 \leq j \leq m$. It is not obvious however that this problem is well-posed.

2. Existence of Solutions and Duality

The basic structure of the problem is characterized by:

Theorem 1: Problems (2) -(6) and (13)-(14) are (possibly constrained) convex optimization problems over a finite dimensional subspace $\tilde{M}$ of products of the space $BV[o,\infty)$ of functions of bonded variation on $[o,+\infty)$. The dimension of the subspace is equal to m^2 if $|Y(t)| \not\equiv 0$.

Proof: Problems (2), (3), (4), (5), (9) and (10) are minimum norm problems in products of $BV[o,\infty)$ as N_∞ is the norm in $BV[o,\infty)$. They take the form

$$\min \{ \; \|Z - Y_A\|_{BV} \; ; \; Z = YK_p \; , \; K_p \in L(R^m) \} \tag{15}$$

together with any other constraints on K_p. These problems are convex provided any extra constraints on K_p are convex. Problems (13) and (14) are similarly convex as, for example,

$$\begin{aligned} N_\infty^p(\lambda Z_1 + (1-\lambda)Z_2 - Y_A) &= N_\infty^p(\lambda(Z_1 - Y_A) + (1-\lambda)(Z_2 - Y_A)) \\ &\leq \lambda \, N_\infty^p(Z_1 - Y_A) + (1-\lambda) \, N_\infty^p(Z_2 - Y_A) \end{aligned} \tag{16}$$

whereever $o \leq \lambda \leq 1$.

Finally

$$\tilde{M} = \{ \; Z \; ; \; Z = YK_p \; \}$$

which is a subspace of products of $BV[o,\infty)$. Writing K_p as

$$K_p = \sum_{i,j} \alpha_{ij} \, e_i e_j^T \tag{17}$$

$$Y(t) = [Y_1(t), \ldots Y_m(t)] \tag{18}$$

then

$$Z = \sum_{i,j} Y_i(t) \, e_j^T \, \alpha_{ij} \tag{19}$$

and $Z = 0$ iff $\alpha_{ij} = 0$, $1 \leq i,j \leq m$, from the nonsingularity of $Y(t)$. $\tilde{M}$ therefore has dimension m^2.

The existence of a solution K_p to many of the optimization problems can be proved by using a dual-space framework. Note that $BV[o,T]$ is the dual of $C[o,T]$ in this context and, in the following development, replace (for technical simplicity) the interval $[o,+\infty)$ by $[o,T]$. This reflects the practical situation where only a finite length of data is available. The following result is pinched from Luenberger p121:

<u>Theorem 2</u>: Let M be a subspace of a real normal space X. Let $x^* \in X^*$ be distance d from $M^\perp$. Then

$$d = \min_{m^* \in M^\perp} \|x^* - m^*\| = \sup_{\substack{x \in M \\ \|x\| \leq 1}} \langle x, x^* \rangle \tag{20}$$

where the mimimum on the left is achieved by $m_o^* \in M^\perp$. If the *supremum* on the right is achieved for $x_o \in M$, then $x^* - m_o^*$ is aligned with x_o.

(Notes: (1) $\langle x,x^* \rangle$ denotes $x^*(x)$.

(2) $M^\perp = \{x^* \in X^* \; ; \; \langle x,x^* \rangle = 0 \}$

(3) x is aligned with x* if $\langle x,x^* \rangle = \|x\| \cdot \|x^*\|$.) .

For application to our problems, let $X^* = (BV[o,T])^{m^2}$ and $X = (C[o,T])^{m^2}$. If the elements are arranged in matrix form

$$x^* = [x^*_{ij}] \quad , \quad x = [x_{ij}] \tag{21}$$

denote

$$\langle x,x^* \rangle = \sum_{i,j} \langle x_{ij}, x^*_{ij} \rangle \tag{22}$$

The dual functional $\langle \cdot, \cdot \rangle$ can be represented as

$$\langle x_{ij}, x^*_{ij} \rangle = \int_o^T x_{ij}(t) \, dx^*_{ij}(t) \tag{23}$$

from the Riesz Representation Theorem on p113 of Luenberger.

<u>The norms chosen in X* and X will depend on the problem!</u>

The existence of solutions to our problems can be verified by setting the problem in the form of theorem 2 where Kp is unconstrained. If K_p is also subjected to boundedness constraints this is unnecessary as the existence of a solution follows from compactness arguments. A potential further use of the results is the possibilities that m_o^* can be found by solving for x* and 'deducing' m_o^* from the alignment condition. The 'primal' problem of finding x* could be simpler than trying to find m_o^* directly!

Our first job is to find M by identifying x* with Y_A , m with YK_p and $M^{\perp}$ with $\check{M}$. Clearly $\check{M}$ is closed as it is finite dimensional. We contend that

$$M = \{z \in X \; ; \; \langle z, z^* \rangle = 0 \quad , \quad z^* \in \check{M} \} \tag{24}$$

which is closed as it is the interaction of a finite number of closed hyperplanes in X. The finite dimensity of $\check{M}$ implies that $\check{M} = M^{\perp}$. Now write $\langle z,z^* \rangle = 0$ with $z^* = YK_p$ as

$$\sum_{i,j,k} \langle z_{ij} , Y_{ik} \rangle \, \alpha_{kj} = 0 \tag{25}$$

As this must hold for all K_p, we obtain the characterization

$$M = \{ z \in X \; ; \; \sum_i \langle z_{ij} , Y_{ik} \rangle = 0 \quad , \quad 1 \leq j,k \leq m \} \tag{26}$$

No further reduction seems to be possible!

Case 1: Consider problem (2) written as the optimization problem in X*

$$\min_{m^* \in M^{\perp}} \| Y_A - m^* \| \tag{27}$$

where the norm in X* is taken as

$$\| m^* \| \triangleq \max_i \sum_{j=1}^{m} \| m^*_{ij} \| \tag{28}$$

which is induced by the norm in X

$$\|x\| \triangleq \sum_i \max_j \|x_{ij}\| \tag{29}$$

as is easily proved by writing

$$|\langle x,m^*\rangle| = |\sum_{i,j} \langle x_{ij},m^*_{ij}\rangle|$$

$$\leq \sum_{i,j} \|x_{ij}\| \; \|m_{ij}^*\| \tag{30}$$

That is

$$|\langle x,m^*\rangle| \leq \sum_i \max_j \|x_{ij}\| \sum_k \|m_{ik}^*\|$$

$$\leq (\max_i \sum_j \|m_{ij}^*\|) \sum_i \max_j \|x_{ij}\| \tag{31}$$

equality being almost achieved as follows: let k be the index in (28) such that

$$\|m^*\| = \sum_{j=1}^{m} \|m_{kj}^*\| \tag{32}$$

and let $x_{ij} = 0$, $i \neq k$, with $\|x_{kj}\| = 1$, $1 \leq j \leq m$, such that, with ε arbitrary,

$$0 \leq (1-\varepsilon) \|m_{kj}^*\| \leq \langle x_{kj},m_{kj}^*\rangle \leq \|m_{kj}^*\| \tag{33}$$

Then (30) and (31) yield

$$(1-\varepsilon) \sum_j \|m_{kj}^*\| \leq |\langle x,m^*\rangle| \leq \sum_j \|m_{kj}^*\|$$

or

$$(1-\varepsilon) \|m^*\| \leq |\langle x,m^*\rangle| \leq \|m^*\| \tag{34}$$

as required.

The structure of theorem 2 is therefore satisfied and we conclude that a solution K_p^* exists to problem (2). The corresponding primal problem is given by the RHS of (20) ie

$$\sup \sum_{i,j} \langle x_{ij}, (Y_A)_{ij} \rangle \tag{35}$$

subject to the constraints (c.f.(26))

$$\sum_{i} -\langle x_{ij}, Y_{ij} \rangle = 0 \quad , \qquad 1 \le j,\ k \le m \tag{36}$$

and

$$\sum_{i} \max_{j} \|x_{ij}\| \le 1 \tag{37}$$

Equivalently, we can write

$$\sup \sum_{i,j} \int_{0}^{T} x_{ij}(t)\ d(Y_A(t))_{ij} \tag{38}$$

subject to

$$\sum_{i} \int_{0}^{T} x_{ij}(t) dY_{ik}(t) = 0 \quad , \quad 1 \le j,\ k \le m \tag{39}$$

and

$$\sum_{i} \max_{j} \max_{0 \le t \le T} |x_{ij}(t)| \le 1 \tag{40}$$

A further simplification is possible if $(Y_A(t))$ has impulse response $h_{ij}^{A}(t)$ when (38) reduces to

$$\sup \sum_{i,j} \int_{0}^{T} x_{ij}(t)\ h_{ij}^{A}(t) dt \tag{41}$$

This is also possible for (39) but, for process control studies the impulse response of Y_{ij} may not be known. We therefore leave it unchanged.

The primal problem is an infinite dimensional linear programming problem. If it has a solution x_o, then the alignment condition reduces to

$$\sum_{i,j} \int_0^T (x_o(t))_{ij} d(Y_A(t) - m_o^*(t))_{ij}$$

$$= (\sum_i \max_j \max_{0 \leq t \leq T} |(x_o(t))_{ij}|)(\max_i \sum_j N_\infty((Y_A(t)-m_o^*(t))_{ij})) \tag{42}$$

If this is solved for $m_o^*(t)$, then K_p^* satisfies

$$m_o^*(t) = Y(t) K_P^* \tag{43}$$

Alternatively, K_P^* could be obtained in principle by substituting (43) into (42) and 'solving' for K_P^*. In essence the optimization problem is replaced by a nonlinear algebraic problem.

It appears that both the 'dual' and 'primal' problems are rather complex. It may be best to attempt a direct numerical solution of the dual problem! Let $t_o = 0 < t_1 < \ldots < t_N = T$ be a partition of $[0,T]$ and consider the finite-dimensional approximating problem

$$\min_{K_P} \max_i \sum_j \sum_{k \geq 1} |E_{ij}^P(t_k) - E_{ij}^P(t_{k-1})| \tag{44}$$

subject to

$$E^P(t_k) = Y(t_k)K_P - Y_A(t_k) \quad , \quad 0 \leq k \leq N \tag{45}$$

<u>Case 2</u>: Consider problem (3) written as the optimization problem in X*

$$\min_{m^* \in M^\perp} ||Y_A - m^*|| \tag{46}$$

or, noting that it can be written in the form of (9) regarded as the problem

$$\min_{m^* \in M_j^{\perp}} \|Y_A^{(j)} - m^*\| \tag{47}$$

in $X_j^* \triangleq (BV[0,T])^m$ with norm

$$\|m^*\| \triangleq \sum_j \|m_j^*\| \tag{48}$$

induced by the norm

$$\|x\| \triangleq \max_i \|x_i\| \tag{49}$$

in $X_j \triangleq (C(0,T))^m$. Here

$$M_j^{\perp} = \{ m^* ; m^* = YK_j \} \tag{50}$$

in the orthogonal complement of

$$\begin{aligned} M_j &\triangleq \{x ; <x,m^*> = 0 , m^* \in M_j^{\perp} \} \\ &= \{x ; \sum_{i,k} <x_i, Y_{ij}> K_{kj} = 0, \forall K_{kj} \} \\ &= \{x ; \sum_i <x_i, Y_{ik}> = 0 , 1 \leq k \leq m \} \end{aligned} \tag{51}$$

The structure of theorem 2 is satisfied and hence a solution K_j^* exists. The primal problem is

$$\sup \sum_i <x_i, (Y_A)_{ij}> \tag{52}$$

subject to the constraints

$$\sum_i <x_i, Y_{ik}> = 0 , \quad 1 \leq k \leq m \tag{53}$$

and

$$\max_i \|x_i\| \leq 1 \tag{54}$$

interpreted as

$$\sup \sum_i \int_0^T x_i(t)\ d(Y_A(t))_{ij} \tag{55}$$

with

$$\sum_i \int_0^T x_i(t)\ dY_{ik}(t) = 0 \quad , \quad 1 \leq k \leq m \tag{56}$$

and

$$\max_{0 \leq t \leq T} |x_i(t)| \leq 1 \quad , \quad 1 \leq i \leq m \tag{57}$$

If the primal problem is solved by x_o, then the alignment condition reduces to

$$\sum_i \int_0^T (x_o(t))_i d((Y_A(t)_{ij} - (m_o{}^*(t))_i)$$

$$= (\max_i \max_{0 \leq t \leq T} |(x_o(t))_i|)(\sum_{\ell=1}^{m} N_\infty((Y_A)_{\ell j} - (m_o{}^*)_\ell) \tag{58}$$

where

$$m_o{}^*(t) = Y(t)K_j \tag{59}$$

Again both primal and dual problems are relatively complex and it may be better to use a discrete approximation to the dual problem directly.

Case 3: Consider problem (13) written as the problem

$$\min_{m^* \in M^{\perp}} \|Y_A - m\| \tag{60}$$

with M as in case 1 and the norm in $X^* = (BV[0,T])^{m^2}$

$$\|m^*\| = \sum_{i,j} \|m_{ij}^*\| \tag{61}$$

induced by the norm

$$\|x\| = \max_{i,j} \|x_{ij}\| \tag{62}$$

in $X = (C(0,T))^{m^2}$.

The structure of theorem 2 is satisfied and hence K_P^* exists!

The primal problem is

$$\sup \sum_{i,j} \langle x_{ij}, (Y_A)_{ij} \rangle$$

$$\sum_i \langle x_{ij}, Y_{ik} \rangle = 0 \quad , \qquad 1 \leq j,k \leq m$$

$$\|x_{ij}\| \leq 1 \quad , \qquad 1 \leq i,j \leq m \tag{63}$$

Again both primal and dual problems are complex unless discretized.