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ROBUST CONTROLLER DESIGN FOR LINEAR DYNAMIC SYSTEMS USING APPROXIMATE MODELS

by

D. H. Owens, B.Sc.,A.R.C.S.,Ph.D.,A.F.I.M.A.,C.Eng.,M.I.E.E.

and

A. Chotai, B.Sc.,Ph.D.

Department of Control Engineering,
University of Sheffield,
Mappin Street, Sheffield S1 3JD

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Abstract

Controller design for scalar or multivariable systems whose models are unknown or highly complex are frequently based in practice upon the use of a highly simplified approximate plant model. In such circumstances it is vital to be able to quantify the degree of uncertainty to be expected from the use of such a model for prediction of closed-loop characteristics. It is shown how frequency-domain design techniques can be simply extended to incorporate information deduced from the observed differences between open-loop plant and approximate model step response to quantify this uncertainty and, in particular, to guarantee closed-loop stability and tracking of step demands. A modificiation of this analysis also yields the possibility of bounding the error in prediction of closed-loop transient performance. The approaches are all graphical in nature and are easily implemented in an interactive computer-aided-design mode.

1. Introduction

Closed-loop control systems design in practice makes frequent and deliberate use of approximate plant models either because the available model is regarded as being too complex for design work or because it is known to possess both structural and data uncertainties or simply because an accurate plant model is not available. An approximate model can be of arbitrary dynamic complexity ranging from, for example, a simple dead-time or first order model as considered by Åström[1] and Owens[2] respectively to more complex models deduced from identification experiments[3] or by analytic modelling techniques[4,5]. In general however, it is desirable that the model is low order with the consequent benefits of reduced computational requirements and the possibility of achieving simple, easily comprehended 'pencil and paper' designs[1,2] to form the basis for further refinement and understanding.

Classically allowance for modelling uncertainties has been attempted by ensuring adequate gain and phase margins for the approximate model in the frequency domain. It would clearly be advantageous, however, to be able to assess the effect of modelling errors in a more precise manner and, in particular, to be able to confidently assess both the stability and transient performance of the implemented feedback scheme in terms of the closed-loop dynamics of the approximate model. This problem is the subject of this paper. The concepts used are similar to those underlying the developing methods of robust stability theory (see eg. refs. (6),(7)) where modelling errors are regarded as being unstructured and characterised in the frequency domain by (assumed known) upper bounds on the singular values (i.e. 'multivariable gains') of the modelling error transfer function matrix at each frequency. Stability is then assessed by using these error bounds in graphical stability criteria. In contrast to this work we concentrate on the situation where the step response of the plant is reliably known and hence the modelling erros are well-known (i.e. structured) and characterized in the time-domain in terms of graphical properties of the error between the unit step responses of the plant and its approximate model. In this sense the paper is in the same spirit on the work of Åstrom[1] and a previous paper by the authors[8]. We consider here however, the more general case with the added bonus that a time-domain characterization of modelling errors generates an easily used graphical stability criterion together with graphical bounds on the deterioration in predicted closed-loop transient performance. Although the derived techniques require only graphical frequency response analysis and simulation calculations for implementation, the proofs of the underlying theorems require the use of functional analytic techniques similar to those found in nonlinear systems analysis as discussed by, for example, Holtzmann[9], Vidyasagar[10] and Harris et al[11] and in textbooks on numerical analysis[12,13]. In order to focus attention on the basic concepts and application of the ideas, the presentation is deliberately fairly informal with proofs only outlined. Detailed proofs can be found in refs. (14), (15).

The frequency domain version of the theory is described in section 2 with particular emphasis on the use of non-interacting system models where the techniques have an intepretation in terms of 'smudging' of inverse Nyquist array[16-19] plots based on error bounds deduced from graphical analysis of step response data. The widths of the resultant 'confidence bands' are related monotonically to the magnitude of the time-domain modelling error and should be regarded as design parameters in the sense that the design engineer has the option of choosing an accurate model to reduce uncertainty at the expense of design complexity or of accepting a highly simplified model (with consequent simplicity in the design process) but a degree of conservation in the design. Stability assessment in the time-domain using simulation data alone is described in section 3 and used to generate easily computed graphical bounds on performance deterioration due to the modelling approximations.

Finally, note that the techniques described require only a knowledge of a plant step response obtained from any source. They can hence be applied using step test data from a plant with unknown model in a similar manner to ref (1), (2).

2. Frequency Domain Design based on Approximate Models

We consider an ℓ-input/m-output, linear system G with output measurements generated from an m-input/m-output linear transducer F and the problem of the design of an m-input/ℓ-output linear forward path controller K to ensure the stability and acceptable dynamic characteristics of the feedback system of Fig. 1(a). All elements are continuous and linear, F is known, K is to be designed and G is either unknown or is regarded as unnecessarily or inconveniently complex for the design exercise under consideration. Following Åstrom[1] and previous work of the authors[2], assume that for each pair of indices (i ,j), the response $Y_{ij}(t)$ from zero initial conditions of the i^{th} output to a unit step in the j^{th} input has been reliably estimated from plant trials or simulations with an available complex model. It is convenient to collect this data together in the plant 'step-response matrix'

$$Y(t) = \begin{pmatrix} Y_{11}(t) & \cdots & Y_{1\ell}(t) \\ \vdots & & \vdots \\ Y_{m1}(t) & \cdots & Y_{m\ell}(t) \end{pmatrix}, \quad t \geq o \qquad (1)$$

Given the data Y(t), suppose that an approximate model G_A of the plant of desired simplicity is devised with a step response matrix $Y_A(t)$. Although the theory to follow trivially covers the choice $G_A = G$ (and hence $Y_A = Y$) obtained by exact identification of G and Y, we will concentrate on the more interesting case of $G_A \neq G$. This assumption more closely describes the practical situation where the design engineer wishes to use an approximate model to simplify conceptual and computational design problems. It is also consistent with the general observation that, although a real plant G (containing possibly series and feedback delays and other complex dynamic effects) can be accurately represented by a model G_A of the form typically used in engineering practice, the representation is rarely exact. This is very important if the model G_A is to be used for feedback controller design when we remember that small open-loop modelling errors can lead to large errors in prediction of closed-loop stability and performance.

Suppose that the model G_A is used to design a controller K to meet the required closed-loop stability and performance specifications applied to the approximating feedback system of Fig. 1(b). The problem now considered is how the 'error function'

$$E(t) \triangleq Y(t) - Y_A(t) = \left[E^{(1)}(t),\ldots,E^{(\ell)}(t)\right] \quad , \quad t \geq 0 \qquad (2)$$

with columns $E^{(j)}(t)$, $1 \leq j \leq \ell$, can be used during the design exercise with the approximate system to guarantee the stability of the real implemented scheme shown in Fig. 1(a).

It is assumed that both G and G_A can be described by input-output relations of the convolution from[10],

$$y(t) = \int_0^t H(t')u(t-t')dt' \ , \ y_A(t) = \int_0^t H_A(t')\, u(t-t')dt' \qquad (3)$$

where the impulse response matrices H and H_A have well-defined Laplace transforms. This assumption covers the use of rational transfer function matrix and state-space systems, but also admits, for example, some distributed systems with time-delays.

Finally, throught the paper, it is assumed that the modelling error

$G-G_A$ is stable. Formally we write this as a requirement that

$$\int_o^\infty ||H(t) - H_A(t)||_m \, dt < +\infty \tag{4}$$

and hence, noting that

$$Y(t) \equiv \int_o^t H(t')dt' \quad , \quad Y_A(t) \equiv \int_o^t H_A(t')dt' \quad , \tag{5}$$

that, for all $t \geq o$,

$$||E(t)||_m \leq \int_o^\infty ||H(t) - H_A(t)||_m dt < +\infty \tag{6}$$

(Note: $||.||_m = \max_i \sum_j |(.)_{ij}|$ is the matrix norm induced by the vector norm $||.||_m = \max_i |(.)_i|$ in C^m). This clearly requires the error $E(t)$ to be bounded or, equivalently, that both G and G_A are open-loop input-output stable in the L_∞ sense or that G_A models the 'unstable past' of G exactly.

2.1 Frequency Domain Stability Theory

If the controller K successfully stabilizes the model G_A we must have[10]

$$|\det (I_m + G_A(s)K(s)F(s))| \geq \alpha, \ \mathrm{Re}\ s > o \tag{7}$$

form some real $\alpha > o$. Clearly K will also stabilize the real plant G if

$$|\det (I_m + G(s)K(s)F(s))| \geq \beta, \ \mathrm{Re}\ s > o \tag{8}$$

form some real $\beta > o$. Using the identity $|I_m + M_1M_2| = |I_\ell + M_2M_1|$ valid for any $m \times \ell$ matrix M_1 and $\ell \times m$ matrix M_2, it is easily verified that

$$\det(I_m + GKF) \equiv \det (I_\ell + KFG)$$

$$\equiv \det(I_\ell + KFG_A) \det(I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A))$$

$$\equiv \det(I_m + G_AKF) \det (I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A)) \tag{9}$$

and hence the following simple result similar to those used in robustness theory[6,7] and nonlinear stability theory[11].

Lemma 1: If K stabilizes the model G_A, it will also stabilize the real plant G if

(a) the composite system GKF is both controllable and observable,

and (b) $$\sup_{s \in D} \gamma(s) < 1 \tag{10}$$

where D is the usual Nyquist 'infinite' semi-circle in the closed-right-half complex plane, $\gamma(s)$ is any available real-valued function satisfying

$$\gamma(s) \geq r(L(s)) \text{ for all } s \in D , \tag{11}$$

and L(s) is the $\ell x \ell$ transfer function matrix

$$L(s) \triangleq (I_\ell + K(s)F(s)G_A(s))^{-1} K(s)F(s)(G(s)-G_A(s)) \tag{12}$$

(Note: the spectral radius r(M) of an $\ell x \ell$ matrix M with eigenvalues m_1, $m_2, \ldots, m_\ell$ is[12,13]

$$r(M) = \max_{1 \leq i \leq \ell} |m_i|) \tag{13}$$

Proof: The controllability and observability assumptions ensure that input/output stability implies asymptotic stability. Note also that L(s) is analytic in Re s > o and hence this region can be replaced by the Nyquist D-contour in both (7) and (8). Equation (10) ensures that $|\det(I_\ell + L(s))| \geq \beta/\alpha$ for all s on D and some $\beta > o$ and the result follows by noting that (8) follows from (7) and (9).

Note that this result is a sufficient condition based on the single numerical measure γ of the effect of errors and hence, in general, contains a degree of conservatism in its predictions. This conservatism will increase depending upon our choice of $\gamma(s)$! The best that we can hope to achieve is to compute $G(s) - G_A(s)$ for $s \in D$ by taking the numerical Fourier transform of E(t), to hence evaluate L(s) 'exactly' for $s \in D$ and to choose $\gamma(s) \equiv r(L(s))$. For the purposes of this paper however, we will reject this possibility and adopt the philosophy of Åstrom[1] and the authors[2,8] by insisting that our choice of γ involves only computations that are simple, robust and preferably graphical in nature even at the expense of increased conservatism Any increased conservatism is not regarded as a problem here as, if the condition (10) of the lemma is violated, it is always possible to retrieve the situation by using the step data Y(t) to construct a more accurate model. The existence of a suitable model is trivially verified by noting that $L(s) \equiv 0$ if we choose $G_A = G$ and that (10) holds with $\gamma(s) \equiv r(L(s))$.

2.2 The Choice of $\gamma(s)$

In frequency-domain based robust control analyses[6,7], it is typical to choose $\gamma(s) = \bar{\sigma}((I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s))\ \ell_a(s)$ where $\ell_a(s)$ is assumed known on the D-contour and satisfies $\ell_a(s) \geq \bar{\sigma}(G(s)-G_A(s))$ for all s on D. The notation $\bar{\sigma}$ (M) denotes the largest singular value[6] of a square matrix M i.e. the positive square root of the spectral radius of M^*M. In this section however we concentrate on bounds that are easily computed from the time-domain data E(t).

It is convenient to introduce the partial ordering[12,13] on $n_1 \times n_2$ matrices by the relation

$$A \leq B \quad \text{iff } A_{ij} \leq B_{ij} \text{ for all } (i,j) \tag{14}$$

and define the 'absolute value' of an $n_1 \times n_2$ matrix to be the $n_1 \times n_2$ real matrix

$$||A||_P = \begin{pmatrix} |A_{11}| & \cdots & |A_{1n_2}| \\ \vdots & & \vdots \\ |A_{n_1 1}| & \cdots & |A_{n_1 n_2}| \end{pmatrix} \tag{15}$$

The importance of this construction lies in its 'norm-like' properties

$$||A||_P \geq 0 \tag{16}$$

$$||\alpha A||_P = |\alpha| \cdot ||A||_P \text{ for all complex scalars } \alpha \tag{17}$$

$$||A + B||_P \leq ||A||_P + ||B||_P \tag{18}$$

$$||AB||_P \leq ||A||_P \, ||B||_P \tag{19}$$

and its use in bounding of spectral radii using the relations

$$O \leq ||A||_P \leq B \Rightarrow r(A) \leq r(||A||_P) \leq r(B) \tag{20}$$

deduced from theorem 2.4.9 in ref. (12). In fact, relations (16),(19) and (20) immediately yield the following important result:

<u>Theorem 1</u>: The conclusions of lemma 1 hold true with γ satisfying

$$\gamma(s) \geq r(\tilde{L}(s)) \text{ for all } s \in D \tag{21}$$

where

$$\tilde{L}(s) \triangleq ||(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)||_P \Delta(s) \tag{22}$$

and $\Delta(s)$ is any available matrix-valued function satisfying

$$\Delta(s) \geq ||(G(s) - G_A(s)||_P \text{ for all } s \in D \tag{23}$$

The result is an applicable stability criterion in its own right given an available bound $\Delta(s)$. Note that (23) can be written in the form $\Delta_{ij}(s) \geq |(G(s) - G_A(s))_{ij}|$ and hence that Δ_{ij} is any available upper bound on the gain of the modelling error of the (i,j) element of G. It clearly therefore has an interpretation in terms of robust control similar to those in ref (6) but with spectral radii of absolute-values of matrices replacing their maximum singular values. Note that this framework is not necessarily more conservative than robust stability techniques as examples are easily constructed of matrices M such that $r(||M||_P) < \bar{\sigma}(M)$. Indeed, this framework may be more convenient as the modelling error $\Delta(s)$ is constructed by bounding the individual errors in $G - G_A$! This is particularly important in our development where we wish to relate $\Delta(s)$ to the time response data $E(t)$.

Remembering our philosophy of computational simplicity and the fact that any increased conservatism in the stability criterion can be offset by choosing a more accurate plant model, we concentrate in this paper on the construction of the simplest possible choice of Δ i.e. a frequency-independent bound. Using the identity

$$G(s) - G_A(s) = \int_o^\infty (H(t) - H_A(t))e^{-st}dt \tag{24}$$

the following result is easily proved by considering elements:

Proposition 1:

$$||(G(s) - G_A(s))||_P \leq \int_o^\infty ||(H(t) - H_A(t)||_P dt$$

$$\text{for all Re } s \geq o \tag{25}$$

Clearly the right-hand-side of (25) is a constant candidate for $\Delta(s)$. It is particularly useful for time-domain studies as its value can be deduced by visual inspection of the elements of the error $E(t)$ and application of the following result proved in Appendix 7.

Proposition 2: If $g \in L_1(O,T)$, d is a real scalar and

$$f(t) \stackrel{\Delta}{=} d + \int_o^t g(t')dt' \tag{26}$$

is bounded and continuous on the infinite open interval $o < t < +\infty$ with local maxima or minima at times $t_1 < t_2 < \ldots$ satisfying $\sup t_j = +\infty$ in the extended half-line $t > o$. Then, taking $t_o = O$. we have

$$|d| + \int_o^T |g(t)|dt = N_T(f) \tag{27}$$

where

$$N_T(f) \triangleq |f(o+)| + \sum_{k=1}^{k^*} |f(t_k) - f(t_{k-1})| + |f(T) - f(t_{k^*})| \qquad (28)$$

and

$$N_\infty(f) \triangleq \sup_{T \geq 0} N_T(f) \qquad (29)$$

where k^* is the largest integer k such that $t_k < T$.

(Notes: (i) $N_T(f)$ is simply the norm of f regarded as a function of bounded variation[20] on the half-open interval $0 < t \leq T$

(ii) for each f, $N_T(f)$ is monotonically increasing and continuous as a function of T so that $N_\infty(f)$ can be obtained as the <u>limit</u> as $T \to +\infty$ of $N_T(f)$.

(iii) $N_T(f)$ is easily computed from graphical inspection of f(t) in the manner illustrated in Fig. 2.

(iv) Noting that $|N_\infty(f) - N_T(f)| \to 0$ as $T \to \infty$ it follows[14] that $N_\infty(f)$ can be accurately estimated using data on a long enough time interval $o < t \leq T$. On such an interval continuity of $N_T(f)$ as a function of the stationary points $t_1, t_2, \ldots$ also indicates[14] that it is insensitive to errors in their estimation.

(v) If f(t) is contaminated by noise n(t), $N_T(f)$ must be evaluated by inspection of f(t) + n(t). If however, the signal to noise ratio is sufficiently large, the stationary points of f(t) can be estimated fairly accurately by visual smoothing of the recorded response f+n. Clearly, together with (iv), we must conclude that estimation of $N_\infty(f)$ is a robust operation in many practical situations).

The importance of proposition 2 lies in its application to the error $E(t) = \int_{o}^{t} (H(t')-H_A(t')dt'$. More precisely, applying the result to the elements of E(t), defining

$$N_T^P(E) = \begin{pmatrix} N_T(E_{11}) & \cdots & N_T(E_{1\ell}) \\ \vdots & & \vdots \\ N_T(E_{m1}) & \cdots & N_T(E_{m\ell}) \end{pmatrix} \qquad (30)$$

and using proposition 1 yields the important result.

Lemma 2:

$$||G(s) - G_A(s)||_P \leq N_\infty^P(E) \quad , \quad \text{for all Re } s \geq o \qquad (31)$$

2.3 A Stability Criterion based on Approximate Models

The following result follows directly from the discussion in Section 2.2 by choosing $\Delta(s) \equiv N_\infty^P(E)$ in theorem 1 and represents a substantial generalization of theorem 1 in ref. (8).

Theorem 2: If K stabilizes the model G_A, it will also stabilize the real plant G if

(a) the composite system GKF is both controllable and observable and

(b) $\lambda_o \triangleq \sup_{s \in D} \gamma(s) < 1$ (31)

where $\gamma(s)$ is any convenient real-valued function satisfying

$$\gamma(s) \geq r(||(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s))||_P N_\infty^P(E))$$

$$\text{for all } s \in D \qquad (32)$$

The application of the result proceeds by verification of the conditions of the theorem and could proceed in a similar manner to the techniques of ref (8):

Step 1: Obtain the plant step response matrix $Y(t)$ from plant trials or from simulations using an available plant model.

Step 2: Choose an approximate plant model G_A with the property that $G-G_A$ is stable in the input/output sense. If the plant G is stable (the most common case) this reduces to ensuring that G_A is stable. Calculate $Y_A(t)$, $E(t)$ and hence estimate $N_\infty^P(E)$.

Step 3: Design the controller K for G_A, by any means available, to obtain the required stability and performance characteristics from the approximating feedback system.

Step 4: Construct a convenient bound $\gamma(s)$ for the spectral radius of $||(I + KFG_A)^{-1}KF||_P \; N_\infty^P(E)$ on the D contour and check the validity of (31) at a selection of frequency points covering the bandwidth of interest. If $\lambda_o \geq 1$, the given approximate model is not accurate enough to provide stability predictions for the implemented scheme. Return to step 2 to construct a more accurate version, or reduce control gains in an attempt to reduce $||(I + KFG_A)^{-1}KF||_P$ and hence λ_o (see section 2.5).

Step 5: Check that GKF is both controllable and observable. As pointed out in ref. (8), this problem requires some structural information about the plant. If, for example, $m = \ell$, $F(s) \equiv I_m$, $K(s)$ is a proportional plus integral controller and the plant G is rational it essentially reduces to the requirement that $G(s)$ is controllable and observable and has no zero at the origin of the complex plane. This can be checked if the plant is stable by checking that $\lim_{t\to+\infty} Y(t)$ is nonsingular. More generally the problem is more complex however! If a plant model is available, it can be checked numerically but, if a model of the plant is not available, the designer could rely on the fact that controllability and observability are generic and hence likely to be present. In all cases, the results guarantee input/output stability however.

Steps 1-3 represent the well-used practical approach to design. Steps 4 and 5 are the important additions derived in this paper. They are not designed to produce <u>direct</u> insight into the design of K being present as a means only of guaranteeing the stability of the implemented control scheme. They can produce insight indirectly however, by releasing the simple model G_A as a vehicle for the design.

The choice of γ depends upon the simplicity of computation required by the designer and the conservatism tolerable for the given model G_A. There are a large number of candidates e.g.

$$\gamma(s) = r(||(I + K(s)F(s)G_A(s))^{-1}K(s)F(s)||_P N_\infty^P(E)) \tag{33a}$$

$$\gamma(s) = r(\sup_{s\in D}\{||(I + KFG_A)^{-1}KF||_P\ N_\infty^P(E)\}) \tag{33b}$$

$$\gamma(s) = r(\{\sup_{s\in D}\ ||(I + KFG_A)^{-1}KF||_P\}N_\infty^P(E)) \tag{33c}$$

(Note: the supremum in (33b) and (33c) is interpreted with respect to the partial ordering on the space of $\ell x\ell$ matrices)

$$\gamma(s) = \bar{\sigma}(||(I + K(s)F(s)G_A(s))^{-1}K(s)F(s)||_P N_\infty^P(E)) \tag{33d}$$

$$\gamma(s) = ||(||(I + KFG_A)^{-1}KF||_P\ N_\infty^P(E))||_m \tag{33e}$$

$$\gamma(s) = ||(I + KFG_A)^{-1}KF||_m \cdot ||N_\infty^P(E)||_m \tag{33f}$$

Note that (33a), (33d) required repeated eigenvalue calculation whereas (33e) and (33f) avoid this complication. The choice of (33b) or (33c) requires the evaluation of the maxima of scalar frequency response with a single eigenvalue calculation.

2.4 <u>Graphical Stability Criteria using Non-interacting Models</u>

If $m = \ell$ and it is conjectured that plant interaction is, on physical grounds, small enough to be neglected during the design, the conceptual and design simplicity of a non-interacting model G_A make this possibility very appealing. In such a case theorem 2 has a useful graphical interpretation similar to that of the inverse Nyquist array technique[16-19] and ref (8). The result applies directly to the single-input/single-output case by setting $m = \ell = 1$.

Theorem 3: The conclusions of theorem 2 are valid if $m = \ell$, G_A, K and F are diagonal (non-interacting) systems and (31) is replaced by the conditions

(i) the inequality

$$\lim_{\substack{\mathrm{Re}\, s \geq o \\ |s| \to \infty}} \sup \left| \frac{K_{kk}(s)F_{kk}(s)}{1 + K_{kk}(s)F_{kk}(s)(G_A(s))_{kk}} \right| < \frac{1}{\sum_{j=1}^{m} N_\infty(E_{kj})} \tag{34}$$

is satisfied for $1 \leq k \leq m$ and,

(ii) the 'confidence bands' generated by plotting the inverse Nyquist locus of $(G_A)_{kk}K_{kk}F_{kk}$, $1 \leq k \leq m$ for $s = i\omega$, $\omega \geq o$ with superimposed 'confidence circles' at each point of radius

$$r_k(i\omega) \triangleq |(G_A(s))^{-1}_{kk}| \sum_{j=1}^{m} N_\infty(E_{kj}) \tag{35}$$

does not contain or touch the (-1,O) point of the complex plane.

(Note: (1) A graphical interpretation of condition (ii) is given in Fig. 3.

(2) The radii of the confidence circles are 'proportional' to the total variation of the modelling error E(t) and are zero if the model is exact. They increase as the chosen modelling error increases in the time-domain. If they are so large as to violate condition (ii), an improvement is always possible by using a more accurate model).

Proof: Applying theorem 2 with γ given by (33e) yields (34) by considering the 'infinite semi-circular' part of D. The imaginary axis component then requires that

$$\left|1 + ((G_A(i\omega))_k K_{kk}(i\omega)F_{kk}(i\omega))^{-1}\right| > \left|(G_A(i\omega))^{-1}_{kk}\right| \sum_{j=1}^{m} N_\infty(E_{kj}) \quad (36)$$

for $-\infty < \omega < +\infty$ and $1 \leq k \leq m$. This is simply condition (ii) as the contribution from the negative imaginary axis is simply the complex conjugate of that from the positive imaginary axis. The result is hence proved.

2.5 Discussion and Robustness Analysis

The application of the results described above is illustrated on the next section and is similar to that described in ref. (8). It is not possible to make any more precise statements unless specific forms of model and/or controller are assumed. For example, if integral action is not included in K, the following result indicates that all the preceeding theorems are of the 'low-gain' type[(21)]:

Proposition 3: If G, G_A, K and F are stable, then the conditions of theorems 1 - 3 are satisfied for all controllers K of 'low-enough' gain.

'Proof': G - G_A is stable, all stable low-gain controllers will stabilize a stable plant and it is trivially verified that $\gamma(s) \to 0$ as the control gains become 'small'.

If integral action is included however, the controller has high low-frequency gains. In such a case the above arguments fail. The following result indicates that the possibility of including integral action is related to the magnitude of the modelling error and the steady state performance of the model G_A:

Proposition 4: Let $m = \ell$ and G, G_A, F and $sK(s)$ be stable, G_A have no zero at the point $s = o$ and $\lim_{|s|\to\infty} sK(s)$ be finite and nonsingular. Then a necessary condition for the existence of a function γ satisfying (31) is that

$$r(||G_A^{-1}(o)||_P N_\infty^P(E)) < 1 \tag{37}$$

(Note: The condition on $K(s)$ represents many practical situations e.g. the choice of $K(s) = K_1 + s^{-1}K_2$ with K_2 nonsingular).

Proof; Simply let $s \to o$ in (32) and use (31) to require that $\gamma(o) < 1$.

Next, we note that designs based on the results are robust in the sense that, if the plant G changes over a period of time to a new plant $\tilde{G}$ with step response matrix $\tilde{Y}$, stability of the implemented scheme will be retained provided $\tilde{G} - G$ is 'small enough'. This is obvious from the observation that 'small' changes in E produce 'small' change in $N_\infty^P(E)$ and hence in the spectral radius in (32). A useful computable measure of the permissible $\tilde{G} - G$ is not obtainable for all choices of γ. Suppose therefore that (33f) is used. The following result follows from the observation that $N_\infty^P(\tilde{Y}-Y_A) \leq N_\infty^P(\tilde{Y}-Y) + N_\infty^P(E)$ using an argument similar to that used in Proposition 7 of ref. (8).

Proposition 5: If the conditions of theorem 2 hold with γ given by (33f) then the controller K will also stabilize all plants $\tilde{G}$ such that $\tilde{G} - G$ is stable, $\tilde{G}KF$ is both controllable and observable and

$$||N_\infty^P(\tilde{Y}-Y)||_m < \frac{1 - \lambda_o}{\sup_{s\in D} ||(I + KFG_A)^{-1}KF||_m} \tag{38}$$

Finally note that when E(t) is monotonic[8], it is trivially verified that $N_T^P(E) = ||E(T)||_P$ and hence that $N_\infty^P(E) = ||E(\infty)||_P$. Choosing γ by (33e) or (33f) then yields previously published results[8]. The work described here is a complete generalization of that case.

2.6 Illustrative Example

To illustrate the application of the above theory is an elementary but representative situation, suppose that a single-input/single-output plant has an unknown transfer function

$$G(s) = \frac{4}{(s^2 + 2s + 4)(s + 1)} \tag{39}$$

and that plant step tests yield the step response Y(t) illustrated in Fig. 4(a). Following a commonly used practice, <u>visual</u> inspection of this response can be used to fit a delay-lag model of the form

$$G_A(s) = \frac{e^{-s\,0.6}}{1+s} \tag{40}$$

with step response again illustrated in Fig. 4(a). The error $E = Y - Y_A$ is clearly stable and shown in Fig. 4(b). The required parameter $N_\infty^P(E) = N_\infty(E)$ (as $m = \ell = 1$) is obtained graphically by the procedure implicit in Proposition 2 to be

$$N_\infty^P(E) = 0.45 \tag{41}$$

The next step is the choice of a proportional plus integral unity feedback controller K for G_A of the form $K(s) = K_1 + s^{-1}K_2$, $K_1 > 0$, $K_2 > 0$. Proposition 4 indicates that the model is capable of including integral action as $r\,(||G_A^{-1}(o)||_P N_\infty^P(E)) = |G_A^{-1}(o)|\; N_\infty^P(E) = 0.45 < 1$. A preliminary gain bound is obtained by invoking condition (i) of theorem 3 i.e.

$$\lim_{\substack{Res>o \\ |s|\to\infty}} \left| \frac{K}{1+KG_A} \right| = |K_1| < \frac{1}{N_\infty(E)} = 2.22 \tag{42}$$

(Note: Taking, for simplicity, the case of $K_2 = 0$, the pessimism in this result is seen by comparing (42) with the real stability range $0 \leq K_1 < 3.5$ As is seen below, this pessimism is not a practical problem in this case. If it were however, we could remove it by attempting to obtain a better model G_A!). Consideration of G_A leads to the choice of $K_1 = 1.0$, $K_2 = 0.5$ to stabilize G_A and produce the acceptable closed-loop characteristic indicated in Fig. 5. This choice clearly satisfies (42) and the inverse Nyquist plot of $G_AKF = G_AK$ with superimposed confidence circles shown in Fig. 6 indicates that the (-1,0) point does not lie in or on the confidence band. In fact all the conditions of theorem 3 are satisfied provided that GK is both controllable and observable when we can conclude that the given controller stabilizes the real plant (39). The only ways that this final condition can be violated is (a) the plant G is uncontrollable and/or unobservable (b) the plant has a zero at s = o or (c) the plant has a pole at $s = -K_2/K_1 = -0.5$. If (a) holds then we can do nothing using control action. Clearly (b) is not valid as $\lim_{t\to\infty} Y(t) = 1 \neq 0$ and even if (c) accidentally did hold it simply means that the closed-loop system has a stable uncontrollable and/or unobservable mode.

Finally, for comparative purposes, the closed-loop response of the real feedback scheme is also shown on Fig. 5. The success of the design is indicated by the similar stability and overall dynamic characteristics and identical steady-states of the real and approximating feedback schemes.

3. Time Domain Techniques based on Approximate Models

The procedures outlined in section 2 have a striking similarity to well-known classical procedures but suffer from the general problem of frequency domain techniques i.e. it is difficult to make predictions about the details of the closed-loop transient performance. In particular, it is not possible to make confident predictions about the response characteristics of the approximating feedback scheme except that it is stable, and, if integrators are present, tracks step demands exactly and rejects constant disturbances. Any design technique capable of resolving this problem must, intuitively, rely heavily on time-domain calculations. The general form of such a design aid is described in this section[14,15]. The use of time-domain data in stability assessment is unusual but it may have a number of advantages over frequency domain calculations, particularly in the multivariable case. For example, checking condition (31) can require the calculation of the inverse of the $\ell x \ell$ matrix $I + KFG_A$ at a large number of frequency points. In contrast, the corresponding time-domain result (see, for example, theorem 4) requires only system simulations and one eigenvalue calculation. Similar techniques can also be used to bound the transient performance deterioration due to the approximation used.

3.1 Mathematical Background

The proofs of the results use an extended version of the contraction mapping theorem[9-13]. Let X be a Banach space (we will take $X = L_\infty(o,t)$ in the following sections) and X^d be the d^{th} Cartesian product of X regarded as the linear vector space of columns $x = (x_1, x_2, \ldots, x_d)^T$ of elements of X. The absolute value of $x \in X^d$ is denoted

$$||x||_P = (||x_1||, \ldots, ||x_d||)^T \in R^d \quad (43)$$

where $||\cdot||$ denotes the norm in X. If L is a bounded linear operator mapping X^{d_2} into X^{d_1}, it can be represented as the operator $y = Lx$ with $y_i = \sum_i L_{ij} x_j$ and L_{ij} bounded, linear operators in X. The absolute value of L is defined to be

$$||L||_P \triangleq \begin{pmatrix} ||L_{11}|| & \cdots & ||L_{1d_2}|| \\ \vdots & & \vdots \\ ||L_{d_1 1}|| & \cdots & ||L_{d_1 d_2}|| \end{pmatrix} \quad (44)$$

where $||.||$ is the operator norm induced by the vector norm in X. It is easily verified that $y = Lu$ implies $||y||_P \leq ||L||_P\, ||u||_P$ and that, if $||y||_P \leq M\, ||u||_P$ for all u, then $||L||_P \leq M$.

Let W be a mapping of X^d into itself, then[12] W is a global P-contraction if there exists a real dxd matrix $P \geq 0$ with the property that $r\,(P) < 1$, and, for all $x,y \in X^d$,

$$||W(x) - W(y)||_P \leq P\, ||x-y||_P \tag{45}$$

For example, if W is the map $x \to Lx + x_o$ with $x_o \in X^d$ and L linear and bounded, W satisfies condition (45) with $P = ||L||_P$ and hence is a P-contraction if $r(||L||_P) < 1$. However, if (45) holds for any other P, we have $||L||_P \leq P$ and hence $r(||L||_P) \leq r(P) < 1$. Clearly the condition $r(||L||_P) < 1$ is both necessary and sufficient for W to be a P-contraction.

We now state a generalized contraction result (ref (12), p. 433):

Lemma 3: Let W be a global P-contraction in X^d. Then, for any $x^{(o)} \in X^d$, the sequence $x^{(k+1)} = W(x^{(k)})$, $k \geq o$, converges to the unique solution x of the equation $x = W(x)$ in X^d. Moreover, we have the error estimate

$$||x-x^{(1)}||_P \leq (I - P)^{-1}P\, ||x^{(1)} - x^{(o)}||_P \tag{46}$$

(Note: the proof in ref(9) is givenfor X = R but it carries through with no change to an arbitrary Banach space. It also follows from Section 12.1 of ref (13)).

Taking norms in R^d yields the following result:

Corollary: If $||P||_m < 1$, then

$$\max_k\, ||x_k - x_k^{(1)}|| \leq \frac{||P||_m}{1 - ||P||_m} \max_k\, ||x_k^{(1)} - x_k^{(o)}|| \tag{47}$$

To complete this section we concentrate on the case of $X = L_\infty(o,t)$ and bounded linear convolutions mappings L of X^d into itself. Suppose that the elements of L (denoted L_{ij}) have the structure

$$(L_{ij}x_j)(t') = d_{ij}x_j(t') + \int_o^{t'} g_{ij}(t'-t'')x_j(t'')dt'' \tag{48}$$

then it is well known[10] that L_{ij} has induced norm

$$||L_{ij}||_\infty = |d_{ij}| + \int_o^t |g_{ij}(t')|dt' \tag{49}$$

in $L_\infty(o,t)$. Comparing with Proposition 2 immediately yields the result:

Lemma 4: If L is a bounded linear convolution map of $L_\infty^d(o,t)$ into itself with elements of the proper form (48), then, denoting the step response matrix of L by $Q(t')$ with elements $Q_{ij}(t')$, we have $||L_{ij}||_\infty = N_t(Q_{ij})$, $1 \leq ij \leq d$, and hence $||L||_P = N_t^P(Q)$.

The value of this result lies in the ability to compute absolute values of system operators from step response data with no need to use, or have available, a detailed system model. This fact is exploited in the proof of the following results.

3.2 Time Domain Stability Test

The following result is a time domain version of theorem 2 that replaces the frequency response calculations of condition (b) of that result by a single eigenvalue calculation based on simulation tests and the time-domain data $E(t)$. A proof is given in Appendix 8.

Theorem 4: Suppose that the controller K stabilizes the model G_A and that simulations are undertaken to reliably calculate the matrix

$$W_A(t) = \left[W_A^{(1)}(t), \; \ldots\ldots, \; W_A^{(\ell)}(t)\right] \tag{50}$$

where $W_A^{(j)}(t)$ is the response from zero initial conditions of the system $(I + KFG_A)^{-1}KF$ to the input vector $E^{(j)}(t)$ defined by (2). Then the controller K will stabilize the real plant G if

(a) the composite system GKF is both controllable and observable and

(b) the following inequality holds

$$\gamma < 1 \tag{51}$$

where γ is any available upper bound for $r(N_\infty^P(W_A))$.

The application of the result is computationally straightforward involving only ℓ simulations of the known (and normally low-order) system $(I + KFG_A)^{-1}KF$ (interpreted as a feedback configuration with forward path KF and feedback G_A) in response to assumed known plant data $E^{(j)}(t)$, $1 \leq j \leq \ell$, followed by graphical analysis of the responses to estimate the real, constant matrix $N_\infty^P(W_A)$ and calculation of its eigenvalues. If $\ell = 1$ then the eigenvalue calculation is trivial and, even if $\ell > 1$, it can be removed if it is possible to estimate an upper bound on the spectral radius that satisfies (51).

The application of the result is illustrated in section 3.5. It is useful at this stage to comment on the reliability of the calculations required to check (51). If E(t) is known accurately (either because the plant model G is known or because identification experiments have been undertaken to estimate Y(t) reliably) then $W_A(t)$ (and hence $N_\infty^P(W_A)$) is computed easily and accurately. If however, the available data E(t) contains noise, so will $W_A(t)$ and some care most be taken as unthinking calculation of the total variation leads (in theory) to infinite values. In general it is probably best to filter noise from the step data before undertaking these calculations. The only situation where this can be avoided has been discussed in note (v) following proposition 2 i.e. the situation where the signal to noise ratio is good enough to allow visual smoothing of the data.

3.3 Performance Assessment: The general case

In general it is unfortunately true that stability and transient performance of the approximating scheme does not necessarily imply acceptable transient performance of the implemented scheme. During the design of K for G_A, it is assumed that the input and output vectors u_A and y_A respectively in response to a given demand r satisfy performance specifications. It is clearly of value therefore to be able to estimate the corresponding inputs and outputs u and y respectively of the real feedback scheme to check that performance deterioration due to the approximation will still lead to acceptable dynamic characteristics. This is particularly obvious if the plant model is unknown or subject to uncertainty but such considerations are also of value if the plant model is known provided that the estimates are easily obtained. The following result characterizes the potential deterioration in input transient magnitudes and is proved in Appendix 9.

Theorem 5: Suppose that the conditions of theorem 4 hold and that

(a) $u^{(o)}(t)$ is the response from zero initial conditions of a freely chosen $\ell \times \ell$ stable, proper convolution system H_2 to the step $\hat{u}(t) = \beta$, $t \geq o$

(b) $\zeta(t)$ is the $\ell \times 1$ vector computed from the convolution

$$\zeta(t) = -\left(\int_o^t W_A(t-t')H_{H_2}(t')dt'\right)\beta \quad , \quad t \geq o \tag{52}$$

where H_{H_2} is the impulse response matrix of H_2,

(c) $u_A(t)$ is the input response from zero initial conditions of the approximating feedback scheme to the demand signal r(t) with

$$u^{(1)}(t) \triangleq u_A(t) + \zeta(t) \quad , \quad t \geq o \tag{53}$$

and

(d)

$$\varepsilon(t) = \begin{pmatrix} \varepsilon_1(t) \\ \cdot \\ \cdot \\ \varepsilon_\ell(t) \end{pmatrix} \triangleq (I_\ell - N_t^P(W_A))^{-1} N_t^P(W_A) \sup_{o \le t' \le t} ||u^{(1)}(t') - u^{(o)}(t')||_P \tag{54}$$

Then the input response of the real feedback scheme from zero initial conditions to the demand r(t) satisfies the bound

$$|u_j(t) - u_j^{(1)}(t)| \le \varepsilon_j(t) \quad , \quad 1 \le j \le \ell, \quad t \ge o \tag{55}$$

The graphical interpretation of (55) is simply that the input $u_j(t)$ lies in the region between the curves $u_j^{(1)}(t) \pm \varepsilon_j(t)$. All the calculations involved in the estimation of $\varepsilon_j(t)$ are also numerically well-conditioned if the signal to noise ratio in E is large. More precisely, $W_A(t)$ is then well-defined and given a choice of H_2 (see below) $u^{(o)}$ is obtained by simulation and $\zeta(t)$ by numerical convolution. The signal $u^{(1)}(t)$ is then trivially computed and $\varepsilon(t)$ follows in a straightforward manner. Many of these calculations can be further simplified. For example, noting that $N_t^P(W_A) \le N_\infty^P(W_A)$ and hence that $||N_t^P(W_A)||_m \le ||N_\infty^P(W_A)||_m$, $t \ge o$, the following result identifies one situation when the inversions necessary for the evaluation of ε can be avoided:

Corollary 5.1: Under the conditions of theorem 5, suppose also that $||N_\infty^P(W_A)||_m < 1$ and define $\lambda_1(t) = ||N_t^P(W_A)||_m$. Then

$$|u_j(t) - u_j^{(1)}(t)| \le \frac{\lambda_1(t)}{1 - \lambda_1(t)} \max_{o \le t' \le t} ||(u^{(1)}(t') - u^{(o)}(t')||_m$$

$$1 \le j \le \ell \tag{56}$$

Proof: Write (55) as $|u_j(t) - u_j^{(1)}(t)| \le ||\varepsilon(t)||_m$ and take norms in equation (54).

The choice of H_2 and β can also lead to considerable simplification although usually at the expense of increased conservatism in the bounds $\varepsilon_j(t)$. Noting that $u^{(o)} = H_2\hat{u}$ is the first guess at u(t) in a successive approximation scheme (see Appendix 9) the following choices immediately suggest themselves:

(1) The choice of $H_2 = 0$ leads to $\zeta(t) \equiv 0$ and $u^{(1)}(t) \equiv u_A(t)$ without the need for numerical calculation of these quantities.

(2) The choice of H_2 and β so that $u^{(o)}(t) = H_2\beta$ is constant and 'representative' of the magnitude of $u_A(t)$ leads to the easily computed form .

$$\zeta(t) = W(t)\ u^{(o)}(t) \tag{57}$$

(3) The choice of $H_2 = (I_\ell + KFG_A)^{-1}KH_o$ where H_o is a stable proper convolution system will lead to $u^{(o)}(t) \equiv u_A(t)$ whenever the demand r(t) is the response of H_o from zero initial conditions to the step input $\hat{r}(t) \equiv \beta$, $t > o$.

(Note: As u_A is our best available estimate of u, the choice (3) is likely to be least conservative).

Although input estimates are useful to avoid excessive input magnitudes, output estimates are also important. In general, however, such estimates do not fall easily from the theory due to a technical problem[15] associated with non-commutation of multivariable convoltuion operators that can preclude the use of the theory for the assessment of controllers containing integral action. This is not a problem in the scalar case (see section (3.4)) but the best general result for multivariable systems appears to be the following:

Corollary 5.2: With the conditions of theorem 5, suppose also that G is stable. Then

$$||y(t) - y^{(1)}(t)||_P \le N_t^P(Y)\ \varepsilon(t) + N_t^P(E) \sup_{o \le t' \le t} ||u^{(1)}(t')||_P \tag{58}$$

where $y^{(1)}(t)$ is the response of G_A from zero initial conditions to $u^{(1)}(t)$.

This result is proved in Appendix 10. It is expected to be somewhat conservative as it is deduced from (55) using norm inequalities. A more accurate estimate is available[15] when K is itself stable but the result cannot always cope with the important practical need to include integral action. It is therefore excluded for brevity.

Finally we note another simplification of the estimate (34) by reducing the number of inversions required (see Appendix 11 for a proof):

Corollary 5.3: If the conditions of theorem 5 hold, the bound (55) holds with $\varepsilon(t)$ replaced by the estimate $\varepsilon^{\mu}(t)$ obtained as in (55) with $N_t^P(W_A)$ replaced by $N_t^{P,\mu}(W_A)$ defined by

$$N_t^{P,\mu}(W_A) \triangleq \begin{pmatrix} N_{\mu_{11}(t)}(W_A) & \cdots & N_{\mu_{1\ell}(t)}(W_A) \\ \vdots & & \vdots \\ N_{\mu_{\ell 1}(t)}(W_A) & \cdots & N_{\mu_{\ell\ell}(t)}(W_A) \end{pmatrix} \tag{59}$$

where, for each pair of indices (i,j), μ_{ij} is some function satisfying $\mu_{ij}(t) \geq t$ for all $t \geq o$.

The simplification made possible by this result can be illustrated by choosing $\mu_{ij}(t) = +\infty$ for all $t \geq o$ when $N_t^{P,\mu}(W_A) = N_{\infty}^P(W_A)$ for all $t \geq o$ and only one matrix inversion is required. Other choices include the choice $\mu_{ij}(t) = t_{ijk}$ for $t_{ijk-1} \leq t < t_{ijk}$ where t_{ijk}, $k \geq o$, are the stationary points of $(W_A(t))_{ij}$. This choice only requires estimates of the value and position of stationary points of signals. This could be of particular importance if the signal has some noise content.

3.4 Stability and Performance Assessment for Single-input/Single-output Systems

Although both theorems 4 and 5 apply to the scalar case, it is possible to obtain better results on output performance as follows (see Appendix 12):

Theorem 6: Suppose that $m = \ell = 1$, that the controller K stabilizes the model G_A and that the response $W_A(t)$ of the system $(1 + KFG_A)^{-1}KF$ from zero initial conditions to the open-loop modelling error E(t) has been computed. Then the controller K will stabilise the real plant G if

(a) the composite system GKF is controllable and observable and

(b) the following inequality holds

$$N_\infty(W_A) < 1 \tag{60}$$

Under these conditions, let $y^{(o)}(t)$ be the response of a stable, proper convolution system H_1 from zero initial conditions to a unit step demand signal r(t). Suppose also that the response $\eta(t)$ of the system $(1 + KFG_A)^{-1}K(1-FH_1)$ to the error E(t) has been obtained by simulation and that

$$y^{(1)}(t) \triangleq y_A(t) + \eta(t) \tag{61}$$

where $y_A(t)$ is the response from zero initial conditions of the approximating feedback system to the step demand r(t). Then the output response y(t) of the implemented feedback scheme from zero initial conditions to the step demand r(t) satisfies the bound

$$|y(t) - y^{(1)}(t)| \leq \varepsilon(t) \triangleq \frac{N_t(W_A)}{1 - N_t(W_A)} \max_{o \leq t' \leq t} |y^{(1)}(t) - y^{(o)}(t)| \tag{62}$$

Finally, (62) holds with $\varepsilon(t)$ replace by $\varepsilon^\mu(t) = N_{\mu(t)}(W_A)$ where $\mu(t)$ is any function satisfying $\mu(t) \geq t$, $t \geq o$.

The interpretation of the result is similar to that of theorem 5, all calculations being simple simulations of low-order systems and graphical analysis of their responses. The system H_1 is specified by the designer to increase the simplicity of the calculations or to increase the accuracy of the predictions by reducing $\varepsilon(t)$. Three choices suggest themselves:

(1) $H_1 = O$ yields $y^{(o)}(t) \equiv O$ and simplifies the calculation of η and when $F = 1$ reduces simulation requirements as $W_A \equiv \eta$.

(2) $H_1 = F^{-1}$ (when F has a proper, stable inverse) yields $\eta(t) \equiv 0$ and hence $y^{(1)}(t) \equiv y_A(t)$ with one simulation avoided.

(3) $H_1 = (1 + G_A KF)^{-1} G_A K$ yields $y^{(o)}(t) \equiv y_A(t)$.

(Note: the choice (3) is likely to be least conservative as $y^{(o)}$ is the first guess in a successive approximation scheme to y).

3.5: Illustrative Example

Consider the scalar example of section 2.6 with the specified model G_A and designed controller K. Applying theorem 6, the response $W_A(t)$ was computed to be as in Fig. 7. Graphical analysis of this response leads to the conclusion that $N_\infty(W_A) = 0.62 < 1$ hence verifying the stability predictions obtained in section 2.6 provided that GK is both controllable and observable. The corresponding performance of the real and approximating feedback schemes to unit step demands has already been seen in Fig. 5. We can verify that the bounds on performance deterioration predicted by (62) anticipate these errors by choosing $H_1 = (1 + G_A KF)^{-1} G_A K$ and computing η to be as shown in Fig. 8. The corresponding bounds $y^{(1)} \pm \varepsilon$ together with y and y_A are illustrated in Fig. 9.

4. Conclusions

With the assumption that the control designer has access to a reliable estimate Y(t) of the step response matrix of a linear system G, the paper has provided systematic techniques for the design of feedback controllers based on a simplified plant model G_A and comparison of Y(t) with its own step response matrix $Y_A(t)$ in the time domain. Although other approaches are possible (Theorem 1), the paper has concentrated on the simplest case where plant modelling errors are measured by a single constant quantity $N_\infty^P(E)$ deduced from graphical analysis of the modelling error $E = Y - Y_A$. This data can be used in a frequency domain stability criterion (Theorem 2) that produces the guarantee that the control system designed on the basis of the approximate model will also stabilize the real plant despite the known modelling errors. At no stage of the design is it necessary to use or even to know the real plant model G i.e. the techniques can be used directly on data from a plant step test provided that either the signal to noise ratio is large or the noise is filtered from the response. In the scalar case and the case where diagonal plant models are used, the techniques generates (theorem 3) a stability criterion based upon inverse Nyquist plots that is very reminiscent of the Gershgorin circle based criterion of the inverse Nyquist array[16-19].

The direct use of time-domain data E(t) also leads to a simple simulation-based method (Theorem 4) for assessing stability that avoids the need for complex frequency domain calculations. This may be of particular value in the multivariable case where manipulation of matrices with complex elements could be a numerical problem. The important bonus from this type of analysis is that, under well-defined conditions, the degradation in input and output transient performance (Theorems 5 and 6) can also be bounded without the need to use or have available a detailed plant model.

The techniques described are very similar in spirit to the techniques of robust control[6,7]. In terms of robust stability analyses (where conservatism in the modelling of modelling errors can be a problem) the constant measure of modelling error $N_{\infty}^{P}(E)$ used here is conservative. However, there is no problem here as the assumption that the data Y is available enables the designer to choose his own compromise between design simplicity based on a highly simplified model (accepting the uncertainty associated with the large error) and the design complexity associated with a more accurate model (with the reduced uncertainty in closed-loop predictions). He simply has to choose a model G_A that is accurate enough to ensure that the conservatism implicit in the criteria is not so large as to make them inapplicable. The example included indicates that highly successful designs are easily achieved even in the presence of substantial modelling errors.

Finally, we note that the work described here is in the same spirit as that of Davison[22], Koivo[23] and Owens[2,8,21]. However, as in Åstrom[1], stability is guaranteed over a computable gain range due to the inclusion of a model in the process. Also the functional analytic treatment made possible by the model carries over[14,15] to consideration of discrete plant and also to inclusion of measurement nonlinearities. These will be reported separately.

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Appendices

7. Proof of Proposition 2

Local maxima and minima of f correspond to points where g changes sign. Write therefore

$$|d| + \int_{o}^{T} |g(t)|\, dt$$

$$= |d| + \sum_{k=1}^{k^*} \int_{t_{k-1}}^{t_k} |g(t)|dt + \int_{t_k^*}^{T} |g(t)|\, dt$$

$$= |d| + \sum_{k=1}^{k^*} \left| \int_{t_{k-1}}^{t_k} g(t)\, dt \right| + \left| \int_{t_{k^*}}^{T} g(t)\, dt \right| \qquad (63)$$

and note that f(o+) = d and

$$\int_{\alpha}^{\beta} g(t)dt = f(\beta) - f(\alpha) \qquad (64)$$

for any $\beta \geq \alpha \geq 0$.

8. Proof of Theorem 4

We regard the stability problem as an input-output stability problem[9-11] in $L_\infty^m(o,+\infty)$ (the m^{th} cartesian product of $L_\infty(o,+\infty)$). Assuming zero initial conditions we follow the techniques of the authors[8] and Harris et al[11] and regard the feedback system of Fig. 1(a) as the input equation

$$u = Kr - KFG\, u \qquad (65)$$

written in the equivalent form

$$u = (I + KFG_A)^{-1}\, Kr - (I + KFG_A)^{-1} KF\, (G-G_A)u \qquad (66)$$

in $L_\infty^\ell(o,+\infty)$. This equation has the form $u = W_\infty(u)$ where W_∞ maps $L_\infty^\ell(o,+\infty)$

into itself whenever $r \in L_\infty^m(o,+\infty)$. Note that the system is input-output stable if, and only if, equation (66) has a unique solution $u \in L_\infty^\ell(o,+\infty)$ whenever $r \in L_\infty^m(o,+\infty)$. A sufficient condition for this is that W_∞ is a P-contraction (lemma 3) i.e. $r(||(I + KFG_A)^{-1}KF(G-G_A)||_P) < 1$. The result follows from lemma 4 which indicates that

$$||(I + KFG_A)^{-1}KF(G-G_A)||_P = N_\infty^P(W_A) \tag{67}$$

where W_A is the step response matrix of $(I + KFG_A)^{-1}KF(G - G_A)$ and noting that

$$W_A(t) = \mathcal{L}^{-1}\{(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)(G(s) - G_A(s))\frac{1}{s}\}$$

$$= \mathcal{L}^{-1}\{(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)\mathcal{L}\{E(t)\}\}$$

$$= ((I + KFG_A)^{-1}KF\ E)(t) \tag{68}$$

where we have used the observation that $\mathcal{L}\{E(t)\} = s^{-1}(G(s) - G_A(s))$.

9. Proof of Theorem 5

Invoking causality, we can regard (66) as an equation $u = W_t(u)$ in $L_\infty^\ell(o, t)$. Note that the conditions of theorem 4 ensure that W_t is a P-contraction with 'P-matrix' $N_t^P(W_A) \le N_\infty^P(W_A)$. Let $u^{(o)}$ be the first guess in the successive approximation scheme $u^{(k+1)} = W_t(u^{(k)})$ and note from lemma 3 that

$$||u - u^{(1)}||_P \le (I_\ell - N_t^P(W_A))^{-1}N_t^P(W_A)||u^{(1)} - u^{(o)}||_P \tag{69}$$

with

$$u^{(1)} = (I + KFG_A)^{-1}K\ r - (I + KFG_A)^{-1}KF\ (G - G_A)u^{(o)}$$

$$= u_A + \zeta \tag{70}$$

where $u_A = (I + KFG_A)^{-1}Kr$ is clearly the input response of the approximating feedback system to the demand r and

$$\zeta(t) = -\mathcal{L}^{-1}\{(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)(G(s)-G_A(s))H_2(s)s^{-1}\beta\}$$

$$= -\mathcal{L}^{-1}\{\mathcal{L}\{W_A(t)\}\ H_2(s)\}\beta \tag{71}$$

which is simply (52). Relation (55) follows directly from (69) from the definitions of the norm in $L_\infty(o,t)$ and noting that (69) is simply

$$||u-u^{(1)}||_P \leq \varepsilon(t)\ ,\ t \geq o \tag{72}$$

10. Proof of Corollary 5.2

Continuing with the argument in Appendix 9, write $y = G\,u$ and $y^{(1)} = G_A\,u^{(1)}$ and

$$||y(t) - y^{(1)}(t)|| \leq ||(y-y^{(1)})||_P = ||G\,u - G_A\,u^{(1)}||_P$$

$$\leq ||G(u - u^{(1)})||_P + ||(G - G_A)u^{(1)}||_P$$

$$\leq ||G||_P \cdot ||u-u^{(1)}||_P + ||(G-G_A)||_P \cdot ||u^{(1)}||_P$$

$$\leq N_t^P(Y)\ \varepsilon(t) + N_t^P(E) \sup_{o\leq t'\leq t} ||u^{(1)}(t')||_P \tag{73}$$

by using (72) and lemma 4 to write $||G||_P = N_t^P(Y)$ and $||G - G_A||_P = N_t^P(E)$.

11. Proof of Corollary 5.3

By monotonicity, $\mu_{ij}(t) \geq t$ $(t \geq o)$ implies $N_\infty^P(W_A) \geq N_t^{P,\mu}(W_A) \geq N_t^P(W_A)$ for $t \geq o$. Noting that

$$(I_\ell - N_t^P(W_A))^{-1}N_t^P(W_A) = \sum_{k=1}^{\infty} (N_t^P(W_A))^k$$

$$\leq \sum_{k=1}^{\infty} (N_t^{P,\mu}(W_A))^k = (I_\ell - N_t^{P,\mu}(W_A))^{-1}N_t^{P,\mu}(W_A) \tag{74}$$

the result follows trivially from (54) and (55).

12. Proof of Theorem 6

Assuming zero initial conditions, write the input-output relations

$$y = GK\,r - GKF\,y \tag{75}$$

in the form

$$y = (I + G_A KF)^{-1} G_A Kr + (I + KFG_A)^{-1} K(G-G_A)(r - Fy) \tag{76}$$

by the standard loop transformation[8,11] and using commutation of scalar convolution operators. Using an argument used in Appendix 8 and elsewhere[8,11], lemma 3 yields stability in the input-output sense if the contraction condition $||(I + KFG_A)^{-1} KF(G-G_A)|| < 1$ holds. The relation (67) immediately requires that $N_\infty(W_A) < 1$. Note also that this condition also implies that a contraction is obtained on any interval $[o,t]$ with contraction constant $N_t(W_A)$.

Let $y^{(o)}$ be the first guess in the successive approximation scheme and note from lemma 3 that, on $[o,t]$, (76) yields

$$||y - y^{(1)}|| \leq \frac{N_t(W_A)}{1 - N_t(W_A)} ||y^{(1)} - y^{(o)}|| \tag{77}$$

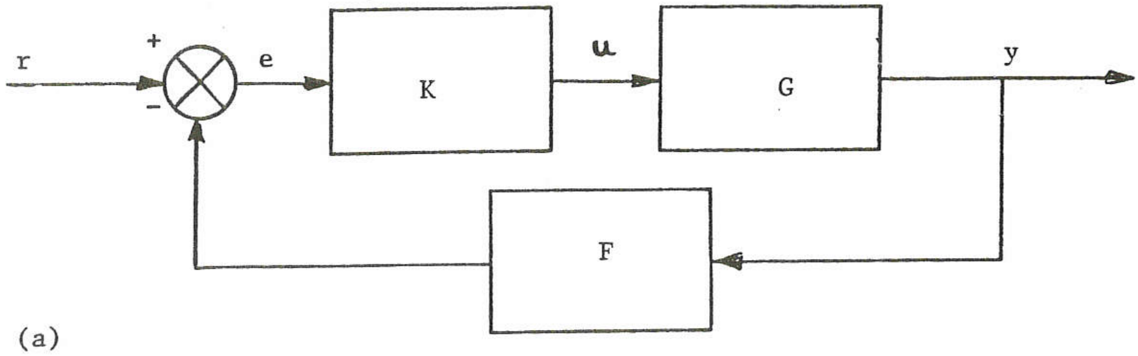
where $y^{(o)} = H_1 r$, r is a unit step and $y^{(1)} = y_A + \eta$ where $y_A = (I + G_A KF)^{-1} G_A K r$ is the unit step response of the approximating feedback scheme and

$$\eta(t) = ((I + KFG_A)^{-1} K(I - FH_1)(G - G_A)r)(t)$$

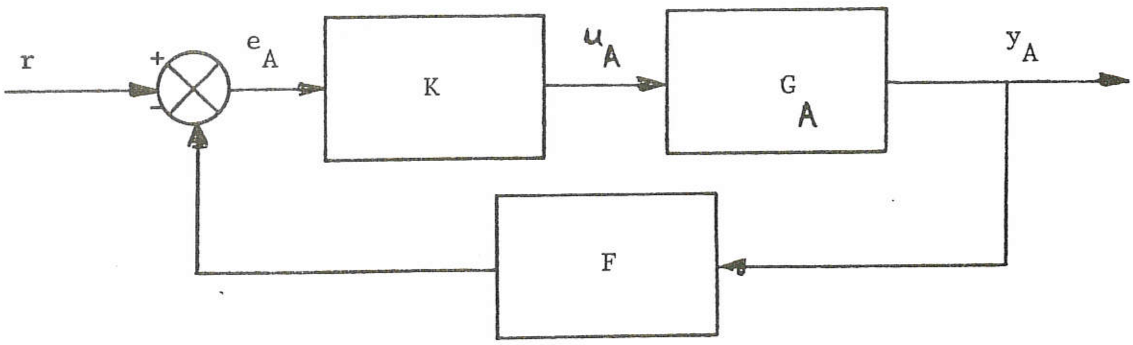
$$= ((I + KFG_A)^{-1} K (I - FH_1)E)(t) \tag{78}$$

as $(G - G_A)r = E$ by definition. Equation (62) follows from (76) by the definition of the norm in $L_\infty(o,t)$.

The final observation of the theorem follows in a similar manner to the proof of Corollary 5.3.



(a)



(b)

Fig. 1 (a) Real and (b) Approxmating Feedback Systems

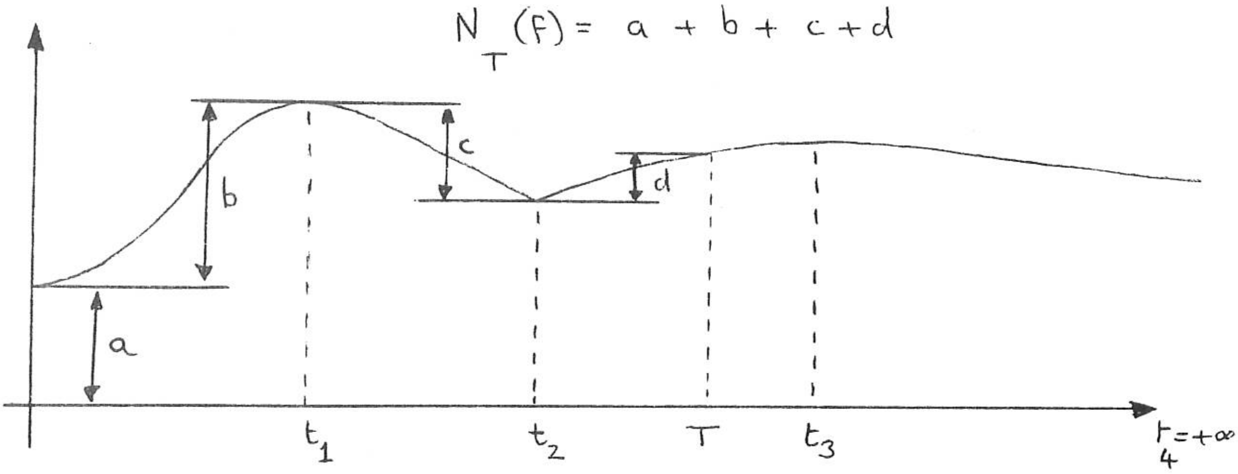


Fig 2.

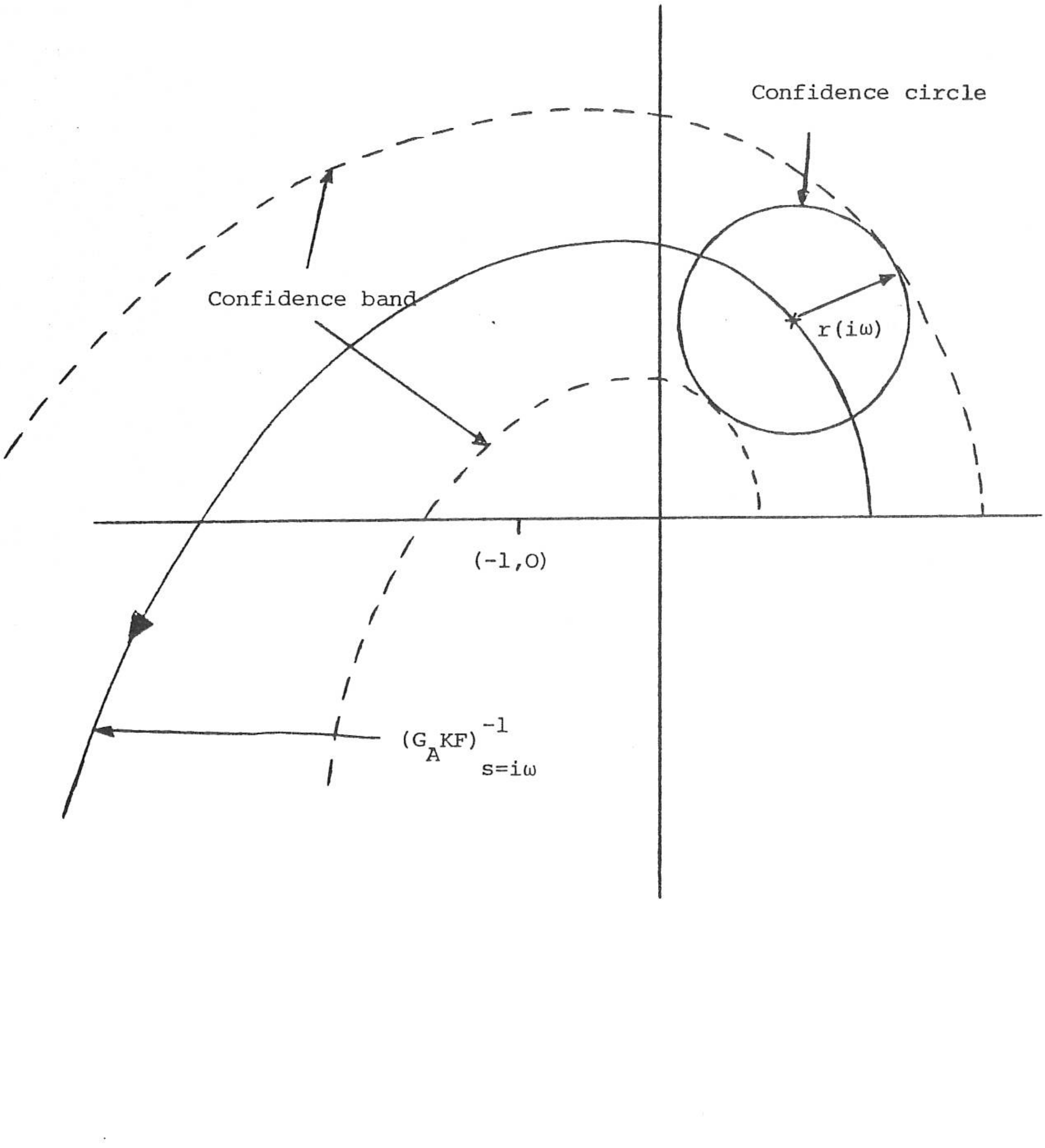


Fig. 3 Confidence band and Confidence circle

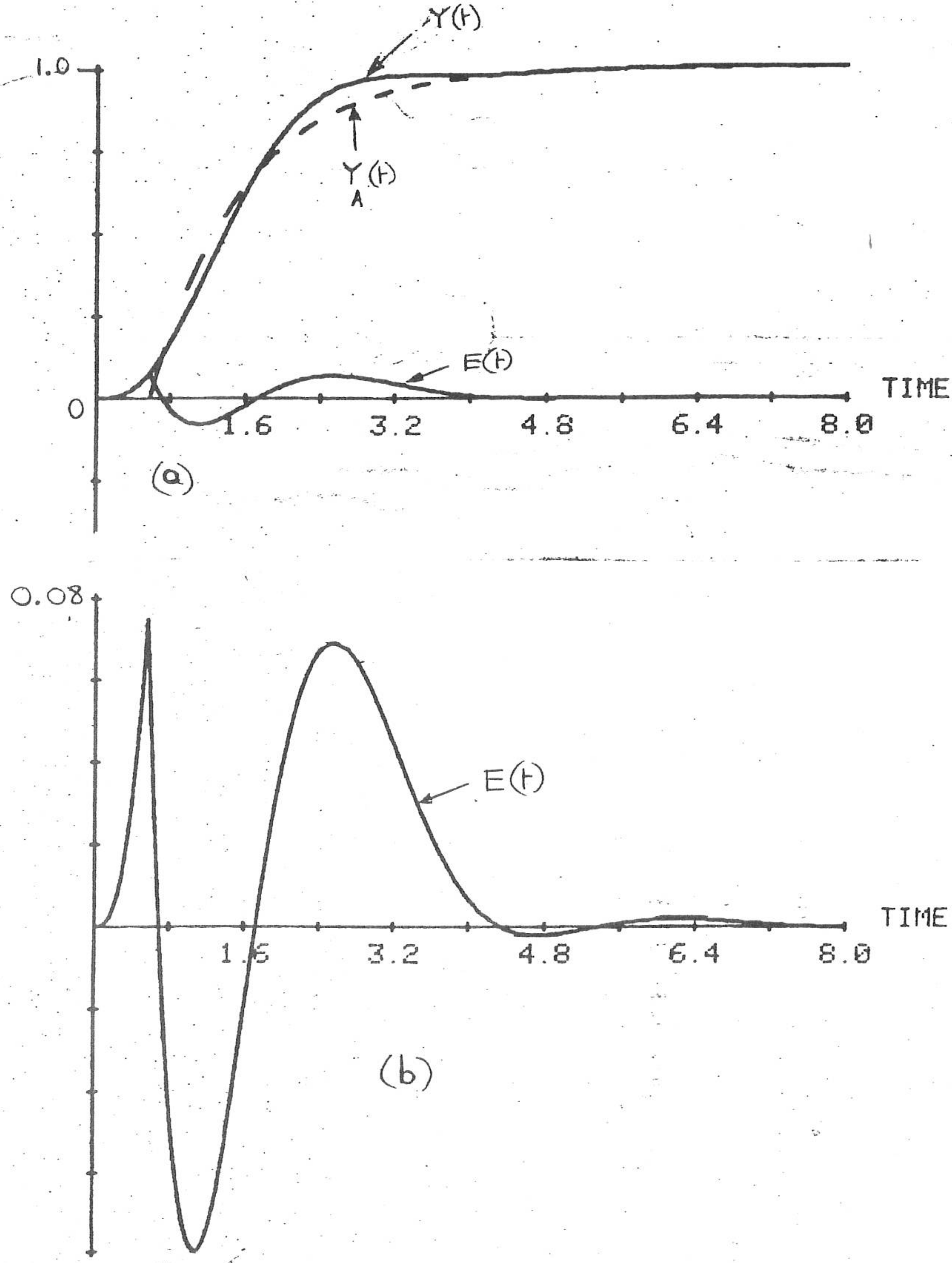


Fig 4

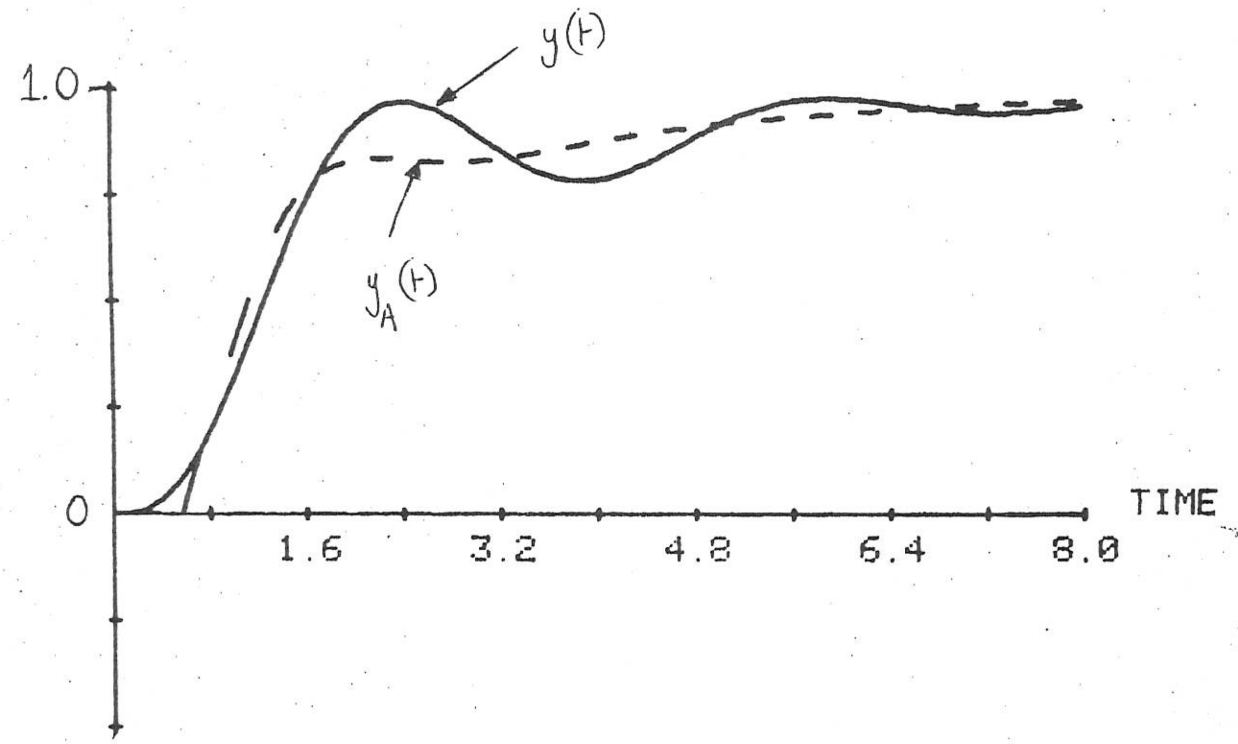


Fig 5.

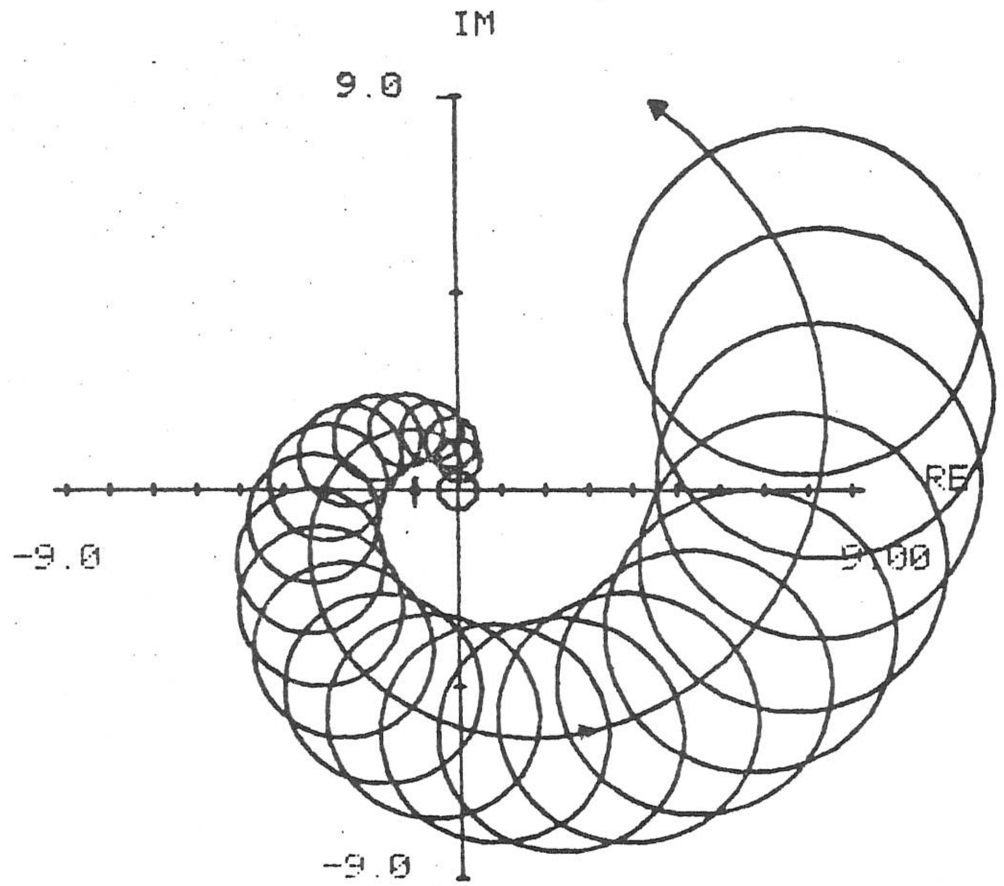


Fig 6

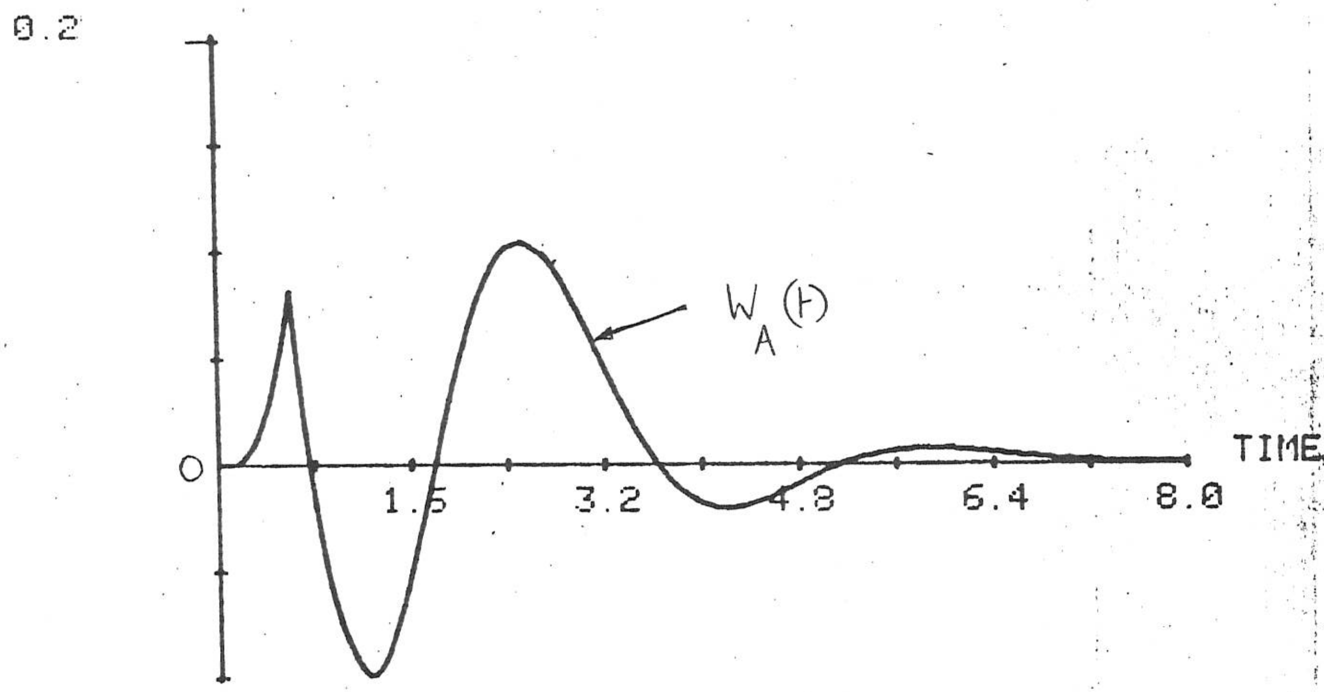


Fig 7

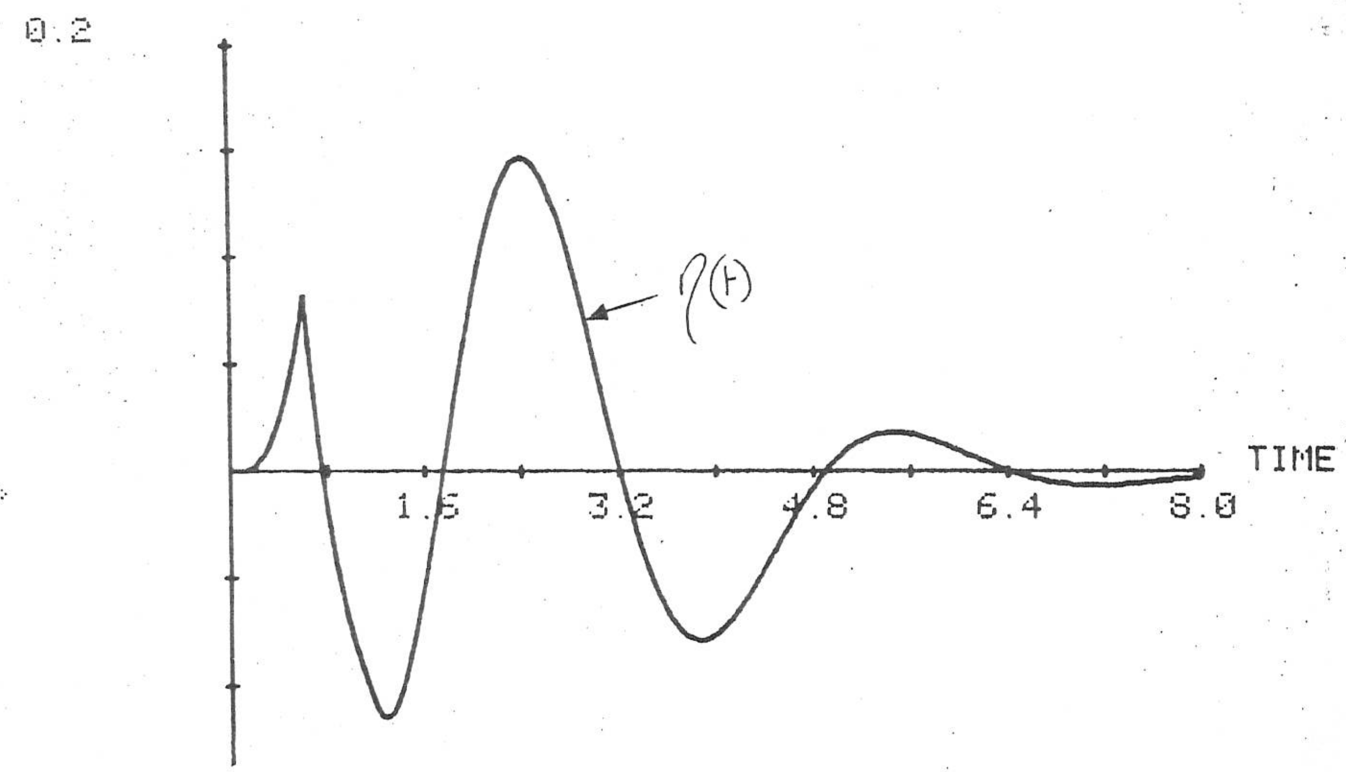


Fig 8

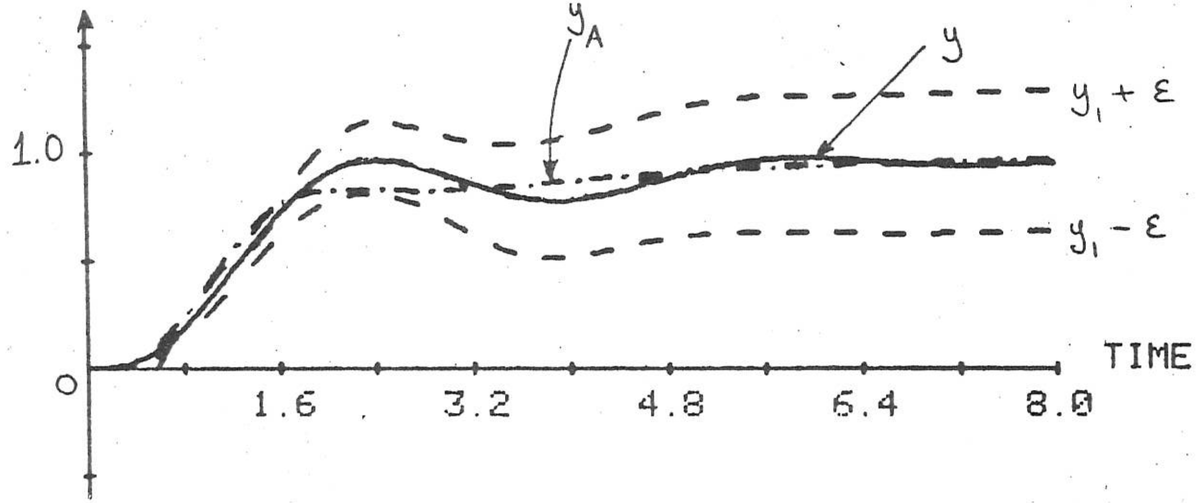


Fig 9