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Banks, S.P and Abbasi-Ghelmansarai, F (1982) Delay Equations, the Left-Shift Operator and the Infinite Dimensional Root Locus. Research Report. ACSE Report 192 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

Delay Equations, the Left-Shift Operator and the Infinite-Dimensional Root Locus

by

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Research Report No. 192

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Abstract

The distributed parameter root locus is considered and a relation between simple delay equations and the left-shift operator is developed. This gives a rigorous explanation of the s-plane behaviour of delay systems and shows that the classical root locus starting on the open-loop poles of the system can become bands swept out by connected components of the spectrum of the system operator in the infinite-dimensional case.

1. Introduction

The classical theory of control in the frequency domain and in the state formulation is now well known and widely applied. The corresponding theory for infinite dimensional systems has been extended to cover most aspects of the state space approach (see for example [Curtain & Pritchard, 1978]). However, the frequency domain methods for distributed systems have not received so much attention; the main area of study has been in the field of stability theory, as in Falb & Freedman, 1969, Banks , 1981. The root locus and classical compensation techniques have barely been touched on in the literature; see, however Pohjolainen, 1981, 1982. Of course, the Laplace transformation technique has been applied to distributed systems in the past, usually involving some form of finite-dimensional approximation, since the transfer function is not rational.

In this paper we shall discuss a simple delay equation and relate it to another equation on a certain Hilbert space, involving a bounded operator. This correspondence will simplify the spectral structure of the system and will show that the root locus can become expanded into bands of the complex plane in the infinite-dimensional case.

2. Terminology

In this paper we shall be concerned with systems of differential equations defined on the Hilbert space ℓ^2 consisting of sequences $x = (x_1, x_2, \ldots)^T$ such that

$$\|x\| = \left(\sum_{i=1}^{\infty} x_i^2 \right)^{\frac{1}{2}} < \infty$$

We recall that the space ℓ^2 has the orthonormal basis $\{e_i\}$ consisting of sequences e_i whose i^{th} element is 1 and all others zero; i.e.

$$e_i = (0,\ldots,0,\overset{i}{1},0,\ldots)^T .$$

If $I \subseteq \mathbb{R}$ is an interval we shall denote by C(I) the space of all real valued functions defined on I with the norm

$$\| f \| = \sup_{t \in I} |f(t)| .$$

The space of all sequences $(x_1, x_2, \ldots)^T$ (with no restriction) will be denoted by c_o. We shall assume that the reader is familiar with the theory of integration of vector - and operator - valued analytic functions. Using the same ideas as in the finite - dimensional case, we use the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(s) ds$$

of such a function, together with the usual inverse transform

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds ,$$

the conditions for existence of the integrals being similar to the classical case.

Finally, we mention the two main results from functional analysis which we shall need. The first is the spectral mapping theorem which states that

$$\sigma(f(T)) = f(\sigma(T)) ,$$

where

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda I - T)^{-1} d\lambda ,$$

and B is a Jordan curve containing $\sigma(T)$ and such that f is analytic on a neighbourhood of B and its interior. ($\sigma(T)$ is the spectrum of the bounded operator T, defined precisely in the next section.) (cf. Yosida, 1974)

The other result we require is the fact that the system

$$\dot{x} = Ax,$$

where A is a bounded operator, is stable iff $\sigma(A)$ is contained in the left half plane. This, of course, generalizes the classical matrix result and is called the spectrum determined growth condition.

3. The Left Shift Operator

In this section we consider some basic properties of a certain bounded operator A defined on ℓ^2. This operator is called the left-shift operator and is defined by

$$Ax = (x_2, x_3, \ldots)^T$$

where $x = (x_1, x_2, x_3, \ldots)^T \in \ell^2$. A is clearly bounded, and in fact $\|A\| = 1$. Recall the definition of the spectrum of an operator B defined on a Banach space X.

Definition 3.1. The complex number λ is said to belong to the spectrum $\sigma(B)$ of the operator B if $(\lambda I - B)$ is not boundedly invertible on X. The spectrum is usually divided into three disjoint sets:

(i) if $(\lambda I - B)$ is not 1-1 then $\lambda \in \sigma_p(B)$ and we say that λ is in the point spectrum of B.

(ii) if $(\lambda I - B)^{-1}$ exists, is unbounded and $\overline{\mathcal{D}(\lambda T - B)^{-1}} = X$, then $\lambda \in \sigma_c(B)$, the continuous spectrum.

(iii) if $\overline{(\lambda I - B)^{-1}} \neq X$ then $\lambda \in \sigma_R(B)$, the residual spectrum.

The spectrum of the left-shift operator A is the closed unit disc $D = \{\lambda : |\lambda| \leq 1\}$. It is instructive to prove this result; in fact, since $\|A\| = 1$ we certainly have $\sigma(A) \subseteq D$. Now, if $|\lambda| < 1$ and $x \in \ell^2$ satisfies

$$(\lambda I - A)x = 0 \tag{3.1}$$

then

$$\lambda x_i = x_{i+1}, \quad i \geq 1$$

and so the sequence $(x_1, \lambda x_1, \lambda^2 x_1 \ldots.)^T$ belongs to ℓ^2 for any x_1. Hence any $\lambda \in D^o$ is an eigenvalue since (3.1) has a nontrivial solution. Suppose that $\lambda \in \partial D$ i.e. $|\lambda| = 1$, and let ℓ_F^2 be the linear subspace of ℓ^2 consisting of finitely-nonzero sequences. Of course, (3.1) has no solution in ℓ^2 if $|\lambda| = 1$ and so such a λ is not in $\sigma_p(A)$. However, if $y = (y_1, \ldots, y_n, 0, 0, \ldots \ldots)^T \in \ell^2_F$, then the equation

$$(\lambda I - A)x = y$$

has the unique solution $x(y)$ given by

$$x_j = \sum_{i=j}^{n} y_i / \lambda^{i-j+1} \quad , \; 1 \leq j \leq n \tag{3.2}$$

$$x_j = 0 \; , \; j > n \; .$$

Clearly $x(y) \in \ell^2$ (in fact to ℓ^2_F) and since $\overline{\ell^2_F} = \ell^2$, it follows that

$$\partial D = \sigma_c(A).$$

Hence $$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) = D.$$

If $\lambda \notin \sigma(B)$, then we say that λ belongs to the resolvent of the operator B and we write $R(\lambda; B) = (\lambda I - B)^{-1}$. It follows from (3.2) that the resolvent of the left-shift operator A is given (in terms of the standard basis of ℓ^2) by the matrix function

$$R(\lambda; A) = \begin{pmatrix} 1/\lambda & 1/\lambda^2 & 1/\lambda^3 & \ldots \\ 0 & 1/\lambda & 1/\lambda^2 & \ldots \\ 0 & 0 & 1/\lambda & \ldots \\ \underline{0} & & & \ddots \end{pmatrix} . \tag{3.3}$$

4. A Simple Delay Equation

We shall now give an interpretation of the simple controlled delay equation

$$\begin{cases} \dot{x}(t) = x(t-\delta) + u(t), \; t \geq 0 \\ x(t) = f(t), \; t \in [-\delta, 0) \; , \; x(o) = x_o \end{cases} \tag{4.1}$$

for some given data f and x_o, where, for technical reasons which will become clear shortly, we shall assume that f(t) is (real) analytic on a neighbourhood of $[-\delta,0)$. We could use the theory of M^2 spaces (Delfour and Mitter, 1972) but we shall find it more convenient here to consider the connection between (4.1) and a certain equation defined on ℓ^2. Consider first the unforced equation

$$\begin{cases} \dot{x}(t) = x(t-\delta) \\ x(t) = f(t) \ , \ t \in [-\delta,0) \ , \ x(0) = x_o \end{cases} \tag{4.2}$$

and, if x(t) is a solution of (4.2) on $[-\delta,\infty)$, define an extension of x(t) on $(-\infty,\infty)$ (also denoted by x(t)) by

$$x(t) = f^{(i)}(t) \ , \quad i \geq 0 \ , \quad t \in [-(i+1)\delta, -i\delta).$$

Of course, x(t) so defined is not necessarily even continous at $t = -i\delta$, $i \geq 0$ and so (4.2) cannot be satisfied for all $t \in (-\infty,\infty)$ by x(t). However, it is clear that x(t) is differentiable on $[-(i+1)\delta, -i\delta)$ and satisfies (4.2) almost everywhere. Now, in order to specify the ℓ^2 interpretation, define

$$\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x(t-\delta) \\ x_3(t) &= x(t-2\delta) \quad , \quad t \in (-\infty,\infty) \\ &\cdots\cdots\cdot \\ x_i(t) &= x(t-(i-1)\delta) \end{aligned} \tag{4.3}$$

Then we have, on each interval $[i\delta,(i+1)\delta)$, $-\infty<i<\infty$

$$\begin{aligned} \dot{x}_1(t) &= \dot{x}(t) = x(t-\delta) = x_2(t) \\ \dot{x}_2(t) &= \dot{x}(t-\delta) = x(t-2\delta) = x_3(t) \\ &\cdots\cdots \\ \dot{x}_i(t) &= x_{i+1}(t) \ , \end{aligned}$$

and so, if $\underline{x} = (x_1,x_2,\ldots)^T$, we have

$$\dot{\underline{x}} = A\underline{x} \tag{4.4}$$

where

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ 0 & & 0 & 1 & \\ & & & \ddots & \ddots \end{pmatrix}$$

is the left shift operator. For the initial condition of (4.4) we have

$$\begin{aligned} x_1(0) &= x(0) = x_o \\ x_2(0) &= x(-\delta) = f(-\delta) \\ x_3(0) &= x(-2\delta) = \dot{f}(-\delta) \qquad (4.5) \\ &\cdots\cdots \\ x_i(0) &= x(-(i-1)\delta) = f^{(i-2)}(-\delta) \end{aligned}$$

For simplicity, let us assume that f has an analytic continuation (again denoted by f) to $(-\infty,\infty)$, and put

$$\mathcal{A} = \{f \in C(-\infty,\infty) : f \text{ is analytic}\}.$$

If $f \in \mathcal{A}$ write

$$f_n = f^{(n)}(-\delta) \ .$$

Then we have

$$f = \sum_{n=o}^{\infty} f_n \frac{(t+\delta)^n}{n!}$$

and by the uniqueness of the Taylor series, we can regard $\mathcal{A}$ as a sunset of c_o under the map.

$$\mathcal{T} : f \rightarrow \{f_n\}_{n \geq o} \ .$$

Recalling that A^* is the right shift operator we define the injection $\mathcal{J} : \mathcal{A} \rightarrow c_o$ by

$$\mathcal{J} f = (x_o,0,0,\ldots)^T + A^* \mathcal{T} f.$$

Then by (4.5), $\mathcal{J}$ defines a one-to-one map between the initial condition of (4.2) and that of (4.4). Put

$$\mathcal{A}^2 = \mathcal{J}^{-1}(c_o \cap \ell^2) \ .$$

Then we have

Lemma 4.1 The systems

$$\dot{x}(t) = x(t-\delta) \;,\; x(t) = f(t) \text{ on } [-\delta,0) \;,\; x(0) = x_o, \qquad (4.6)$$

and

$$\dot{\underline{x}}(t) = A\underline{x}(t) \;,\; \underline{x}(0) = \mathcal{J} f \qquad (4.7)$$

have solutions related by

$$x_i(t) = x(t-(i-1)\delta) \;,\quad i\geq 1 \;,\quad t \in [0,\delta) \qquad (4.8)$$

provided that $f \in \mathcal{A}^2$.

Proof We have already seen that , for $f \in \mathcal{A}^2$, a solution $x(t)$ of (4.6) defines a solution of (4.7) on $[0,\delta)$ by (4.8) , which is uniquely defined by

$$\underline{x}(t) = e^{At}\,\underline{x}(0) \;,\quad t \in [0,\delta)$$

since $\underline{x}(0) \in \ell^2$. Conversely, if $\underline{x}(0) \in \ell^2$ is given, write

$$\underline{x}(0) = (x_o, f_o, f_1, \ldots)$$

and define $f = \mathcal{T}^{-1}\{(f_o, f_1, \ldots)\}$. Of course, $f \in \mathcal{A}^2 \subseteq \mathcal{A}$ and so (4.6) has a unique solution with initial data f and x_o. The functions x_i defined by (4.8) satisfy (4.7) on $[o,\delta)$ as before and so these functions must coincide on $[o,\delta)$ with the components of the unique solution $e^{At}\underline{x}(0)$ of (4.7). □

This lemma only allows us to relate the solutions of the delay equation and the infinite dimensional equation on ℓ^2 on the time interval $[0,\delta)$. However, let us note that the initial conditions (4.5) suggest that we consider the values of $\underline{x}(t)$ at each point $i\delta$ $(0\leq i<\infty)$. Then, we have

$$x_1(\delta) = x(\delta)$$

$$x_2(\delta) = x(o) = x_o$$

$$x_3(\delta) = x(-\delta) = f(-\delta)$$

$$\cdots\cdots$$

$$x_i(\delta) = f^{(i-3)}(-\delta)$$

and generally,

$$x_{j+1}(j\delta) = x_o$$

$$x_{j+2}(j\delta) = f(-\delta)$$

. . . . . .

$$x_i(j\delta) = f^{(i-j-2)}(-\delta) \quad , \quad i \geq j+2$$

Now, if $\underline{y} \in \ell^2$, let P_i denote the projection of $\underline{y}$ onto the subspace generated by the basic elements $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_i$, i.e.

$$P_i \underline{y} = P_i\{(y_1, y_2, \ldots)\} = (y_1, \ldots, y_i, 0, 0, \ldots).$$

Then, as a simple corollary of lemma 4.1 we have

Corollary 4.2 The solution of (4.6) can be obtained from the system (4.7) (via the formulae (4.3)) by defining

$$\underline{x}(t) = e^{At}\underline{x}(o) \quad , \quad t \in [0,\delta)$$

$$\underline{x}(t) = e^{A(t-\delta)}\{P_1\underline{x}(\delta) + A^*\underline{x}(0)\} \quad , \quad t \in [\delta, 2\delta) \tag{4.9}$$

. . . . . .

$$\underline{x}(t) = e^{A(t-i\delta)}\{P_i\underline{x}(i\delta) + (A^*)^i\underline{x}(0)\} \quad , \quad t \in [i\delta, (i+1)\delta)$$

i.e. we let the flow of $\dot{\underline{x}} = A\underline{x}$ proceed from $\underline{x}(o)$ for a time δ , then we jump to the point $P_1\underline{x}(\delta) + A^* \underline{x}(o)$ and follow the flow for another time interval δ , etc. □

Consider now the controlled equation (4.1) when $u(t) = 0$, $t<0$. Then, as above, we have

$$\dot{x}_1(t) = \dot{x}(t) = x(t-\delta) + u(t) = x_2(t) + u(t)$$

$$\dot{x}_2(t) = \dot{x}(t-\delta) = x(t-2\delta) + u(t-\delta) = x_3(t) + u(t-\delta)$$

. . . . . .

$$\dot{x}_i(t) = x_{i+1}(t) + u(t-(i-1)\delta).$$

This leads is to consider the controlled equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{u}(t) \tag{4.10}$$

with $\underline{u}(t) = (u_1(t), u_2(t), \ldots) \in \ell^2$. Of course, the control u(t) in (4.10) is, in general, not the same as the control $(u(t), u(t-\delta), \ldots)$ in (4.1). However, if we apply (for example) a simple gain feedback in (4.1), i.e. $u(t) = -kx(t)$, then (4.10) becomes

$$\dot{\underline{x}}(t) = (A-kI)\underline{x}(t) \qquad (4.11)$$

and the solution of (4.1) with such a control and that of (4.11) are again related by (4.9), with A replaced by A-kI.

Let us recall now the definition of stability of a delay equation, as given, for example, in Hale (1971).

Definition 4.3 Let $C_H = \{f\varepsilon C : \sup_{t\varepsilon[-\delta,o]} |\phi|<H\}$, and denote by $x(t;f)$ the solution of (4.1) with $x(t)=f(t)$, $t\varepsilon[-\delta,o]$, and $x(o) = f(o)$

Then (4.1) is stable if $\exists$ b>o such that

(i) $f \varepsilon C_b \Rightarrow x_t(f) \varepsilon C_H$ for some H>o

(ii) $\forall \varepsilon>o, \exists \delta>o$ such that

$$f \varepsilon C_\delta \Rightarrow x_t(f) \varepsilon C_\varepsilon \quad , \quad t\geq o.$$

The system is asymptotically stable if it is stable and $\exists$ H>o such that $f \varepsilon C_H \Rightarrow \lim_{t\to\infty} |x_t(f)| = 0$.

Here we have used the standard notation

$$x_t(f)(\theta) = x(t+\theta;f) \quad , \quad \theta \varepsilon[-\delta,o] .$$

Then we have the following result.

Theorem 4.4 (Hale 1971) The System

$$\dot{x}(t) = -ax(t) - bx(t-\delta)$$

is stable in the region shown in fig. 4.1 . □

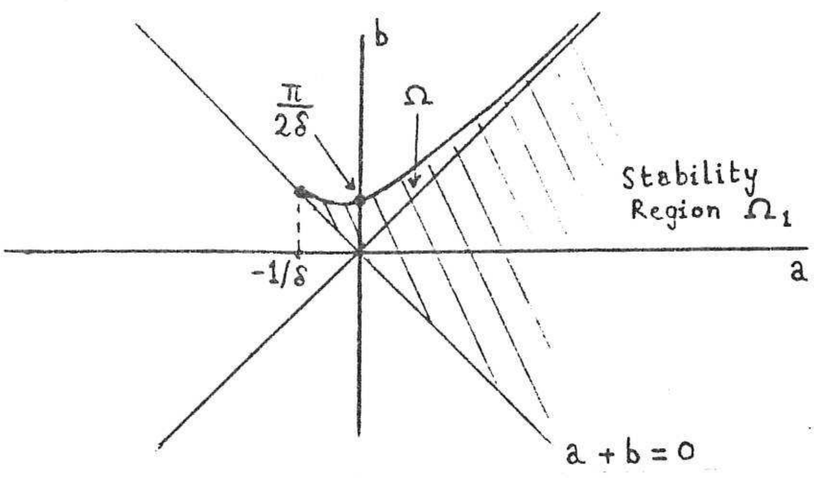

Fig. 4.1

This leads to the following result.

Theorem 4.5 The system (4.11) is stable for $k \geq o$ iff the delay system

$$\dot{x}(t) = x(t-\delta) - kx(t) \qquad (4.12)$$

is stable (in the above sense).

Proof. We first remark that by stability of (4.11) we mean, of course, that $\| \exp(A-kI)t \| \leq \mathcal{M} e^{\omega t}$ for some $\mathcal{M} \geq o$ and $\omega < o$. Now $\sigma(A-kI) = \{ \lambda-k : \lambda \varepsilon \sigma(A) \}$ which equals the closed unit disc shifted to the left by k, and this is in the left half plane iff $k \geq 1$. However, since A-kI is a bounded operator, it must satisfy the spectrum determined growth assumption and so the system (4.11) is stable iff $k \geq 1$ and this is equivalent by theorem 4.4 to the stability of (4.12). □

Note however that if we consider the equation

$$\dot{x}(t) = -x(t-\delta) - kx(t) \qquad (4.13)$$

and the associated equation

$$\dot{\underline{x}} = -A\underline{x} - k\underline{x} \qquad (4.14)$$

on ℓ^2, then a similar argument to that in theorem 4.5 shows that (4.14) is again stable iff $k \geq 1$. Finally, the equation

$$\dot{\underline{x}} = -A\underline{x} + k\underline{x}$$

is not stable for any $k \geq o$. It follows therefore that the equation

$$\dot{\underline{x}} = -aA\underline{x} - b\underline{x}$$

is stable in the region shown in fig 4.2

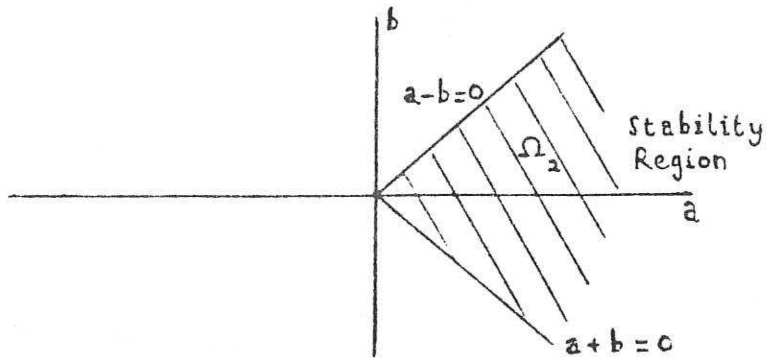

Fig. 4.2

Hence, replacing a delay equation by the associated equation on ℓ^2 is likely to produce conservative stability regions since the latter system does not predict stability in the region $\Omega = \Omega_1 \backslash \Omega_2$. This can be understood intuitively, since as the delay becomes large the region Ω shrinks to the empty set, and equation (4.9) (which is equivalent to the delay equation) becomes identical to equation (4.11) (over the large interval $[o,\delta)$). Conversely, as $\delta \to o$, the region Ω expands to a whole quadrant and the solutions of (4.9) and (4.11) equates only on the small interval $[o,\delta)$. Of course, the purpose of introducing the equation on ℓ^2 is to replace the unbounded operator and transcendental characteristic equation of the delay equation (cf Hale, 1971) by a bounded operator with simple spectrum consisting of the closed unit disc.

Remark 4.6 Although the proof of theorem 4.5 is easy it is instructive to prove the asymptotic stability of equation (4.9) for k>1 directly since this method may prove useful for nonlinear delay systems. To do this we merely note that since A is an isometry (i.e. $\|A\| = 1$) we have

$$\| \exp(A-kI)t \| \leq \| e^{At} \| \, \| e^{-kIt} \| \leq e^{-\alpha t}$$

for $\alpha = k-1 > o$. Hence, we have

$$\| x(t) \| \leq e^{-\alpha t} \| \underline{x}(o) \| \quad , \quad t \in [o,\delta)$$

$$\| \underline{x}(t) \| \leq e^{-\alpha(t-\delta)} \{ e^{-\alpha\delta} \| \underline{x}(o) \| + \| \underline{x}(o) \| \} \quad , \quad t \in [\delta, 2\delta)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\| \underline{x}(t) \| \leq e^{-\alpha(t-i\delta)} \{ e^{-\alpha i\delta} + e^{-\alpha(i-1)\delta} + \ldots + 1 \} \| \underline{x}(o) \| \quad , \quad t \in [i\delta,(i+1)\delta)$$

$$= e^{-\alpha(t-i\delta)} \left(\frac{1-e^{-\alpha(i+1)\delta}}{1-e^{-\alpha\delta}} \right) \| \underline{x}(o) \| .$$

If follows that $\underline{x}(t)$ is bounded in ℓ^2 as $t \to \infty$, and it is an easy consequence of the definition of $\underline{x}$ that

$$x_1(t) = x(t) \to o \quad \text{as} \quad t \to \infty$$

where x is the solution of the delay equation. Note that $\underline{x}(o) \in \mathcal{A}^2$. However $\mathcal{A}^2$ is dense in C and by the continuing of the solution of the delay equation with respect to the initial function f, we have shown stability in the sense of definition 4.3. □

5. The Root Locus Method for Infinite Dimensional Systems.

We come now to discuss the root locus for infinite dimensional systems in general and also to give an example involving the theory developed in § 4. Consider then the system

$$\dot{x} = Ax + Bu \qquad (5.1)$$

$$y = Cx$$

where A, B and C will be assumed to be bounded operators on a Hilbert space H. (The more general case of unbounded operators can be dealt with in a similar manner. We restrict consideration here to bounded operators because of our intended application and to clarify the ideas involved). For simplicity we shall consider the case of gain feedback of y, and thus obtain the system

$$\dot{x} = Ax + B(v-ky) \qquad (5.2)$$

$$y = Cx$$

Taking the Laplace transform of this equation, which one can do because of the boundedness of the operators (in the unbounded case it would be necessary to introduce further assumptions) it follows that

$$sX(s) = AX(s) + B(V(s) - kY(s))$$

$$Y(s) = CX(s) ,$$

assuming, as usual, that $x(0) = 0$. Hence, if $s \notin \sigma(A)$, we have

$$X(s) = R(A;s)B(V(s)-kY(s)) ,$$

where we recall that

$$R(A;s) = (sI-A)^{-1}$$

is the resolvent of A. Hence,

$$Y(s) = CX(s) = CR(A;s)B(V(s)-kY(s)). \qquad (5.3)$$

In analogy with the classical case we call

$$G(s) = CR(A;s)B \quad (s \notin \sigma(A))$$

the <u>transfer function</u> of the system, and then by (5.3)

$$Y(s) = G(s)\ (V(s)-kY(s)).$$

Hence,

$$(I+kG(s))Y(s) = G(s)V(s),$$

and so if s is such that $0 \notin \sigma(G(s))$,

$$G^{-1}(s)(I+kG(s))Y(s) = V(s) \qquad (5.4)$$

Now, in order to solve (5.4) for Y in terms of V, we require to find those values of s such that

$$0 \notin \sigma(G^{-1}(s)(I+kG(s)) . \qquad (5.5)$$

Let us define

$$F(s) = G^{-1}(s)(I+kG(s)) ,$$

which we know exists as a bounded operator-valued analytic function of s for all s such that $0 \notin \sigma(G(s))$ and $s \notin \sigma(A)$. Denote the set of all such values s by Ω. In order to satisfy the condition (5.5), we shall apply the spectral mapping theorem. Then we have (if $\lambda \neq 0$).

$$\lambda \varepsilon \sigma(G(s)) \Longleftrightarrow \frac{1}{\lambda}(1+k\lambda) \varepsilon\ \sigma(F(s)) . \qquad (5.6)$$

For each $k \geq o$ denote by $\Gamma(k)$ the set of complex values s such that

$$0 \notin \sigma(F(s)).$$

Then we call the locus of the sets $\mathbb{C} \backslash \Gamma(k)$ the <u>root locus</u> (or spectral locus) of the system. Thus, for fixed k, if $s \in \Gamma(k)$, then we can write

$$Y(S) = F^{-1}(s)V(s) ,$$

and $F^{-1}(s)$ is an analytic bounded operator valued function on $\Gamma(k)$. Note that although F(s) was defined only for $s \notin \Omega$, it may be that $\Gamma(k) \backslash \Omega \neq \phi$; all this means is that $F^{-1}(s)$ is defined by analytic continuation from $\Gamma(k) \backslash \Omega$ into $\Gamma(k)$.

Let us now return to the example of the system

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad (5.7)$$

where $x \in \ell^2$ and A is the left shift operator. Suppose, for now that B=C=I , from which we have

$$(G(s) = (sI-A)^{-1} = R(s;A)$$

provided $s \notin \sigma(A)$. However,

$$\mu \varepsilon \sigma(A) \Leftrightarrow \frac{1}{s-\mu} \varepsilon\ \sigma(R(s;A)) \quad (s \neq \mu)$$

and so by (5.6)

$$\mu \varepsilon \sigma(A) \Leftrightarrow (s-\mu)\left(1 + \frac{k}{s-\mu}\right) \varepsilon\ \sigma(F(s)) \quad (s \neq \mu)$$

Hence, for each $k \geq o$,

$$O \varepsilon \sigma(F(s)) \quad \text{iff} \quad s = \mu - k$$

for some $\mu \varepsilon \sigma(A)$. It follows that the spectral locus of the system (5.7) is as shown in fig 5.1 (a).

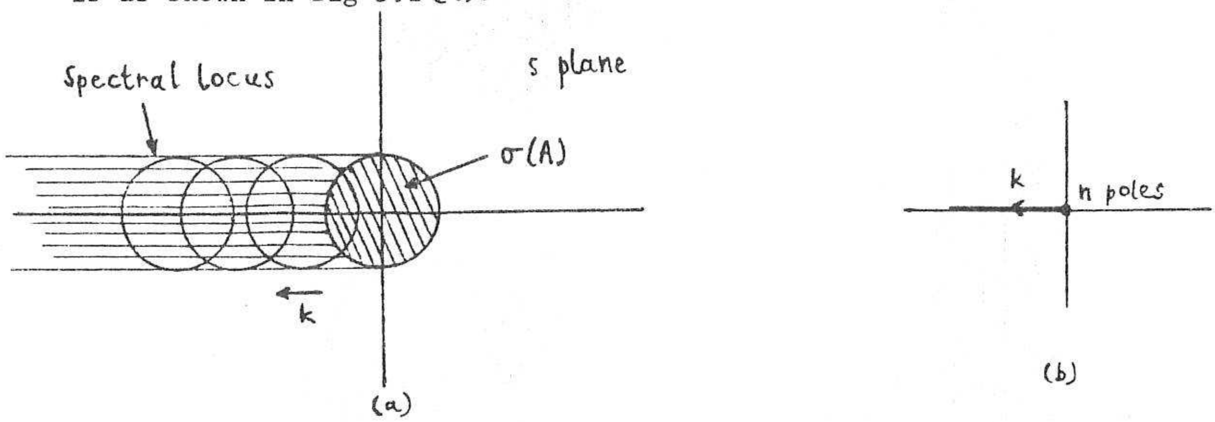

Fig 5.1

Remark 5.1

If we recall that

$$R(s;A) = (sI-A)^{-1} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^3} & \cdots \\ & \frac{1}{s} & \frac{1}{s^2} & \cdots \\ & & \frac{1}{s} & \cdots \\ & \underline{0} & & \ddots \end{pmatrix}$$

it is interesting to note that any finite-dimensional approximation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ \cdot \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \underline{0} \\ & 0 & 1 & \\ & & \ddots & \ddots & \\ \underline{0} & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ \cdot \\ \vdots \\ x_n \end{pmatrix}$$

to the system $\dot{\underline{x}} = A\underline{x}$ has spectrum consisting of just n poles at the origin (i.e. an n^{th} order integrator) which can be stabilized by an arbitrarily small gain feedback as in fig. 5.1(b), whereas fig. 5.1(a) shows that we require at least a gain of $1+\varepsilon$ in the feedback loop, in the case of the infinite dimensional system. This explains why finite dimensional approximations to the delay equation do not correctly predict stability.

Remark 5.2

We see from this example that we can expect the root locus for distributed systems to consist of bands swept out by connected components of $\sigma(A)$.

To illustrate the comments in remark 5.2 in more detail we shall consider a delay analogue of the trivial two dimensional system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

for which $(sI-A)^{-1} = \begin{pmatrix} 1/s & 1/s(s+3) \\ 0 & 1/(s+3) \end{pmatrix}$

and

$$G(s) = C(sI-A)^{-1}B = \frac{1}{s(s+3)} .$$

The root locus of this system, as is well-known, is shown in fig. 5.2.

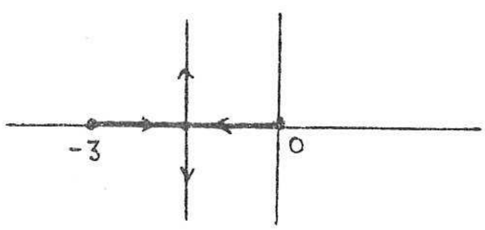

Fig. 5.2

Consider then the coupled delay system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t-\delta) \\ x_2(t-\delta) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \quad 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and replace it by the corresponding system

$$\begin{pmatrix} \dot{\underline{x}} \\ \dot{\overline{\underline{x}}} \end{pmatrix} = \begin{pmatrix} A & I \\ 0 & A-3I \end{pmatrix} \begin{pmatrix} \underline{x} \\ \overline{\underline{x}} \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u = \mathcal{A} \begin{pmatrix} \underline{x} \\ \overline{\underline{x}} \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u, \text{ say}$$

$$y = (I \quad 0) \begin{pmatrix} \underline{x} \\ \overline{\underline{x}} \end{pmatrix},$$

where A is again the left-shift operator and $\underline{x}$, $\overline{\underline{x}} \in \ell^2$ are related to $x_1(t)$, $x_2(t)$, respectively, as in (4.3). Now,

$$R\left\{ s\,;\begin{pmatrix} A & I \\ 0 & A-3I \end{pmatrix} \right\} = \begin{pmatrix} (sI-A)^{-1} & (sI-A)^{-1}(sI-A+3I)^{-1} \\ 0 & (sI-A+3I)^{-1} \end{pmatrix}$$

(if $s \notin \sigma(A) \cap \sigma(A-3I)$), and so the transfer function for this system is

$$G(s) = (sI-A)^{-1}(sI-A+3I)^{-1} ,$$

and by the spectral mapping theorem,

$$\lambda \in \sigma(A) \Longleftrightarrow \frac{1}{s-\lambda} \cdot \frac{1}{s-\lambda+3} \in \sigma(G(s)) \quad , \quad s \neq \lambda, \lambda-3$$

Hence, by (5.6), the root locus of this system is given by those values of s which satisfy

$$(s-\lambda)(s-\lambda+3)\left\{ 1 + k\,\frac{1}{s-\lambda} \cdot \frac{1}{s-\lambda+3} \right\} = 0 \,, \quad \text{for } \lambda \in \sigma(A)$$

and is shown in fig. 5.3 (cf. fig. 5.2).

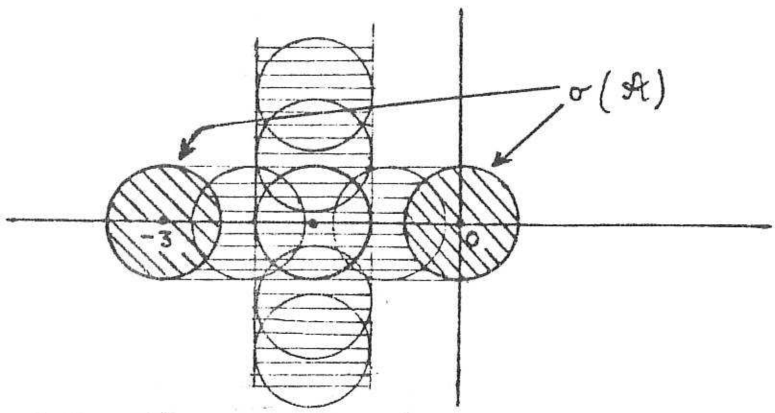

Fig. 5.3

Remark 5.3 It is clear from the above results that to each finite-dimensional system there corresponds an infinite-dimensional system whose root-locus is a band of unit width about the root locus of the finite-dimensional system.

Let us note finally that from

$$Y(s) = F^{-1}(s)V(s)$$

it follows that

$$y(t) = \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F^{-1}(s)ds\right) * v(t)$$

and since $F^{-1}(s)$ defines a linear bounded system, if we cloose k so that the spectrum of this system is in the left-half plane, we can take $c<o$ and the system is stable (by the spectrum determined growth condition).

6. Conclusions

In this paper we have discussed the connection between simple delay systems and the left-shift operator and related them to the infinite-dimensional root (or spectrum) locus. It has been seen that, for systems defined by bounded operators, the root locus behaves in a similar way to that of a finite-dimensional system, the connected components of the spectrum of the former taking the counterpart of the poles of the latter.

We have seen that if the initial function $f(t)$, $t\varepsilon[-\delta,0]$ of a delay equation is analytic, with $\{f^{\ell}(-\delta)\}_{\ell\geq o} \;\varepsilon \ell^2$, then we can identify the solutions of the delay equation with those of a corresponding equation involving the left-shift operator, at least on $[o,\delta]$. This led to a stability theory for delay systems and demonstrated that finite dimensional approximations do not correctly predict stability.

7. References

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