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Realisation Theory and the Infinite Dimensional Root Locus

by

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Abstract

The theory of realisation of linear input-output maps is applied to the study of root locus for distributed systems with bounded defining operators .

(1) Introduction

In this paper we shall study the root locus of single input single output infinite dimensional systems which have a bounded realisation . The frequency domain approach to finite dimensional single input single output systems , which was developed by Bode, Nyquist and others gave way to the state space theory which is discussed in the book of Lee and Markus (1967) . The generalisation of the classical frequency domain methods was then given a new impetus with the theory of multivariable control developed largely by Rosenbrock (1974), Postlethwaite and MacFarlane (1979) , and Owens(1978) . On the other hand the control theory of distributed systems has been studied mainly from the state space viewpoint as discussed by Balakrishnan (1976) and Curtain and Pritchard (1978); the frequency domain methods have been neglected in infinite dimensional systems theory , apart from the realisation theory of Baras and Brockett (1975) and Fuhrmann and Brockett (1976) and work on systems with discrete spectrum (the heat equation , for example) by Pohjolainen(1981,1982) .

In a recent paper , Banks and Abbasi-Ghelmansarai (1983) have studied the root locus of an approximation to a simple delay equation involving the left shift operator . In the present work we shall attempt to give a classification of the root loci for infinite dimensional systems with a bounded realisation , and we shall see some interesting phenomena related to the cuts in the plane which must be introduced to ensure the single-valuedness of the transfer functions . This will lead to a definition of 'generalised pole' which is , essentially , a connected component of the set of singularities of the transfer function , including the cuts . If a similar definition is given for 'generalised zeros' then we can recover the classical criterion that if an open loop system has n generalised poles and m generalised zeros , then the root locus has n connected branches (including cuts) , m of which are attracted by the generalised zeros and n-m of which diverge to ∞ . Hence the correct objects to regard as poles are <u>not</u> the branch points , but the branch points which are 'glued together' by the connected cuts .

We shall start by introducing some notation and then recall the basic realisation theory of distributed systems ; the latter is then related to the classical 'canonical ' realisation of a finite dimensional system . The root locus for a system with a bounded realisation is then studied and we end with some simple examples .

(2) Notation and Terminology

In this paper we shall use the letters H,V and $\mathcal{H}$ to denote certain Hilbert spaces and , in particular , we shall use the right-shift operator U_r with the matrix representation

$$U_r = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & \underline{0} & \\ \underline{0} & & \cdot & \vdots & \\ & & & \cdot & \cdot \end{bmatrix}$$

in the standard basis of ℓ^2 (the Hilbert space of square summable sequences). If G is a function from $\mathbb{R}^+$ to $\mathbb{R}$, then $\tilde{G}$ or $\mathcal{L}G$ will denote the Laplace transform of G . We shall also use some elementary complex function theory (the complex plane being denoted , as usual , by $\mathbb{C}$). In particular we shall find it convenient to discuss the theory of analytic functions on the Riemann sphere S_∞ , which is just the space $\mathbb{C}\cup\{\infty\}$ with the one point compactification topology . If V is a real Hilbert space ,

$V \oplus \mathbb{C}$ will denote the complexification of V and $\mathcal{L}(H_1, H_2)$ will denote the space of bounded operators from the Hilbert space H_1 to the Hilbert space H_2 .

Finally , we shall use the spectral theory of bounded operators , which can be found in Dunford and Schwartz (1959) . To summarise this theory , note that if A is a bounded operator on a Banach space X then $\sigma(A)$ (the spectrum of A) is compact and any set σ which is open and closed in $\sigma(A)$ is called a spectral set . Assume that

$$\sigma(A) = \bigcup_{i=1}^{n} \sigma_i \quad ,$$

i.e. $\sigma(A)$ is the union of a finite number of spectral sets σ_i.

Then we define the operator

$$P_i = \frac{1}{2\pi i} \int_\Gamma e(\lambda)(\lambda I - A)^{-1} d\lambda$$

where Γ is a Jordan curve containing σ in its interior , and

$$e(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \varepsilon \sigma \\ 0 & \text{if} \quad \lambda \varepsilon \sigma(A) \setminus \sigma \end{cases}$$

It is easy to see that P_i is a projection operator . Let

$$X_{\sigma_i} = P_i X \qquad \text{and} \qquad A_{\sigma_i} = A\big|X_{\sigma_i} \quad .$$

Then

$$X = \bigoplus_{i=1}^{n} X_{\sigma_i} \quad , \quad AX_{\sigma_i} \subseteq X_{\sigma_i} \; ;$$

and

$$\sigma(A_{\sigma_i}) = \sigma_i \quad .$$

Hence A has a diagonal representation $A = \text{diag}\left[A_{\sigma_1}, \ldots\ldots, A_{\sigma_n}\right]$ in X .

(3) Realisation Theory

In this section we shall recall the realisation theory of input output maps due to Baras and Brockett (1975) and Fuhrmann (1974), and relate the bounded infinite dimensional realisations to limits of finite dimensional (i.e. rational) transfer functions. Consider, then, the scalar input-output map $G:[0,\infty) \to \mathbb{R}$. The (bounded) realisation problem is to find a bounded operator A on some Hilbert space H and vectors $b, c \in H$ such that

$$G(t) = \langle c, e^{At} b\rangle_H \quad .$$

It is well known that this problem has a solution if and only if G (respectively $\tilde{G} \triangleq \mathcal{L}G$) is entire and of exponential order (or G is analytic at infinity and vanishes there). In fact, G can be realised on ℓ^2 in the following way :

If $\tilde{G}(s) = \Sigma_{i=0}^{\infty} a_i s^{-(i+1)}$ for $|s|>\rho$ (say), then we can take

$$A = k\begin{bmatrix} 0 & & & & \\ 1 & 0 & \underline{0} & & \\ & 1 & 0 & & \\ \underline{0} & & \cdots & & \\ & & & \cdots & \\ & & & & \cdots \end{bmatrix} = kU_r \qquad (3.1)$$

$$b = \{1,0,0,\ldots\} \quad ,$$

$$c = \{a_0, a_1/k, a_2/k^2, \ldots\}$$

for any $k>\rho$. Defining

$$\sigma(\tilde{G}) = \{s \in \mathbb{C}: \tilde{G} \text{ is not analytic at } s\} \quad , \qquad (3.2)$$

we have

$$\sigma(\tilde{G}) \subseteq \sigma(A) = k\{ s \in \mathbb{C}: |s| \leq 1\} \quad . \qquad (3.3)$$

and so k must be chosen large enough so that (3.3) holds .If $\sigma(\tilde{G}) \subseteq \{ s \in \mathbb{C}: |s-s_0| \leq 1 \}$, then we may realise G with (A,b,c) where

$$A = \begin{bmatrix} s_0 & & & \\ 1 & s_0 & \underline{0} & \\ & 1 & s_0 & \\ \underline{0} & & & \\ & & \ldots\ldots & \end{bmatrix} = s_0 I + U_r \tag{3.4}$$

$$b = \{1,0,0,\ldots.\}$$

$$c = \{a_0, a_1, \ldots.\}$$

where

$$\tilde{G}(s) = \sum_{i=0}^{\infty} a_i (s-s_0)^{-(i+1)} \quad , \quad |s-s_0| > 1.$$

We can easily generalise the above result to the case where $G\colon [0,\infty) \to \mathcal{L}(V,V)$ for some Hilbert space V ; we have

Theorem 3.1 The weighting function G has a bounded realisation iff $\tilde{G}(s)$ is analytic at infinity and vanishes there .($\tilde{G}$ is an analytic function with values in $\mathcal{L}(V\oplus\mathbb{C}, V\oplus\mathbb{C})$.)

Proof The necessity is obvious . For sufficiency , write

$$\tilde{G}(s) = \sum_{i=0}^{\infty} \mathcal{A}_i s^{-(i+1)} \quad , \quad |s| > \rho$$

Then we define $\mathcal{H} = \oplus_{i=0}^{\infty} V$ and the operator A on $\mathcal{H}$ by

$$A = k \begin{bmatrix} 0_{\mathcal{H}} & & & \\ I_{\mathcal{H}} & 0_{\mathcal{H}} & \underline{0}_{\mathcal{H}} & \\ & I_{\mathcal{H}} & 0_{\mathcal{H}} & \\ \underline{0}_{\mathcal{H}} & & \ldots\ldots & \\ & & \ldots\ldots & \end{bmatrix} \quad , \quad k > \rho \tag{3.5}$$

$$B = \{I_{\mathcal{H}}, 0_{\mathcal{H}}, \ldots\ldots\}$$

$$C = \{\mathcal{A}_0, \mathcal{A}_1/k, \mathcal{A}_2/k^2, \ldots.\} \quad ,$$

where $0_{\mathcal{H}}$, $I_{\mathcal{H}}$ are respectively the zero and identity operators on $\mathcal{H}$. We then have

$$G(t) = Ce^{At}B \ .$$

A is again a unilateral shift (of multiplicity τ , where τ is the dimension of V). In the case where $V = \mathbb{R}^n$ (n inputs,n outputs) we have

$$A = k\begin{bmatrix} 0_n & & \underline{0} \\ I_n & 0_n & \\ \underline{0} & \cdots\cdots & \\ & & \cdots\cdots \end{bmatrix} .$$

Returning to the single input-single output case recall that if the system has finite dimensional realisation , then the classical 'canonical' realisation is of the form $c^T e^{At} b$, where

$$A = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \underline{0} & \\ & & \cdots & & & \\ & & & \cdots & 0 & 1 \\ b_0 & -b_1 & \cdots\cdots & & b_{n-2} & b_{n-1} \end{bmatrix} \tag{3.6}$$

$$b = \{0,0,\ldots.,1\}$$

$$c = \{a_0,a_1,\ldots,a_m,0,\ldots,0\} .$$

It will be important for us to relate this representation to that given by (3.1) . Suppose that the transfer function of a finite dimensional system is given by

$$\tilde{G}(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n-1} b_i s^i + s^n} , \quad m < n \tag{3.7}$$

$$= \sum_{i=0}^{\infty} c_i s^{(m-n)-i} ,$$

where we find inductively ,

$$c_0 = a_m$$

$$c_k = a_{m-k} - \left(\sum_{\substack{0 \le i < n \\ n-k-i \le 0}} b_i c_{-n+k+i} \right) , \quad k \geq 1 \tag{3.8}$$

Hence , by (3.1) , we have the realisation defined by

$$A = kU_r$$

$$b = \{1,0,0,\ldots.\} \tag{3.9}$$

$$c = \{\underbrace{0,0,\ldots.,0}_{n-m-1},c_0/k^{n-m-1}, c_1/k^{n-m},\ldots\}$$

where $k > \max_{s\in\sigma(\tilde{G})} |s|$. Of course , in this case , $\sigma(\tilde{G})$ consists of the poles of $\tilde{G}$. The important point to notice , however , is that whereas the representation (3.6) is s-minimal (i.e. $\sigma(A) = \sigma(\tilde{G})$) that given by (3.9) is not . Nevertheless , it is precisely the form (3.9) which provides the connection with infinite dimensional realisation theory . This connection is brought out more clearly if we use the following well-known theorem of complex function theory (cf. for example , Rudin 1966):

Theorem 3.2 (Runge's Theorem) Let Ω be open in $\mathbb{C}$, and let A be a set which has one point in each component of $S_\infty - \Omega$, and assume that f is analytic in Ω . Then there exists a sequence $\{R_n(s)\}$ of rational functions , with poles in A , such that $R_n \to f$ uniformly on compact subsets of Ω . □

In this theorem , $S_\infty = \mathbb{C} \cup \{\infty\}$ refers to the Riemann sphere (i.e. the compactified complex plane) . Before applying the theorem , we rewrite (3.9) in a different form assuming this time that $\tilde{G}$ is given in terms of poles and zeros ;i.e.

$$\tilde{G}(s) = \frac{\prod_{i=1}^{m} (s-z_i)}{\prod_{i=1}^{n} (s-p_i)} \quad . \tag{3.10}$$

In this case ,

and

$$\begin{aligned} a_m &= b_n = 1 \\ a_i &= (-1)^{m-i}\sigma_{m-i}(z_1,\ldots,z_m) \\ b_i &= (-1)^{n-i}\sigma_{n-i}(p_1,\ldots,p_n) \end{aligned} \tag{3.11}$$

where σ_i is the i^{th} elementary symmetric function . Hence , the realisation is again given by (3.9) with

$$c_0 = 1$$

$$c_k = (-1)^{m-k}\sigma_{m-k}(z_1,\ldots,z_m) - \sum_{\substack{0\leq i<n \\ n-k-i\leq 0}} \left\{(-1)^{n-i}\sigma_{n-i}(p_1,\ldots,p_n)c_{-n+k+i}\right\} \tag{3.12}$$

where $k > \max_{1 \leq i \leq n} \{|p_i|\}$.

Consider again a single input single output system with transfer function $\tilde{G}(s)$ and suppose this has a bounded realisation. Then $\sigma(\tilde{G}(s))$ contains poles and branch cuts , the latter necessarily being in the finite plane since $\tilde{G}(s)$ is analytic at $s = \infty$. $\sigma(\tilde{G})$ is closed and so we may apply Runge's theorem with $A = \sigma(\tilde{G}) \cup \{\infty\}$. However , since $|\tilde{G}(s)| \to 0$ as $|s| \to \infty$, we may obviously assume that the rational approximations just have poles in $\sigma(\tilde{G})$. Therefore , taking $A = \sigma(\tilde{G})$, theorem 3.2 implies that there exists a sequence of rational functions R_n such that $R_n \to \tilde{G}(s)$ uniformly on compact sets . Suppose that

$$R_n = \frac{\prod_{i=1}^{m_n} (s-z_i^n)}{\prod_{i=1}^{n} (s-p_i^n)} ,$$

where we have assumed that R_n has n poles (it is easy to see that there is no loss of generality in this assumption) which are at the points p_i^n $(1 \leq i \leq n)$ and m_n $(< n)$ zeros at z_i^n $(1 \leq i \leq m_n)$. Then we see that the function $\tilde{G}(s)$ has a realisation of the form (3.1) which can be obtained as a limit of a sequence of realisations of the form (3.9) , where

$$c_0^n = 1$$

$$c_k^n = (-1)^{m_n - k} \sigma_{m_n - k}(z_1^n, \ldots, z_{m_n}^n) - \sum_{\substack{0 \leq i < n \\ n-k-i \leq 0}} \left\{ (-1)^{n-i} \sigma_{n-i}(p_1^n, \ldots, p_n^n) c^n_{-n+k+i} \right\}$$

for n = 1,2,.... (cf (3.12)).

If we consider the example given by Baras and Brockett (1975), defined by the transfer function

$$\tilde{G}(s) = \frac{1}{\sqrt{(s^2 + 1)}} ,$$

then $\sigma(\tilde{G}) = \{s \in \mathbb{C}: \mathrm{Re}\, s = 0, |\mathrm{Im}\, s| \leq 1\} \triangleq L$, i.e. $\sigma(\tilde{G})$ consists of the points $\pm i$ together with the cut along the imaginary axis joining these points . Then by the above theory , we find a sequence $\tilde{G}_n(s)$ of rational functions with n poles on L such that

$$\tilde{G}_n \to \tilde{G} \text{ , uniformly on compacta .}$$

(It should be noted that the authors cited above give an S-minimal realisation using the Laurent operator ; the same theory as that above could be given using this operator in place of U_r.)

Note finally that we have expressed the realisations above in terms of the right shift operator U_r ; we could have used instead the left shift $U_\ell = U_r^*$, since

$$\langle c, e^{U_r t} b\rangle = \langle e^{U_\ell t} c, b\rangle \quad .$$

(4) Root Locus Theory

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + kbu \\ y &= cx \end{aligned} \tag{4.1}$$

where A,b,c are bounded operators belonging , respectively , to the spaces $\mathcal{L}(H,H)$, $\mathcal{L}(R,H)$, $\mathcal{L}(H,R)$ (we shall consider the single-input single output case for simplicity). Suppose that

$$\sigma(A) = \bigcup_{i=1}^{n} \sigma_i \tag{4.2}$$

where each σ_i is a spectral set of A . Then we can write

$$H = \bigoplus_{i=1}^{n} H_{\sigma_i} \tag{4.3a}$$

$$A_{\sigma_i} = A\big|_{H_{\sigma_i}} \tag{4.3b}$$

and

$$bu = (b_1 u, \ldots., b_n u) \text{ , } cx = c_1 x_1 + \ldots. + c_n x_n \text{ ,}$$

where

$$x_i \varepsilon H_{\sigma_i} \;,\; b_i u \varepsilon H_{\sigma_i} \text{ and } c_i = c\big|_{H_{\sigma_i}} \;.$$

Hence ,

$$\dot{x}_i = A_i x_i + kb_i u$$
$$y = \Sigma c_i x_i \tag{4.4}$$

where we have abbreviated A_{σ_i} to A_i .

Taking Laplace transforms in the usual way , we obtain

$$sX_i(s) = A_i X_i(s) + kb_i U(s)$$
$$Y(s) = \Sigma c_i X_i(s) \;,$$

where

$$X_i(s) = \mathcal{L}(x_i(t)), U(s) = \mathcal{L}(u(t)) \;,\; Y(s) = \mathcal{L}(y(t)) \;.$$

If the control u is just the error from feedback of y , then for any input v to the overall system we have

$$(1+k\sum_{i=1}^{n} c_i R(s;A_i)b_i)Y(s) = k(\sum_{i=1}^{n} c_i R(s;A_i)b_i)V(s) \;, \tag{4.5}$$

provided $s \notin \sigma(A_i) = \sigma_i$, where $R(s;A_i) \triangleq (sI-A_i)^{-1}$ is the resolvent of A_i . Hence , if

$$s \,\varepsilon \{\lambda \,\varepsilon \mathbb{C}: 1+k\sum_{i=1}^{n} c_i R(\lambda;A_i)b_i = 0\}$$

then it follows that

$$Y(s) = \left\{ \frac{k(\sum_{i=1}^{n} c_i R(s;A_i)b_i)}{1 + k\sum_{i=1}^{n} c_i R(s;A_i)b_i} \right\} V(s)$$
$$= \tilde{G}(s)V(s)$$

where

$$\tilde{G}(s) \triangleq \frac{k(\sum_{i=1}^{n} c_i R(s;A_i)b_i}{1 + k\sum_{i=1}^{n} c_i R(s;A_i)b_i} \tag{4.6}$$

is called the <u>closed-loop transfer function</u> . If we wish to emphasize the dependence of $\tilde{G}$ on k we shall write $\tilde{G}(s;k)$. The <u>root locus</u> L of the system is defined by

$$L = \bigcup_{k \geq 0} \{s \in \mathbb{C}: \tilde{G}(s;k) \text{ is not analytic at } s\} \tag{4.7}$$

<u>Remark 4.1</u> In the case of finite dimensional realisations , the representation (4.1-4.3) of a system corresponds to the Jordan decomposition of A . In fact , the spectral sets of the matrix A are just the (isolated) eigenvalues and the subspaces H_{σ_i} correspond to the generalised eigenspaces . The decomposition of the open loop transfer function

$$\tilde{G}_0(s) = \sum_{i=1}^{n} c_i R(s;A_i) b_i$$

is then just the partial fraction expansion .

In order to demonstrate that the classical properties of the root locus can be generalised to the present situation , it is important to define transmission zeros for our systems . Pohjolainen (1981) has defined such zeros,in the case where A has compact resolvent (and so has isolated eigenvalues) , as the finite eigenvalues of A+kBC as $k \to \infty$. It is then easy to show that this is equivalent to the condition

$$\lambda \notin \sigma(A) \text{ and } \det[C(A-\lambda I)^{-1}B] = 0 \tag{4.8}$$

However , these systems are spectral minimal , as shown by Brockett and Fuhrmann (1976) and the definition of transmission zeros in terms of $\sigma(A+kBC)$ is then reasonable .

In the case where the original system (A,b,c) and the transfer function are not spectral minimal , it is necessary to define the transmission zeros directly by (4.8) ; ie. for single input single output systems by

$$\lambda \notin \sigma(A) \quad \text{and} \quad c(A-\lambda I)^{-1}b = 0 \tag{4.9}$$

It then follows directly from (4.6) that the finite singularities of $\tilde{G}(s;k)$ tend to the transmission zeros as $k \to \infty$. However ,even the definition (4.9) is not quite correct since we must not only include the zeros of $c(A- I)^{-1}b$ but also any 'cut' in the plane which is necessary to make $c(A- I)^{-1}b$ single valued . Hence , by comparison with (3.2) we introduce

Definition 4.2 The transmission zero set ζ for the system (A,b,c) is defined by

$$\zeta(\tilde{G}_0) = \{ s \in \mathbb{C} : s \notin \sigma(A) \ , \ \tilde{G}_0^{-1} \text{ is not analytic at } s \} \tag{4.10}$$

where $\tilde{G}_0(s) = c(A-sI)^{-1}b$, the open loop transfer function .

Just as in (3.2) , this does not uniquely specify $\sigma(\tilde{G}_0)$. However , it is to be understood in the same sense as (3.2) , i.e. $\sigma(\tilde{G}_o)$ consists of the solutions of (4.9) together with any cuts which are necessary for the single-valuedness of $\tilde{G}_0$.(Note that this ambiguity can be removed by interpreting the root locus on an appropriate Riemann surface.)

For reasons which will become clear shortly , we also introduce

Definition 4.3 A connected component of $\sigma(\tilde{G}_0)$ is called a generalised open-loop pole . Similarly , a component of $\sigma(\tilde{G}(s;k))$ (where G is defined by (4.6)) is called a generalised closed-loop pole . A connected component of $\zeta(\tilde{G}_0)$ is called a generalised open-loop zero .

The transfer function $\tilde{G}(s)$ of a finite dimensional system is , of course , rational and can be factorised in the form

$$\tilde{G}(s) = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

where each zero z_i and pole p_i represents a single connected component of $\sigma(\tilde{G})$. We would like to obtain a similar factorisation when $\tilde{G}(s)$ is irrational , of the form

$$\tilde{G}(s) = \frac{\prod_{i=1}^{m} \Psi_i(s)}{\prod_{i=1}^{n} \Phi_i(s)} \tag{4.11}$$

where Ψ_i (Φ_i) corresponds to a generalised zero (pole) . We shall prove this in the case when $\tilde{G}$ has a finite number of algebraic singular points . Let us assume for simplicity that in the spectral factorisation of (4.2)-(4.4) each operator A_{σ_i} is such that σ_i $(= \sigma(A_{\sigma_i}))$ contains a single connected component of $\sigma(\tilde{G}(s))$. Then such a spectral representation of the system (4.1) will be called <u>simple</u> .

<u>Theorem 4.4</u> Suppose that all the branch points for the system (4.1) are algebraic . Then we may write the open loop transfer function $\tilde{G}_0(s)$ in the form

$$\tilde{G}_0(s) = \left(\prod_{i=1}^{m_1} \Psi_i(s) \middle/ \prod_{i=1}^{n_1} \Phi_i(s) \right) R(s) \tag{4.12a}$$

where R is a rational function with m_2 zeros and n_2 poles , $m_1+m_2 < n_1+n_2$ and Ψ_i (Φ_i) correspond to generalised zeros (poles). Moreover , we have

$$\Psi_i(s)-s \text{ , } \frac{s}{\Phi_i(s)} - 1 \text{ are analytic at } \infty \text{ and vanish there .} \tag{4.12b}$$

<u>Remark</u> The term R(s) corresponds to those operators A_{σ_i} for which σ_i contains a single isolated pole and consequently which have a finite dimensional realisation . We shall therefore ignore the term R (ie. set R=1) since this result is well known in this case . Hence we shall prove the result with R=1, $m=m_1, n=n_1$.

<u>Proof of theorem (4.4)</u> Since we are considering only components

of $\sigma(\tilde{G}_0(s))$ which are nontrivial (ie. which are not just isolated poles) we can suppose that the singularities (which are assumed to be algebraic) occur at isolated branch points . Now at such a point b it is well known that $\tilde{G}_0(s)$ must have branches of the form

$$F(\sqrt[n]{s-b}) \tag{4.13}$$

for some n , where F is holomorphic in a neighbourhood of 0 except possibly at 0 where F can have at most a pole . (See , for example , aks and Zygmund ,1971) . We can assume without loss of generality that if 0 is a pole then it is simple . Consider , then , two cases :

Case 1 0 is a pole of F .

If $n > 1$, then in order that $F(\sqrt[n]{s-b})$ be well-defined we must cut the plane from b to ∞ . Hence , in this case , since we know that $\tilde{G}_0$ is analytic at ∞ we must be able to find another term F_1 in $\tilde{G}_0$ so that

$$FF_1 = F_2(\sqrt[n]{s^n+a_{n-1}s^{n-1}+....+a_0})$$

for some new function F_2 , holomorphic except at 0 . This term will then correspond to the generalised pole of $\tilde{G}_0(s)$ containing b. Also , if $\tilde{G}_0$ contains a factor of the form $1/(\sqrt{s^4-a^4})$, for example , then this will not represent a single component of $\sigma(\tilde{G}_0(s))$ since it can be factorised as $1/(\sqrt{s^2+a^2}\sqrt{s^2-a^2})$. Hence , if we collect together the representations (4.13) for each branch point we must have a function of the form

$$P(s) = \phi_1(s)....\phi_n(s)$$

which satisfies (4.12b). Now write $Z(s) = \tilde{G}_0(s)P(s)$. Then it is clear that Z(s) has branch points which satisfy

Case 2 0 is not a pole of F . The same argument as in case 1 now completes the proof . ∎

Corollary 4.5 Under the conditions of theorem 4.4 , if the system (4.1) has a simple spectral representation in the form (4.4) , we may write

$$\tilde{G}_0(s) = \sum_{i=1}^{n} \frac{h_i(s)}{\Phi_i(s)} \tag{4.14}$$

where each $h_i(s)$ is holomorphic in $\mathbb{C}$.

The expression (4.14) is a generalised partial fraction expansion . We can state an obvious conjecture :

Conjecture 4.6 A decomposition of the form (4.12a) exists even for transcendental singularities . □

Returning to the case of algebraic singularities , in view of corollary 4.5 and the importance of the partial fraction expansion of a rational function , it is of interest to determine when we can obtain a simple spectral representation of a given system . It turns out that if the system has a bounded realisation and only algebraic singularities , we can always do this . In order to prove this statement , we must first recall the theory of G-C Rota (1960) which says , essentially , that the left-shift operator (on a certain Hilbert space) is a 'universal model ' for any bounded operator whose spectrum is contained in the unit disc . In fact , if A is a bounded operator with $\sup|\sigma(A)| \lesssim 1$ defined on a Hilbert space H , then we define the new Hilbert space $H^\infty \triangleq \oplus_{k=1}^{\infty} H$ with the inner product

$$<\underline{x},\underline{y}> = \sum_{k=1}^{\infty} <x_k, y_k>_H \quad , \; x_k, y_k \in H \tag{4.15}$$

for all $\underline{x},\underline{y} \in H^\infty$ for which the sums $\sum_{k=1}^{\infty} x_k^2$, $\sum_{k=1}^{\infty} y_k^2$ exist .Here,

$$\underline{x} = (x_1, x_2, \ldots.) \;,\; \underline{y} = (y_1, y_2, \ldots.) \;.$$

Now define the left-shift U_ℓ on H^∞ by

$$U_\ell(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \,,$$

and the operator $I_A : H \to H^\infty$ given by

$$I_A x = (x, Ax, A^2 x, \ldots) \,.$$

Then it follows that I_A is invertible and

$$I_A^{-1} U_\ell I_A x = Ax \,,$$

and so A is similar to U_ℓ.

Now consider again the general spectral decomposition of a system as in (4.1-4.3), and suppose that a particular projected subsystem defined by A_i, b_i, c_i as in (4.4) has a transfer function $\tilde{G}_i(s) = c_i R(s; A_i) b_i$ which has several singularities (which are generalised poles) contained in $\sigma(A_i)$. (cf. fig. 4.1) . Assume without loss of generality that $\|A\| \leq 1$.

Consider the term $\Phi_1(s)$ (say) in (4.11) due to the generalised pole p_1. Then we can draw a Jordan curve γ in $\sigma(A_i)$ containing p_1 in the interior and with p_2, p_3 in the exterior of γ, as in fig . 4.2 . Since γ is a Jordan curve , the Riemann mapping theorem (Rudin , 19) states that there exists a conformal mapping f which takes γ in the s-plane into the unit circle in the f(s)-plane. However , f^{-1} is analytic and so we can define $A_{i,1} = f^{-1}(U_\ell)$, where U_ℓ is the left shift operator in H . Then $A_{i,1}$ has spectrum which consists of γ and its interior (by the spectral mapping theorem) . However , $A_{i,1}$ has the model U_ℓ on H^∞, i.e.

$$A_{i,1} = I^{-1}_{A_{i,1}} U_\ell I_{A_{i,1}}$$

and we can realise $(\Phi_1(s))^{-1}$ on H^∞ by the results of §3 in terms of the triple (U_ℓ, b, c) ,say; i.e. by the system

$$\dot{\xi} = U_\ell \xi + bv$$
$$y = c\xi$$

If we change variables to $\Xi = I_{A_{i,1}}^{-1} \xi$, we obtain

$$\dot{\Xi} = I_{A_{i,1}}^{-1} U_\ell I_{A_{i,1}} \Xi + I_{A_{i,1}}^{-1} bv$$
$$y = cI_{A_{i,1}} \Xi$$

or

$$\dot{\Xi} = A_{i,1}\Xi + b'v$$
$$y = c'\Xi \tag{4.16}$$

where

$$b' = I_{A_{i,1}}^{-1} b \quad , \quad c' = cI_{A_{i,1}} .$$

(4.16) is a system defined on H with transfer function $(\Phi_1(s))^{-1}$ whose system operator $A_{i,1}$ has spectrum γ and its interior . The stated result now follows by induction .

We now return to the general structure of the root locus for a system with bounded realisation having algebraic singularities . It follows by an elementary spectral continuity argument that generalised poles remain generalised poles as k increases . (Of course , at some values of k a generalised pole consisting of , say , a pair of singularities joined by a cut may coalesce into a single ordinary pole .)

<u>Proposition 4.7</u> Suppose that the open loop transfer function $\tilde{G}_0(s)$ of the system (4.1) is written in the form (4.12a) with $m = m_1+m_2$ and $n = n_1+n_2$, then we have

$$\tilde{G}_0(s) = k\left(\frac{a_{n-m}}{s^{n-m}} + \frac{a_{n-m+1}}{s^{n-m+1}} + \cdots\right)$$

where $a_{n-m} \neq 0$, in a neighbourhood of $s = \infty$.

<u>Proof</u> This follows easily from (4.12a) since each term $\Psi_i(s)$, $\Phi_i(s)$ has an expansion $\sum_{i=1}^{\infty} \alpha_i s^i$ with $\alpha_1 \neq 0$, by (4.12b). □

Corollary 4.8 If the feedback system (4.5) has m generalised zeros and n generalised poles, then m of the poles converge, as $k \to \infty$ to the generalised zeros and the remaining n-m generalised poles tend to ∞ with asymptotic directions given by the angles

$$\theta = \ell\pi/(n-m)$$

and which intersect the x-axis at $\sigma_0 = \left(\frac{a_{n-m+1}}{a_{n-m}}\right) \Big/ (n-m)$. □

It is clear from the above results, therefore, that an infinite dimensional system which has a bounded realisation with only algebraic singularities has a root locus in which the generalised poles and zeros behave in essentially the same way as the corresponding root locus of a finite dimensional system. In the next section we shall give some examples to clarify these remarks. However, before presenting these examples, we shall first mention briefly the generalised pole assignment problem for our systems.

Consider again a system og the form (4.1) which is canonical (i.e. controllable and observable) and suppose that the open loop transfer function $\tilde{G}_0(s)$ is of the form (4.12a) with only algebraic singularities. The generalised pole assignment problem for this system is to move the poles (generalised or ordinary) with state feedback to any other finite set of generalised poles. The next result shows that we can solve this problem arbitrarily closely.

Theorem 4.9 The generalised pole assignment problem is approximately soluble for the system (4.1) in the sense that there exists a sequence of finite dimensional systems $S_{0i} = (A_{0i}, b_{0i}, c_{0i})$ approximating $\tilde{G}_0(s)$, with transfer functions $\tilde{G}_{0i}(s)$, $1 \leq i < \infty$ such that, if $\tilde{G}_d(s)$ is the desired closed loop transfer function of (4.1), then we can assign the poles of S_{0i} with state feedback so that

$$\overline{\bigcup_{i=1}^{\infty} \sigma(\tilde{G}_i)} = \sigma(\tilde{G}_d) \quad .$$

i.e. the union of all the poles of the systems S_i is dense in $\sigma(\tilde{G}_d)$ where S_i is the closed loop system of S_{0i} and $\tilde{G}_i$ is its transfer function .

<u>Proof</u> The proof of this result follows easily from the above remarks and Runge's theorem . Indeed , using Runge's theorem we can find a sequence of transfer functions $\tilde{G}_{0i}(s)$ of finite dimensional systems such that $\tilde{G}_{0i}(s) \to \tilde{G}_0(s)$ uniformly on compact subsets of $\mathbb{C}\setminus\sigma(\tilde{G}_0)$. Similarly , we can find a sequence $\tilde{G}_i(s)$ such that $\tilde{G}_i(s) \to \tilde{G}_d(s)$ uniformly on compact subsets of $\mathbb{C}\setminus\sigma(\tilde{G}_d)$. Without loss of generality we may assume that $\tilde{G}_{0i}$ and $\tilde{G}_i$ each have i poles (counted with multiplicity) . However , the classical result now implies that the poles of S_{0i} can be arbitrarily assigned . If we assign them precisely to $\sigma(\tilde{G}_i)$ then the result follows .□

(5) <u>Examples</u>

<u>Example 5.1</u> Consider the transfer function

$$\tilde{G}_0(s) = \frac{s+1}{\sqrt{(s^2+1)}} \quad ,$$

with one zero at s=-1 and a generalised pole consisting of the branch points $s=\pm i$ joined by a cut . (This transfer function does not have a bounded (A,b,c) realisation , since $\tilde{G}_0(s) \to 1$ as $|s| \to \infty$. However , it is the simplest nontrivial case and will show very well some of the peculiarities of irrational transfer functions .)

The root locus is given by

$$1+k\frac{s+1}{\sqrt{(s^2+1)}} = 0$$

i.e.

$$s^2+1 = k^2(s+1)^2 \quad .$$

(Note that in squaring we have effectively lost the cut between $+i$ and $-i$, but we must bear in mind that this cut is still present .) Hence ,

$$s^2 - \frac{2k^2}{1-k^2} s + 1 = 0$$

and so the root locus consists of the loci of the poles

$$s = \frac{k^2 \pm \sqrt{(2k^2-1)}}{1 - k^2} \quad .$$

Of course , when $k=0$ the locus starts at $s=\pm i$ and has complex roots until $k=1/\sqrt{2}$, when the roots become real at the point $s=1$. It is easy to see that for $k\varepsilon[0,1/\sqrt{2}]$ we obtain the locus of a semicircle . Now consider $k> 1/\sqrt{2}$. If $k\varepsilon[1/\sqrt{2},1)$ then we obtain two branches ; one tends to $s=0$ as $k\to 1$ and one tends to ∞ as $k\to 1$. When $k\varepsilon(1,\infty)$ we obtain branches which tend to $s=-1$ as $k\to\infty$. The root locus is therefore as shown in fig.5.1(a),(b),(c). The poles $s=\pm i$ are shown at various positions on the root locus as s_1,s_2 connected by a cut (shown by a dotted line). Note that the cut disappears when $k=1/\sqrt{2}$ and that when $k=1$ one pole (s_2) is at ∞ . The behaviour of the generalised pole as $k\to\infty$ with respect to the zero at $s=-1$ can be shown more easily on the Riemann sphere as in fig.5.2(a),(b) where the branch points and cuts are shown with a continuous line for various values of k . The point at infinity is marked ∞ .

<u>Example 5.2</u> Consider the system with open loop transfer function

$$\tilde{G}_0(s) = \frac{(s+i)}{s\sqrt{(s^2 - 1)}}$$

i.e. $\tilde{G}_0$ has a zero at $-i$, a pole at 0 and a generalised pole (± 1). The root locus is shown in fig.5.3.

Note that the apparent splitting of the zero at $s=0$ is caused by squaring $\tilde{G}_0(s)$ to find the root locus . Alternatively , we can regard the pole at $s=0$ as a 'degenerate' generalised pole which splits into two singularities joined by a cut for $k > 0$ and coalesces at $s=-i$ when $k = \infty$.

Example 5.3 Consider finally the system

$$\tilde{G}_0(s) = \frac{\sqrt{\{(s+1)^2 + \frac{1}{4}\}}}{\sqrt{(s^2+4)}\sqrt{(s^2+1)}}$$

which has two generalised poles $(\pm i)$, $(\pm 2i)$ and a generalised zero $(-1 \pm i/2)$. The root locus is shown in fig.5.4.

As expected , one generalised pole diverges while the other $(\pm i)$ is attracted by the generalised zero .

(6) Conclusions

In this paper we have discussed the application of realisation theory to the root locus of infinite dimensional systems . In particular we have generalised the classical theory to the case of systems with bounded realisations whose transfer functions have a finite number of isolated algebraic singularities . A complete theory for such systems has been developed in terms of generalised poles and zeros which must be regarded as the appropriate counterparts of the classical poles and zeros . Indeed , we have seen that the latter behave in many ways as degenerate generalised singularities .

We leave two problems for further research : firstly , of course , we could use the theory developed here to generalise the classical compensator theory and secondly it is important to attempt to extend the theory to unbounded system operators A . The latter problem is complicated by the fact that the spectrum of A may now accumulate at infinity ; i.e if we compactify $\mathbb{C}$ to the Riemann sphere S_∞ then any neighbourhood of ∞ contains infinitely many singularities . This suggests that we should study such systems on S_∞ and regard ∞ as just another point ; we must then extend the theory of this paper to the case of transfer functions $\tilde{G}_0$ with accumulation points in $\sigma(\tilde{G}_0)$.

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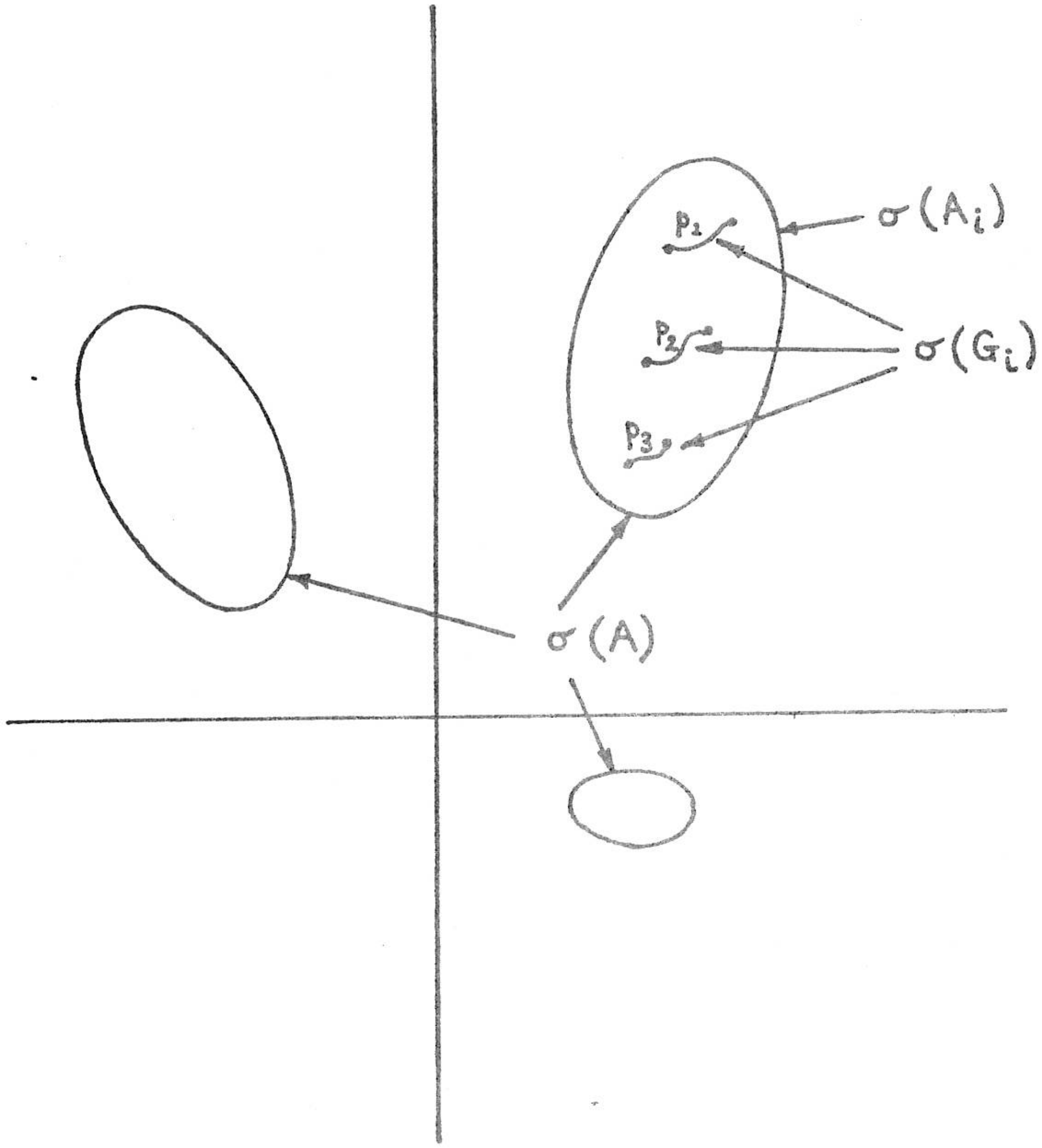


fig. 4.1.

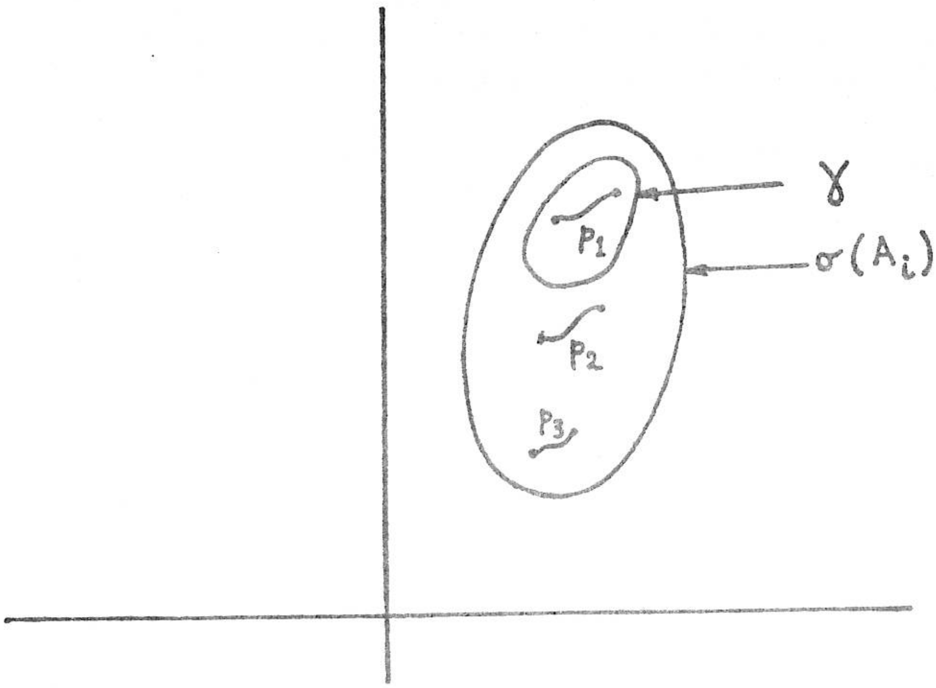
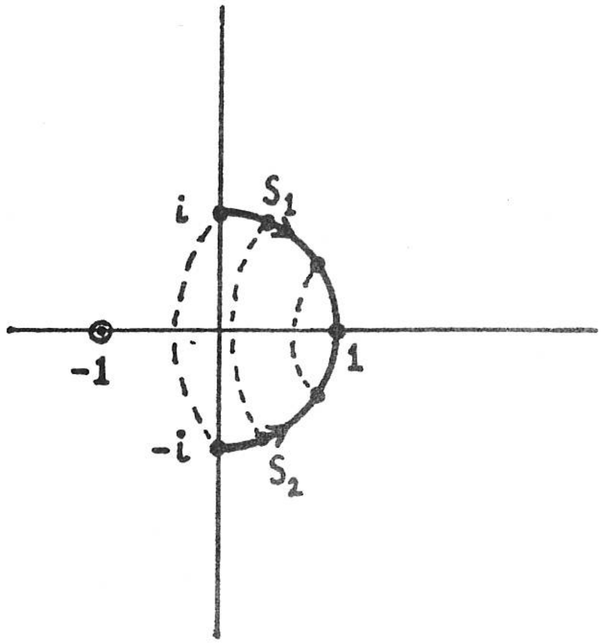
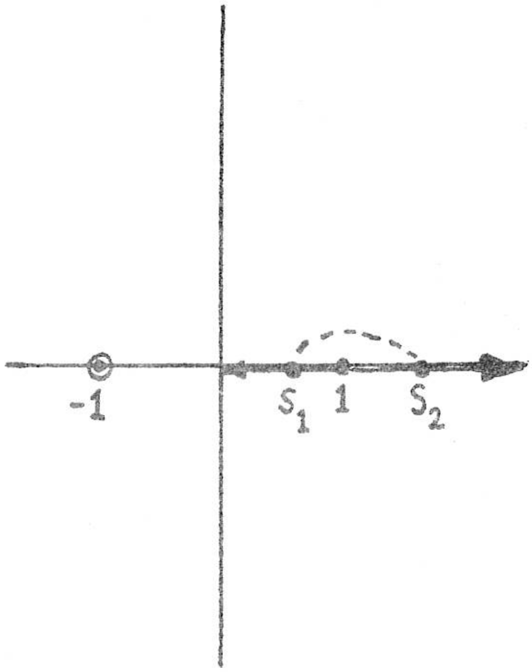


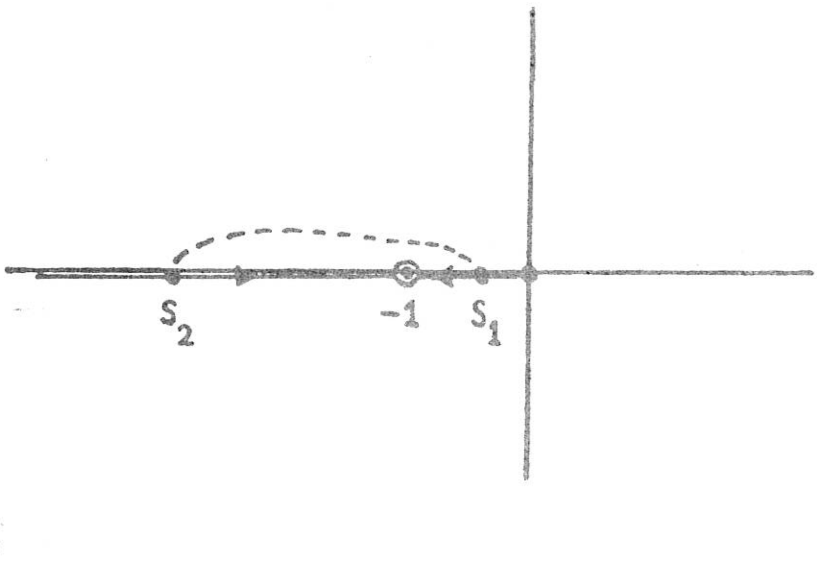
fig. 4.2.



(a). $0<k<1/\sqrt{2}$

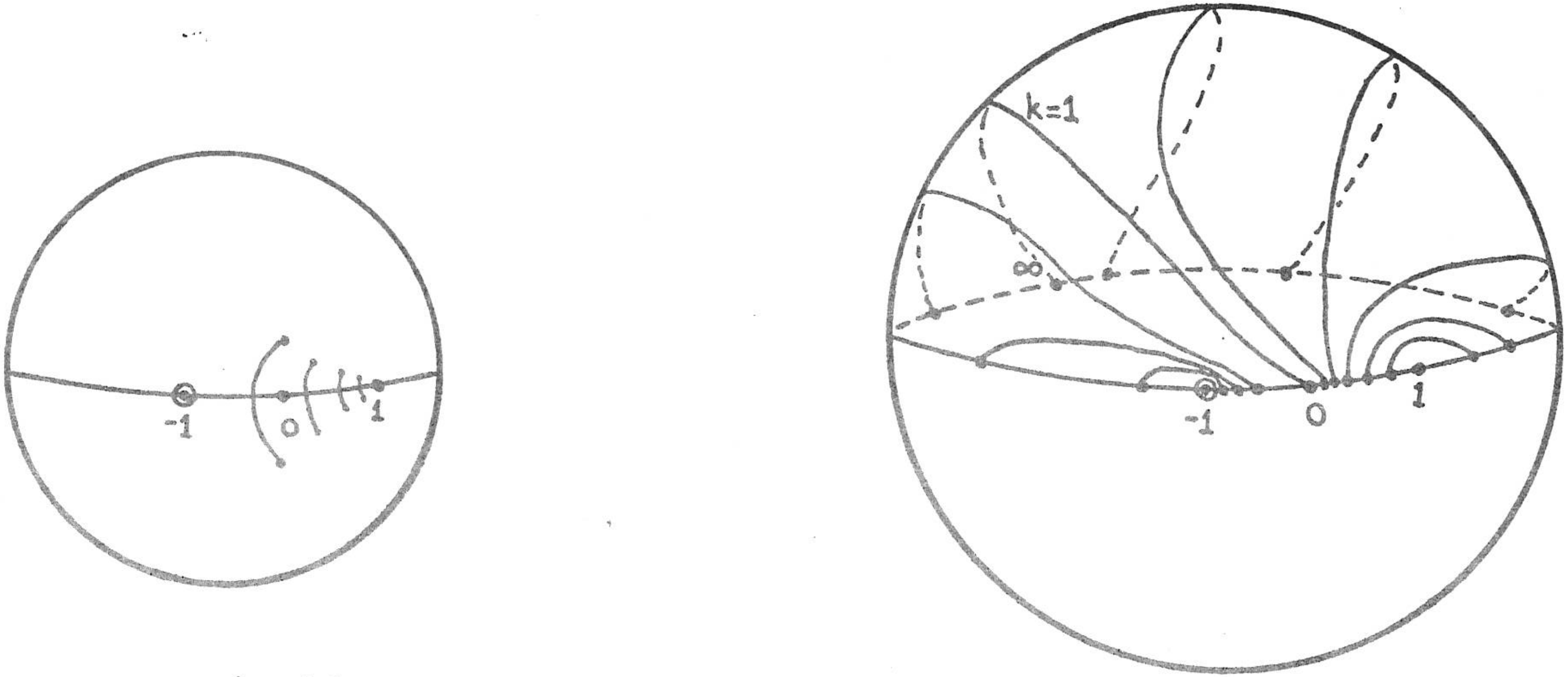


(b). $1/\sqrt{2} \leq k < 1$



(c). $k>1$

fig. 5.1



(a). $k < 1/\sqrt{2}$

(b). $k \geq 1/\sqrt{2}$

fig 5.2

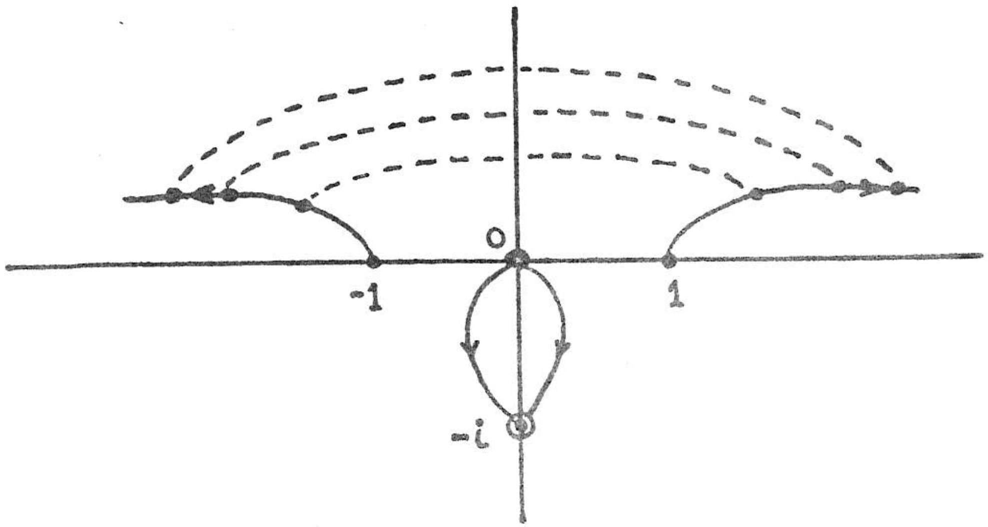


fig. 5.3

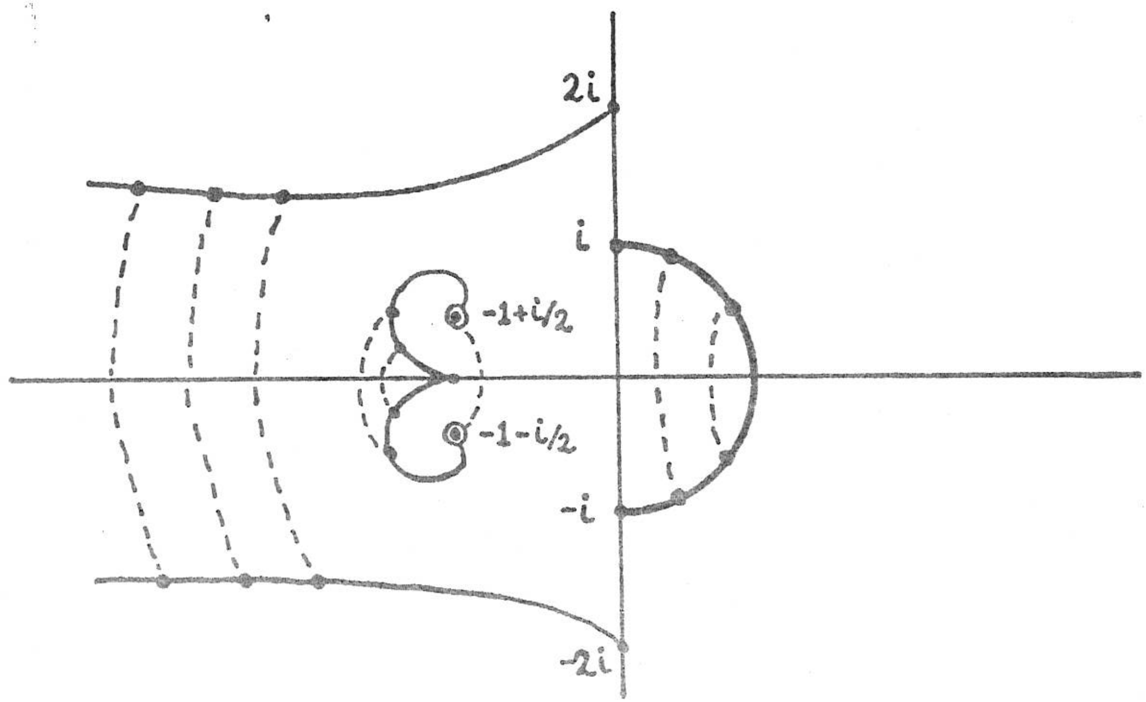


fig. 5.4

Legends for the figures

fig.4.1 A typical spectral decomposition into spectral sets

fig.4.2 Surrounding a generalised pole by a Jordan curve.

fig.5.1 Root locus for example 5.1

fig.5.2 Root locus for example 5.1 shown on S_∞ .

fig.5.3 Root locus for example 5.2.

fig.5.4 Root locus for example 5.3.