ON NONLINEAR PERTURBATIONS OF NONLINEAR DYNAMICAL SYSTEMS AND APPLICATIONS TO CONTROL

S.P. Banks

Department of Control Engineering,
University of Sheffield,
Mappin Street, Sheffield S1 3JD.

Research Report No. 199

October 1982

ABSTRACT

The nonlinear variation of constants formula is generalised to infinite dimensional systems and applied to the stability problem and to obtain bounds on the states of a nonlinear control system when the controls are bounded. The theory is illustrated by a simple model from nuclear reactor dynamics.

1. Introduction

In this paper we shall consider the nonlinear generalisation of the familiar variation of constants (or parameters) formula to infinite dimensional systems. The latter formula, of course, is obtained as an integrated version of the system.

$$\dot{x} = Ax + f(t) \quad , \quad x \in X(\text{a Banach space}) \tag{1.1}$$

where A generates a semigroup $T(t)$ in X, and f satisfies suitable hypotheses so that (1.1) is soluble. Then we have

$$x(t) = T(t)\,x(o) + \int_o^t T(t-s)f(s)ds \tag{1.2}$$

The formula (1.2) has many important applications in systems theory and so it is reasonable to suppose that a similar formula when A is nonlinear may also have some uses in nonlinear systems theory.

The finite dimensional generalisation is due to Alekseev[1] and is applied extensively to stability theory by Brauer[6]. If we consider the nonlinear system

$$\dot{x} = f(x,t) \tag{1.3}$$

and the perturbation

$$\dot{y} = f(y,t) + g(y,t) \tag{1.4}$$

then the formula relating the solutions $x(t;t_o,x_o)$, $y(t;t_o,x_o)$ (with the same initial condition x_o) of (1.3) and (1.4) is[6]

$$y(t;t_o,x_o) - x(t;t_o,x_o) = \int_{t_o}^{t} \Phi(t,s,y(s;t_o,x_o))g(s,y(s;t_o,x_o))ds \tag{1.5}$$

and corresponds to (1.2). Here,

$$\Phi(t;t_o,x_o) \triangleq \frac{\partial}{\partial x_o}\left[x(t;t_o,x_o)\right]$$

is the fundamental matrix of the variational system

$$\dot{z} = f_x\left[x(t;t_o,x_o),t\right]z \tag{1.6}$$

The main difficulty with (1.5) is that if g is independent of y then, whereas in the linear case (1.2) gives the solution of (1.1), the formula (1.5) is only an integral equation in y since $\Phi(t,s,y)$ depends (nonlinearly) on y. This will restrict the usefulness of (1.5) to some extent.

However, as stated above the formula (1.5) does have important applications to stability theory and has also been used to obtain state bounds for bounded controls and with some success, in the theory of nonlinear observers and stability (Banks (2,3,4)). In this paper we shall be particularly interested in generalising some results in stability and state bounds for nonlinear distributed control systems and we shall exemplify the theory with the system

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \lambda\phi - \rho\phi^2 \quad , \ \phi(o,t) = \phi(1,t) = 0, \tag{1.7}$$

which occurs as a simple model in nuclear reactor dynamics[11].

We shall begin by finding conditions under which the distributed system corresponding to (1.5) generates an evolution operator. (In the present paper we shall discuss, for simplicity, the case when f is independent of t; the general case is treated similarly). We must require f to be Frechet differentiable, of course. The generalisation of (1.5) will then follow easily and we shall use the formula to prove an asymptotic stability result for a perturbation of (1.7). We shall finally consider the system

$$\dot{x} = Ax + Bu + \psi(x,u,t) \quad , \quad x \in X \ , \ u \in U$$

and obtain bounds on the state x when the control u is restricted to the ball

$$\{u: ||u||_U \leq k\}$$

in U.

2. Notations and Terminology

In this paper capitals E,F,X etc. will denote general Banach spaces and H will be, in particular, a Hilbert space. The space of bounded operators from E into F will be denoted by $\mathcal{L}(E,F)$. The special Hilbert spaces $H^1([0,1])$, $H_o^1([0,1])$, $L^2([0,1])$ etc will have their usual meanings (the first two being the familiar Sobolev spaces), and if there is no danger of confusion, we shall omit the domain of definition of the functions and write simply H^1, H_o^1, L^2 etc.

If A is an operator (linear or nonlinear), D(A) will denote the domain of A and if A is linear the resolvent $(\lambda I-A)^{-1}$ for λ not in the spectrum of A will be written $R(\lambda;A)$. Generally, $S(t)$ will denote a nonlinear semigroup and $T(t)$ a linear one. Finally, C(or C_i for some integer i) will denote a generic constant. Other notations will be introduced as we proceed.

3. The Nonlinear Variation of Constants Formula

(3.1) Fréchet Derivatives

Definition 3.1.1 If E and F are Banach spaces, we say that a function $f: E \to F$ is (Fréchet) differentiable at $x \in E$ is there exists an operator $B(x) \in \mathcal{L}(E,F)$ such that

$$\lim_{||p||_E \to 0} \frac{||f(x+p) - f(x) - B(x)p||_F}{||p||_E} = 0$$

The Fréchet derivative of f at x_o will be denoted by $\mathcal{F}f(x_o)$. If $f: E_1 \times E_2 \to F$ is a function of two variables $(x,y) \in E_1 \times E_2$, then we write the partial Fréchet derivatives at (x_o,y_o) by $\mathcal{F}_x f(x_o,y_o)$, $\mathcal{F}_y f(x_o,y_o)$.

We shall frequently be in the situation where A is a nonlinear (single-valued) operator defined in a Hilbert space H with domain D(A). If D(A) is a linear subspace of H and is a Banach space with norm $||.||_{D(A)}$, then we can regard A as a nonlinear map from D(A) into H and define the derivative with respect to these spaces.

Example 3.1.2 As a simple example, consider the function $f: H^{1,4} \to L^2$ defined by

$$f(\phi) = \left(\frac{d\phi}{dx}\right)^2, \quad \phi \in H^{1,4}.$$

Then,

$$\left|\left|\left(\frac{d}{dx}(\phi + p)\right)^2 - \left(\frac{d\phi}{dx}\right)^2 - 2\,\frac{d\phi}{dx}\,\frac{dp}{dx}\right|\right|_{L_2} \Big/ ||p||_{H^{1,4}}$$

$$= \left|\left|\left(\frac{dp}{dx}\right)^2\right|\right|_{L_2} \Big/ ||p||_{H^{1,4}}$$

$$= ||p||_{H^{1,4}} \to o \text{ as } p \to o \text{ in } H^{1,4}$$

Hence, $f(\phi) = \frac{2d\phi}{dx}\,\frac{d}{dx} \quad : \quad H^{1,4} \to L^2$

(3.2) Nonlinear Semigroups and Evolution Operators

We shall now recall some basic properties of nonlinear semigroups, which may be found in Barbu[(5)].

Definition 3.2.1 If C is a closed subset of a Hilbert space H, then a semigroup of type ω on C is a function $S : [o,\infty[\times C \to C$ such that the following properties are satisfied:

(i) $S(t+s)x = S(t)S(s)x, \quad \forall\, x \in C\ ,\ t,s \geq 0$

(ii) $S(o)x = x \quad , \quad \forall\, x \in C$

(iii) $S(t)x$ is continuous in $t \geq o$, $\forall\, x \in C$.

(iv) $||S(t)x - S(t)y|| \leq e^{\omega t}||x - y||, \ \forall\, t \geq o, \quad x, y \in C$.

Lemma 3.2.2 Let S be a semigroup of type ω on C and $x \in D(A)$, where $D(A) = \{x \in C: \liminf_{t \to o} ||S(t)\,x - x\,||/t < \infty\}$; then $S(t)\,x$ is Lipschitz on every interval $[o,T]$, $\quad T > 0$. □

We wish to consider semigroups $S(t)$ for which $S(t)x$ is Frechet differentiable with respect to x $(\in D(A))$. In order to do this we shall need the following result of Tanabe (c.f. Yosida[(12)], Friedman[(9)]):

Theorem 3.2.3 Let $A(t)$ $(t\in[o,T])$ be a set of linear operators in H with the following properties:

(a) The domain $D(A(t))$ of $A(t)$ is independent of t and dense in H.

(b) For each $t \in [o,T]$, the resolvent $R(\lambda;\, A(t))$ of $A(t)$ exists for all λ with Re $\lambda \geq o$ and

$$||R(\lambda;\ A(t))|| \leq \frac{C}{|\lambda|+1} \quad , \quad \text{Re}\ \lambda \geq 0$$

for some constant $C > 0$

(c) For any $t,s,\tau \in [0,T]$,

$$||(A(t) - A(\tau))A^{-1}(s)|| \leq C|t - \tau|^{\alpha},\ 0 < \alpha \leq 1,$$

where C is a constant independent of t,s,τ.

Then $A(t)$ generates a unique evolution operator $U(t,\tau)$, i.e.

$$\frac{\partial U}{\partial t}(t,\tau) = A(t)U(t,\tau) \quad , \quad \tau < t \leq T$$

$$U(\tau,\tau) = I,$$

where the derivative of U with respect to t is taken in the sense of the strong topology. □

In the case of finite dimensional ordinary differential equations we know that if $x(t;\ t_o,\ x_o)$ is the solution of the equation

$$\dot{x} = f(x,t) \quad , \ x(t_o) = x_o,$$

then $\Phi(t;t_o,x_o) \triangleq \frac{\partial}{\partial x_o}\left[x(t;\ t_o,\ x_o)\right]$ is the fundamental solution of the system

$$\dot{z} = f_x\left[x(t;\ t_o,\ x_o),t\right]z.$$

We now wish to generalise this result to the case when we have a nonlinear system given by

$$\dot{x} = Ax \ , \ x(o) = x_o \in D(A)$$

where A is an operator with domain $D(A)$ dense in H, which is a linear subspace of H. (We shall consider the autonomous case for simplicity; the results can be extended to nonautonomous systems).

<u>Theorem 3.2.4</u> Let $A: D(A) \rightarrow H$ be a nonlinear operator which generates a semigroup $S(t)$, and suppose that $\mathcal{F}A(x)$ ($\in \mathcal{L}(D(A),H)$) exists uniformly for each $x \in D(A)$. Assume that $\mathcal{F}A$ satisfies the following conditions along a solution $S(t)x$ of $\dot{x} = Ax$;

(i) The domain of $\mathfrak{F}A(S(t)x)$ is independent of t and dense in H.

(ii) For each $t \in [0,T]$, the resolvent $R(\lambda;\mathfrak{F}A(S(t)x))$ exists for all Re $\lambda \geq 0$ and

$$||R(\lambda;\mathfrak{F}A(S(t)x))|| \leq \frac{C}{|\lambda|+1} \quad , \quad \mathrm{Re}\ \lambda \geq 0$$

for some $C > 0$.

(iii) For any $t,s,\tau \in [0,T]$,

$$||(\mathfrak{F}A(S(t)x) - \mathfrak{F}A(S(\tau)x))\mathfrak{F}A^{-1}(S(s)x)|| \leq C\ |t - \tau|.$$

Then the semigroup $S(t)x$ is differentiable with respect to x(as a map from H into H) and $\mathfrak{F}_{x_o} S(t)x_o = U(t,o)$, where U is the evolution operator which is the fundamental solution of

$$\dot{y} = \mathfrak{F}_x A(S(t)x_o)y.$$

<u>Proof</u> We shall follow essentially, the method of proof of Coddington and Levinson (7, p.25). Define $x = x_o + h \in D(A)$ and set

$$M(t;x_o,h) = S(t)x - S(t)x_o .$$

Since the semigroup $S(t)x$ satisfies the equation $\dot{x} = Ax$, we have

$$\frac{dM}{dt}(t;x_o,h) = AS(t)x - AS(t)x_o \quad , \quad t \geq o$$
$$= (\mathfrak{F}_x A(S(t)x_o) + \Gamma)M(t;x_o,h) \qquad (3.1)$$

where

$$||\Gamma||_{\mathcal{L}(D(A),H)}/||h||_{D(A)} \to 0 \text{ as } ||h|| \to 0$$

uniformly for $t \geq o$.

Hence, if $\overline{M} = M/||h||$, then

$$\frac{d}{dt}\overline{M}(t;x_o,h) = (\mathfrak{F}_{x_o} A(S(t)x_o) + \Gamma)\overline{M}(t;x_o,h). \qquad (3.2)$$

Consider now the system

$$\frac{dy}{dt} = \mathfrak{F}_x A(S(t)x_o)y \quad , \quad t \geq 0 \qquad (3.3)$$

It follows from theorem 3.2.3 and the conditions satisfied by $\mathfrak{F}A(x)$ that this system defines a unique evolution operator $U(t,s;x_o)$, such that the solution of (3.3) is given by

$$y(t) = U(t,o;x_o) \quad , \quad t \geq t_o$$

$$y(o) = I_H \ .$$

Hence, $M_1 = y(x-x_o) - \bar{M}(t;x_o,h)$ satisfies the equation

$$\frac{dM_1}{dt} = \mathcal{F}_x A(S(t)x_o)M_1 - \Gamma\bar{M}(t;x_o,h) \ , \ t \geq o,$$

i.e.

$$M_1 = -\int_0^t U(t,s;x_o)\ \Gamma\bar{M}(s;x_o,h)ds,$$

and so

$$||M_1|| \leq t \sup(||U(t,s;x_o)||\ \frac{||\Gamma||}{||h||} \quad ||M(t;x_o,h)||)$$

$$\to 0 \qquad \text{as} \qquad ||h|| \to 0,$$

for all $t \in [0,T]$. Thus $\lim_{||h|| \to o} M(t;x_o,h)$ exists and equals $U(t,o;x_o)(x-x_o)$. $\square$

Remark 3.2.5 It is easy to see that theorem 3.2.6 is true if we only assume that A generates a semigroup on some subset C of H and that C is contained in a dense linear subspace V of H such that

$$\mathcal{F}A(x) \in \mathcal{L}(V,H), \quad \forall x \in C,$$

and $A(x)$ satisfies the same conditions along a solution $S(t)x$ $(x \in C)$ as in the theorem. $\square$

We shall now consider an example to illustrate the above theory.

Example 3.2.6 Consider the nonlinear diffusion equation

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \lambda\phi - \rho\phi^2 \ , \quad \rho,\lambda \geq o, \ \lambda < \pi^2, \ x \in [0,1] \qquad (3.4)$$

and let

$$C = \{\phi \in H_0^1([0,1]) \ : \ \phi(x) \geq o, \quad \forall\, x \in [0,1]\}.$$

Then it is well known (10) that the equation (3.4) has a unique solution in C for any initial value in C. Moreover, if we define the operator A by

$$A\phi = \frac{d^2\phi}{dx^2} + \lambda\phi - \rho\phi^2 \ , \ \phi \in D(A) = H^2([0,1]) \cap H_0^1([0,1])$$

then we have, for any $\phi_1, \phi_2 \in C \cap D(A)$,

$$< A\phi_1 - A\phi_2, \phi_1 - \phi_2 >_{L^2([0,1])} = - ||\phi'||^2_{L^2} + \lambda||\phi||^2_{L^2} - \rho\int_0^1 (\phi_1+\phi_2)\phi^2 dx$$

$$\leq - ||\phi'||^2_{L^2} + \lambda||\phi||^2_{L^2}$$

$$\leq (\lambda - \pi^2)||\phi||^2_{L^2},$$

($\phi \triangleq \phi_1 - \phi_2$, $\phi' = d\phi/dx$), where we have used the unequality

$$\pi^2||\phi||^2_{L^2} \leq ||\phi'||^2_{L^2} \quad , \quad \forall\, \phi \in H_0^1([0,1]).$$

A is therefore dissipative on $D_d(A) \triangleq C \cap D(A)$ (the dissipative domain of A) and is the generator of a semigroup S(t) of type $-(\pi^2 - \lambda)$ on $\overline{C}$ (closure in $L^2([0,1])$).

Consider now $\mathcal{J}A(S(t)\phi)$ for some $\phi \in D_d(A)$. The only term of any difficulty in A is $f(\phi) \triangleq \phi^2$. Now,

$$\frac{||f(\phi_1+\phi) - f(\phi_1) - 2\phi_1\phi||_{L^2}}{||\phi||_{D(A)}} \leq C\frac{||\phi^2||_{L^2}}{||\phi||_{L^4}} = C||\phi||_{L^4}$$

(Since $D(A) \subseteq L^4$) for some constant C and so $\mathcal{J}A(S(t)\phi)$ exists uniformly in t for $\phi \in D_d(A)$ and is given by

$$(\mathcal{J}A(S(t)\phi)\psi)(x) = \frac{d^2\psi}{dx^2}(x) + \lambda\psi(x) - 2\rho(S(t)\phi)(x)\psi(x) \tag{3.5}$$

for $\psi \in D(A)$.

We would now like to show that the conditions of theorem 3.2.4 are satisfied by $\mathcal{J}A$. First, recall the well known Sobolev imbedding theorem

$$D_d(A) \subseteq H^2([0,1]) \cap C^1([0,1]). \tag{3.6}$$

If $\phi \in D_d(A)$, then $S(t)\phi \in D(A) \subseteq H^2 \subseteq C^1$ and so $(S(t)\phi)(x)$ is continuous. It follows that the operator $\mathcal{J}A(S(t)\phi)$, for each fixed $t \in [0,T]$ and $\phi \in D_d(A)$, is a uniformly strong elliptic operator on D(A) and the domain is clearly independent of t. Also, since $S(t)$ is defined on C we have $(S(t)\phi)(x) \geq 0$ and so $\mathfrak{F}A(S(t)\phi)$ is also dissipative, for each $t \in [0,T]$; in fact, as above

$$< \mathfrak{F}A(S(t)\phi_1) - \mathfrak{F}A(S(t)\phi_2), \phi_1 - \phi_2 >_{L^2} \leq (\lambda - \pi^2)||\phi||^2_{L^2} \quad (3.7)$$

It therefore remains to verify condition (iii) of the theorem. From the classical theory of elliptic operators[9], we have

$$||\{\mathfrak{F}A(S(t)\phi)\}^{-1}\psi||_{D(A)} \leq C\,||\psi||_{L^2}, \quad \psi \in L^2,$$

for some $C \geq 0$. By (3.6) we therefore have

$$||\{\mathfrak{F}A(S(t)\phi)\}^{-1}\psi||_{C([0,1])} \leq C_1||\psi||_{L^2},$$

for a new constant C_1. The first two terms in (3.5) are constant in t, and so we can ignore them. As for the third term, we have

$$||((S(t)\phi) - (S(\tau)\phi)).\,\{\mathfrak{F}A(S(s)\phi)\}^{-1}\psi||_{L^2}$$

$$\leq ||S(t)\phi - S(\tau)\phi||_{L^2}\,||\{\mathfrak{F}A(S(s)\phi)\}^{-1}\psi||_{C([0,1])}$$

$$\leq C_2|t - \tau|\,||\psi||_{L^2},$$

since $S(t)\phi$ is Lipschitz on every interval $[0,T]$.

(3.3) <u>The Nonlinear Variation of Constants Formula</u>

We now consider the equation

$$\dot{x} = Ax \quad (3.8)$$

where A is a nonlinear operator satisfying the conditions of the previous section and the nonlinear perturbation

$$\dot{y} = Ay + B(y,t) \quad (3.9)$$

where B is a nonlinear map from $D(B) \times [0,\infty)$ into H for some subset $D(B)$ of H. We assume that (3.8) and (3.9) have unique solutions through some point $(y_o,t_o) \in H\times[o,\infty)$ and we denote these solutions respectively by $x(t;t_o,y_o)$ and $y(t;t_o,y_o)$. Of course, in the context of section 3, $x(t;t_o,y_o) = S(t-t_o)y_o$, where $S(t)$ is the semigroup generated by A. We shall assume that A satisfies the conditions of theorem 3.2.4 and then we can write

$$\Phi(t,o;x_o) = \mathcal{F}_{x_o}\left[x(t;o,x_o)\right]$$

and $\Phi(t,s;x_o)$ is the fundamental solution of the equation

$$\dot{\psi} = \mathcal{F}_x A(S(t)x_o)\psi\,.$$

Then we have the following theorem due to Alekseev[1] (c.f. Brauer[6]) whose proof in the distributed case is formally the same as in the finite-dimensional situation.

Theorem 3.3.1 Let A satisfy the conditions of theorem 3.2.4 on $D_d(A)$. Then, for all $t \geq t_o$ such that $x(t;t_o,y_o), y(t;t_o,y_o) \in D_d(A) \cap D(B)$ we have

$$y(t;t_o,y_o)-x(t;t_o,y_o)= \int_{t_o}^{t} \Phi(t,s;y(s,t_o,y_o))B(y(s;t_o,y_o),s)ds \tag{3.10}$$

(The Nonlinear Variation of Constants Formula .) □

4. Applications to Systems Theory

(4.1) Nonautonomous Linear Systems

We have proved in section 3 the nonlinear variation of constants formula for distributed systems and we would like to apply this result to some aspects of stability theory of nonlinear systems, following the results of Brauer[6]. We begin by proving the following theorem, which generalises the familiar finite-dimensional result.

Theorem 4.1.1 Consider the linear system

$$\dot{x} = A(t)x$$

where, for each t, A(t) is a strongly elliptic operator on a region $\Omega(\subset \mathbb{R}^n)$ (independent of t). If the order of A(t) is 2p and we have $\left.\frac{\partial^i}{\partial n^i}\right|_\Gamma x = 0$,

$\Gamma = \partial\Omega, 1 \leq i \leq p$, then there exists a function $f(t)$ such that

$$||x(t)||_{L^2(\Omega)} \leq ||x(t_o)||_{L^2(\Omega)} \exp\left(\int_{t_o}^{t} f(s)ds\right), \quad t \geq t_o$$

Proof. Since $A(t)$ is strongly elliptic with zero boundary conditions, so is $A(t) + A^*(t)$. The latter operator is self-adjoint and has compact resolvent (Dunford and Schwartz[8]) and so has a sequence of eigenvalues $\{\lambda_n(t)\}$ with finite multiplicity and a corresponding complete orthonormal sequence of eigenvectors $\{\phi_n(t)\}$ such that of

$$x = \sum_{i=1}^{\infty} < x, \phi_i(t) > \phi_i(t),$$

then,

$$(A(t) + A^*(t))x = \sum_{i=1}^{\infty} \lambda_i(t) < x, \phi_i(t) > \phi_i(t).$$

Of course, in the above summations, each λ_i is counted according to its multiplicity. It follows that, since we can order the $\lambda_i(t)$ so that

$$\cdots \leq \lambda_n(t) \leq \cdots \leq \lambda_1(t) \leq \lambda(t)$$

where $\lambda(t)$ is independent of n, we have

$$\lambda(A(t)) \triangleq \sup_{x \neq o} \frac{< x, (A+A^*)x >_{L^2(\Omega)}}{||x||^2}$$

$$= \frac{\sum_{i=1}^{\infty} \lambda_i(t) < x, \phi_i(t) >^2}{\sum_{i=1}^{\infty} < x, \phi_i(t) >^2}$$

$$\leq \lambda(t).$$

Hence,

$$\frac{d}{dt} ||x(t)||^2_{L^2(\Omega)} = \frac{d}{dt} < x(t), x(t) >$$

$$= < x, A(t)x > + < A(t)x, x >$$

$$= < x, (A(t) + A^*(t))x >$$

$$\leq \lambda(t) ||x||^2_{L^2(\Omega)}$$

i.e. $$||x(t)||^2_{L^2(\Omega)} \leq ||x(o)||^2_{L^2(\Omega)} \exp\left(\int_o^t \lambda(s)ds\right) . \square$$

Corollary 4.1.2 Under the conditions of theorem 4.1.1, if we have that $A(t)$ is $-\omega(t)$ dissipative on a real Hilbert space (i.e. $< A(t)x,x > \leq - \omega(t)||x||^2$, $x \in D(A)$) and $D(A^*) = D(A)$, then we have

$$||x(t)||^2_{L^2(\Omega)} \leq ||x(o)||^2_{L^2(\Omega)} \exp\left(- \int_o^t 2\omega(s)ds\right)$$

Proof. Clearly,

$$< A(t)x,x > = < x,A^*(t)x > = < A^*(t)x,x > ,$$

and so

$$< A(t) + A^*(t))x,x > \leq - 2\omega(t) ||x||^2$$

Now the largest eigenvalue of $A(t) + A^*(t)$ is given by

$$\lambda(t) = \sup < \frac{(A(t) + A^*(t))x,x}{||x||^2} > \leq - 2\omega(t),$$

and so the result follows. $\square$

Example 4.1.3 Consider again the system discussed in example 3.2.6, i.e.

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \lambda\phi - \rho\phi^2 \quad , \rho , \lambda \geq o, \lambda < \pi^2, x \in [0,1]$$

If A is defined as before by

$$A\phi = \frac{d^2\phi}{dx^2} + \lambda\phi - \rho\phi^2 , \phi \in D_d(A)$$

then

$$(\mathcal{F}_\phi A(S(t)\phi)\psi)(x) = \frac{d^2\psi}{dx^2} + \lambda\psi(x) - 2\rho(S(t)\phi)(x)\psi(x)$$

and $\mathcal{F}_\phi(S(t)\phi)$ satisfies

$$\dot{\psi} = \mathcal{F}_\phi A(S(t)\phi)\psi.$$

However, we have seen that $\mathcal{F}_\phi A(S(t)\phi)$ is dissipative on $D(A)$ and so by corollary 4.1.2 it follows that

$$||\mathcal{F}_\phi(S(t)\phi)||_{L^2([0,1])} \triangleq ||\Phi(t,o;\phi)||_{L^2([0,1])}$$
$$\leq \exp(-2(\pi^2 - \lambda)t).$$

(4.2) Stability Theory

The next result follows from corollary 4.1.2 as in Brauer[6].

Lemma 4.2.1. Under the conditions of theorem 3.2.4 we have, for any $x_o, y_o \in H$,

$$||x(t;y_o) - x(t;x_o)||_H \leq ||y_o - x_o||_H \exp\left(-\int_o^t 2\omega(s)ds\right)$$

where $x(t;x_o)$ is the solution of the nonlinear equation

$$\dot{x} = A\,x \ , \ x(o) = x_o. \ \square$$

We also have the following generalisation of theorem 2 of Brauer[6] to the distributed case, the proof being formally the same.

Theorem 4.2.2. Let the conditions of theorem 3.2.4 hold for the system

$$\dot{x} = A\,x$$

where A is a nonlinear operator in H and suppose that the assumptions of corollary 4.1.2 hold with $A(t) = \mathcal{F}_x A(S(t)x_o)$. Then if the origin is an equilibrium point of the perturbed system

$$\dot{y} = Ay + g(y,t) \qquad (4.1)$$

where $g(y,t) = o(||y||_H)$ as $||y|| \to 0$ uniformly in t, the system (4.1) is asymptotically stable at $y = 0$. $\square$

Example 4.2.3. It follows from the above results that the system

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \lambda\phi - \rho\phi^2 + g(\phi,t)$$

with $g(\phi,t) = o(||\phi||_{L^2([0,1])})$ as $||\phi||_{L^2} \to 0$ uniformly in t is asymptotically stable at the origin.

(4.3) Application to nonlinear distributed control systems

In this section we shall consider a nonlinear control system of the form

$$\dot{x} = Ax + Bu + \Psi(x,u,t) \quad , \quad x = x_o \qquad (4.2)$$

where A is a (nonlinear) operator defined in the Hilbert space H, B is a linear operator defined on the control space U with values in a Hilbert space V and the controls are assumed to belong to the ball

$$U_k = \{u : ||u|| \leq k\} \subseteq U$$

We shall also assume that $B \in \mathcal{L}(U,V)$ and $\psi : H \times U \times \mathbb{R}^+ \rightarrow V$ and satisfies an inequality of the form

$$||\psi(x,u,t)||_V \leq \alpha + \beta ||x||_H \quad , \qquad \forall\, u \in U_k$$

for some constants α, β, and for $||x||_H \leq \gamma$.

The object here is to determine bounds on the state x when the control is bounded by k. Of course, for real physical systems the control input will always be bounded and so a consideration of the effects of such bounds on the set of reachable states is important; amond other things, this approach will show that the system (4.2) cannot be controllable (in certain times) when the control input is bounded. We shall consider (4.2) as a perturbation of the nonlinear system

$$\dot{y} = Ay \quad , \quad y = x_o \tag{4.3}$$

where A is assumed to possess the properties specified in theorem 3.2.4 . If A generates the semigroup $S(t)$, then we denote, as before, $\Phi(t,s;x_o)$ to be the fundamental solution of the variational system

$$\dot{z} = \mathcal{F}_y A(S(t)x_o)z. \tag{4.4}$$

Then we have the relation (c.f. (3.10))

$$x(t;t_o,x_o) - y(t;t_o,x_o) = \int_{t_o}^{t} \Phi(t,s;x(s;t_o,x_o)) \times (Bu + \psi(x,u,s))ds \; . \tag{4.5}$$

Let us write

$$A(t) \triangleq \mathcal{F}_y A(S(t)x_o)$$

and consider the linear (nonautonomous) equation

$$\dot{\xi} = A(t)\xi, \; \xi(t) \in D(A(t)).$$

Suppose that there exists a linear operator $\bar{A}$ (independent of t) such that $||(A(t) - \bar{A})\xi||_H \leq C_1 ||\xi||_H$ for each $\xi \in H$ and a constant C_1, and assume that $\bar{A}$ generates a semigroup $T(t)$ which is smoothing in the sense that

$$||T(t)\xi||_H \leq \frac{C_2}{t^\alpha} e^{-at} ||\xi||_V$$

for each $\xi \in V$ and some constants C_2, $a > o$ and $o < \alpha < 1$. Then, we have,

$$\dot{\xi} = A(t)\xi$$
$$= \bar{A}\xi + (A(t) - \bar{A})\xi$$

and so

$$\xi(t) = T(t-s)\xi(s) + \int_s^t T(t-\tau)(A(\tau) - \bar{A})\xi d\tau .$$

Hence,

$$||\xi(t)||_H \leq \frac{C_2}{(t-s)^\alpha} e^{-a(t-s)} ||\xi(s)||_V + \int_s^t e^{-a(t-\tau)} C_1 ||\xi||_H d\tau$$

and so, by a generalisation of the Gronwall inequality (c.f. Henry[10]) we have

$$||\xi(t)||_H \leq \frac{C}{(t-s)^\alpha} e^{-a(t-s)} ||\xi(s)||_V,$$

for some new constant $C(=C(C_1,C_2,\alpha,T))$ on some compact interval $[0,T]$. The next lemma follows early from the above remarks.

Lemma 4.3.1 The solution $\Phi(t,s;x_o)$ of (4.4) satisfies (under the above assumptions) the inequality

$$||\Phi(t,s;x_o)||_{(V,H)} \leq \frac{C}{(t-s)^\alpha} e^{-a(t-s)}$$
$$\triangleq g(t-s), \text{ say }. \quad \square$$

It now follows from (5.4) that

$$||x(t;t_o,x_o)|| \leq e^{-a(t-t_o)} ||x_o||_H$$

$$+ \int_{t_o}^t \frac{C}{(t-\tau)^\alpha} e^{-a(t-\tau)} \{k||B||_{\mathcal{L}(U,V)} + \alpha + \beta||x(\tau;t_o,x_o)||_H\} d\tau$$

provided $||x(\tau;t_o,x_o)||_H \leq \gamma$.

The state x can now be bounded as in the following theorem, whose proof is similar to the proof of lemma 3.1 in Banks[3].

Theorem 4.3.2 Suppose that, for $t_o \leq t \leq t_o + \delta$, $\delta > 0$, we have

$$1 - \beta\int_{t_o}^{t} g(t-s)ds > 0$$

and that

$$\left\{\sup_{t\varepsilon[t_o,t_o+\delta]} (1 - \beta\int_{t_o}^{t} g(t-s)ds)\right\}^{-1} \left\{\sup_{t\varepsilon[t_o,t_o+\delta]} \left(e^{-a(t-t_o)}||x_o||_H\right.\right.$$

$$\left.\left. +(k||B||_{\mathcal{L}(U,V)} + \alpha)\int_{t_o}^{t} g(t-s)ds\right)\right\} < \gamma.$$

Then, if $||x_o||_H < \gamma$, we have

$$||x(t;t_o,x_o)||_H < \gamma \text{ for } t \ \varepsilon \ [t_o,t_o + \delta]. \ \square$$

Note that

$$\int_{0}^{t} g(t-s)ds \leq \int_{0}^{\infty} g(t-s)ds \leq C(\frac{1}{1-\alpha} + \frac{e^{-a}}{a}),$$

and so we have

Corollary 4.3.3 Suppose that T is such that $1 - \beta C(\frac{1}{1-\alpha} + \frac{e^{-a}}{a}) > 0$

and that

$$\eta \triangleq \gamma - C(\frac{1}{1-\alpha} + \frac{e^{-a}}{a})(\gamma\beta + k||B||_{\mathcal{L}(U,V)} + \alpha) > 0.$$

Then, if $||x_o||_H < \eta$ we have

$$||x(t;t_o,x_o)||_H < \gamma, \text{ for } t \ \varepsilon \ [t_o,T]. \ \square$$

It follows that the system (4.2) with bounded controls for which the conditions of corollary 4.3.3 hold can never be (approximately) controllable. In fact the controlled states must always be less that γ for sufficiently small initial conditions.

Example 4.3.4 We return to the system

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} + \lambda\phi - \rho\phi^2, \ \rho, \lambda \geq 0, \ \lambda < \pi^2, \ \phi(0,t)=\phi(1,t)=0 \qquad (4.6)$$

and consider the nonlinear control system

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \lambda\phi - \rho\phi^2 + \delta(x-x_1)u^2 \qquad (4.7)$$

(i.e. point control at $x=x_1$). As before, let S(t) denote the semigroup generated by the system (4.6), and let T(t) denote the linear semigroup generated by

$$\bar{A}\phi = \frac{\partial^2\phi}{\partial x^2} + \lambda\phi.$$

Then we have

$$||T(t)\phi||_{L^2([0,1])} \le \frac{C_2}{t^{\frac{1}{4}+\frac{1}{2}\varepsilon}} e^{-(\pi^2-\lambda)t} ||\phi||_{H^{-\frac{1}{2}-\varepsilon}([0,1])} ,$$

for any $\varepsilon > 0$. Now, by (3.5) we have

$$A(t)\psi \triangleq (\mathfrak{J}A(S(t)\phi_o))\psi = \frac{d^2\psi}{dx^2} + \lambda\phi - 2\rho(S(t)\phi_o).\psi$$

and so

$$||(A(t) - \bar{A})\xi|| = 2\rho\ ||S(t)\phi_o||_{C([0,1])} ||\xi||_{L^2([0,1])} \triangleq C_1 ||\xi||_{L^2}$$

for $\phi_o \in D_d(A)$. We let

$$V = H^{-\frac{1}{2}-\varepsilon}([0,1]) \ , \quad U = R$$

and then

$$\psi(x,u,t) = \delta(x-x_1)u^2 \ : U \to V$$

has norm

$$||\psi(x,u,t)||_V \le ||\delta||_{H^{-\frac{1}{2}-\varepsilon}([0,1])} k^2 \triangleq \alpha,$$

if $|u| \lesssim k$. ($\beta = 0$). Let $C(=C(C_1,C_2,\alpha,T))$ be the constant determined as above. Then by corollary 4.3.3 it follows that the state of the system (4.7) is bounded by γ (for any $\gamma \ge 0$) on $[0,T]$ provided $\eta = \gamma - C\alpha\bar{\varepsilon} > 0$ ($\bar{\varepsilon} \triangleq (\frac{3}{4} - \frac{1}{2}\varepsilon)^{-1} + e^{-(\pi^2-\lambda)}/(\pi^2 - \lambda)$) and $||\phi_o||_{L^2([0,1])} \le \eta$.

5. Conclusions

We have considered in this paper the nonlinear variation of constants formula and its application to two aspects of infinite dimensional systems theory. In view of the immense importance of the linear version of this formula, it is to be expected that the nonlinear generalisation should also prove to be important for nonlinear systems. The utility of the integrated form of a nonlinearly perturbed system is however somewhat restricted by the fact that the variational equation generates an evolution operator $\Phi(t,t_o;x_o)$ which depends on x_o (and which in the linear case is just the semigroup $T(t)$). When Φ appears in the integrated form of the equation

$$\dot{y} = Ay + f(t) \tag{5.1}$$

where A is nonlinear, this Φ dependence on x_o appears in terms of the solution y, i.e.

$$y(t) = S(t)y_o + \int_o^t \Phi(t,s;y(s))f(s)ds \tag{5.2}$$

where S is the semigroup generated by A. (Compare this with the linear version

$$y(t) = T(t)y_o + \int_o^t T(t-s)f(s)ds).$$

We therefore obtain only a nonlinear integral equation in y. Also, we have shown that if y satisfies (5.1) then it also satisfies (5.2). The converse implication has not been shown and is more difficult.

In spite of these difficulties we have seen that the formula does have applications in stability and nonlinear control theory, and an example from nuclear reactor theory has been given. Finally, we leave the reader with the speculation that the formula should have other applications in system theory, possibly, for example, to nonlinear controllability.

6. References

1. V.M. Alekseev. An estimate for the perturbations of the solutions of ordinary differential equations (Russian). Vestnik Moskov. Univ. Ser. I. Mat. Mek. No. 2 (1961), 28-36.

2. S.P. Banks. A note on nonlinear observers, Int. J. Control, 34, (1981) 185-190.

3. S.P. Banks. Nonlinear perturbations of dynamical systems with bounded inputs. Int. J. Systems Sci., 12, (1981), 917-926.

4. S.P. Banks and P.C. Collingwood. Stability of nonlinearly interconnected systems and the small gain theorem, Int. J. Control, 30, (1979), 901-916.

5. V. Barbu. Nonlinear semigroups and differential equations in Banach spaces, Noordhoff Int. Publ., (1976).

6. F. Brauer. Perturbations of nonlinear systems of differential equations, J. Math. Analysis Appls, 14, (1966), 198-206.

7. E.A. Coddington and N. Levinson. Theory of ordinary differential equations, Mc.Graw-Hill, (1955).

8. N. Dunford and J.T. Schwartz. Linear operators, Interscience, (1963).

9. A. Friedman. Partial differential equations, Holt Rinehart and Winston, (1969).

10. D. Henry, Geometric theory of semilinear parabolic equations, Lecture notes in mathematics V840, Springer-Verlag, (1980).

11. W.E. Kastenberg. Stability analysis of nonlinear space dependent reactor kinetics, Adv. in Nucl. Science and Tech.,Acad. Press, 5, (1969).

12. K. Yosida, Functional analysis, Springer-Verlag, (1974).