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CONTROLLABILITY, OPTIMAL CONTROL AND RECEDING HORIZON CONTROL OF DISTRIBUTED MULTIPASS PROCESSES

by

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Abstract In this paper the control of linear distributed multipass processes is considered using the semigroup formulation. A controllability condition is proved, and requires a certain integral operator to have zero kernel. Both the optimal linear - quadratic and receding horizon approaches are studied, and finally a simple example is presented.

1. Introduction

The idea of processing engineering systems repeatedly, such that the state (or 'pass profile') on the k operation depends in some way on the previous states has been around for many years. Examples can be found in metallurgy; multipass welding and rolling, in chemical reactor control and longwall coal cutting[1]. The theoretical development and control of such processes is much more recent - properties such as stability and controllability have been studied[2,3] in the finite-dimensional case and the theory of existance, uniqueness and stability of infinite-dimensional (i.e. distributed) nonlinear multipass processes has also appeared in the literature[4]. In this paper we shall be concerned with the controllability, optimal and receding horizon control in the case of distributed multipass processes. Receding horizon control has recently been extended to the distributed case[5] and allows one to introduce nonlinear feedback controls which react more quickly than the linear optimal control to large errors and more slowly to small errors, which may be due to noise disturbances.

The paper is composed as follows. We first discuss the notation and give the basic definitions required in the paper in section 2 and then in section 3 the equations defining the system are shown to take the form of an evolution equation on a certain Hilbert space $\mathcal{H}$. The operator defining this system is then shown to generate a sequence of semigroups on appropriate subspaces of $\mathcal{H}$. The controllability of the system is discussed in the next section and in section 5 the linear quadratic solution is found, firstly as a non causal operator and then in terms of a causal representation. In section 6 we consider the receding horizon control of this system and in the last section a simple example is given to illustrate the theory.

2. Notations and Definitions

In this paper, we shall consider a differential equation on a Hilbert space.

The state of each pass will be assumed to belong to a Hilbert space H and the whole system will be defined on the Hilbert space $\mathcal{H}$ which is the direct sum of a countable number of copies of H, and which will be denoted by $\overset{\infty}{\underset{k=o}{\oplus}} H$. A finite direct sum of copies of H will be denoted by $\overset{m}{\underset{k=o}{\oplus}} H \triangleq \mathcal{H}_m$.

We shall introduce an operator $\mathcal{T}_t$ on $\mathcal{H}$ and show that $\mathcal{T}_t$ has the properties

(i) $\mathcal{T}_o = I_{\mathcal{H}}$

(ii) $\mathcal{T}_{t+s} = \mathcal{T}_t \mathcal{T}_s$

However, in order to show that

(iii) $\lim\limits_{t \to o} \mathcal{T}_t x = x$, for each $x \in \mathcal{H}$

we must project $\mathcal{T}_t$ onto $\mathcal{H}_m$. The conditions (i),(ii) and (iii) then show that $\mathcal{T}_t^m$ (the projection of $\mathcal{T}_t$ onto $\mathcal{H}_m$) is a <u>semigroup</u> of operators on $\mathcal{H}_m$.

The space of measurable maps $f:[o,\tau] \to H$ such that

$$\int_o^\tau ||f(s)||_H^2 \, ds < \infty$$

will be denoted, as usual, by $L^2([0,\tau];H)$, and if X and Y are Hilbert spaces, the space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If X=Y we write $\mathcal{L}(X)$.

If L is a linear operator in $\mathcal{L}(X,Y)$, then

ker L $\triangleq$ $\{x \in X : Lx = 0\}$

Range L $\triangleq$ $\{y \in Y : \exists x \in X \text{ such that } Lx = y\}$.

The dual or adjoint operator L^* of L is defined by

$$\langle x, L^* y\rangle_X = \langle Lx, y\rangle_Y ,$$

for each $x \in X$, $y \in Y$. It should be noted that we have identified X with X^* and Y with Y^* as usual.

If A is an unbounded operator defined in a Hilbert space X with values in Y, then we denote the domain of A by $\mathcal{D}(A)$. If $\mathcal{D}(A)$ is dense in X, then[6] we can define the dual of A again by

$$\langle x, A^* y\rangle_X = \langle Ax, y\rangle_Y.$$

3. System Equations

Consider the linear distributed multi-pass process with finite memory of length ℓ given by the equation

(3.1) $\frac{dx_k}{dt}(t) = A_o x_k(t) + A_1 x_{k-1}(t) + \ldots + A_\ell x_{k-\ell}(t) + Bu_k(t), \ k \geq 0$

where $x_k(t) \in H$, a Hilbert space, for each $k \geq -\ell$, $u_k(t) \in U$ a Hilbert space of controls for $k \geq 0$, A_i is an (unbounded) operator defined on $\mathcal{D}(A_i) \subseteq H$ for $0 \leq i \leq \ell$ and $B \in \mathcal{L}(U,H)$. For the system (3.1) to be well-defined we must specify the states $x_{-1}(t), \ldots, x_{-\ell}(t)$ for all $t \in [0,\tau]$, where τ is the pass length, and also the initial values $x_k(0)$, $k \geq 0$.

We would like to replace the system (3.1) by an equation of the form

(3.2) $\dot{x} = \mathcal{A}x + \mathcal{B}u + f$

which is defined on an appropriate Hilbert space, $\mathcal{H}$. The space $\mathcal{H}$ in the present situation will be

(3.3) $\mathcal{H} = \bigoplus_{k=0}^{\infty} H$,

i.e. the direct sum of a countable number of copies of H, with inner product

$$\langle x,y \rangle_{\mathcal{H}} = \sum_{k=0}^{\infty} \langle x_k, y_k \rangle_H ,$$

with

$$x = (x_o, x_1, \ldots)^T \in \mathcal{H}$$

$$y = (y_o, y_1, \ldots)^T \in \mathcal{H} .$$

Consider now the operator $\mathcal{A}$ on $\mathcal{H}$ defined by

$$\mathcal{A}x = (A_o x_o, A_o x_1 + A_1 x_o, A_o x_2 + A_1 x_1 + A_2 x_o, \ldots, A_o x_\ell + \ldots + A_\ell x_o,$$

$$A_o x_{\ell+1} + \ldots + A_\ell x_1, A_o x_{\ell+2} + \ldots + A_\ell x_2, \ldots)^T ,$$

for $x \in (x_o, x_1 \ldots)^T \in \mathcal{D}(\mathcal{A}) = \bigoplus_{k=o}^{\infty} \{ \bigcap_{i=0}^{\ell} \mathcal{D}(A_i) \}$.

In terms of infinite matrices of operators,

$$\mathcal{A} = \begin{pmatrix} A_o & & & & & & \\ A_1 & A_o & & & & & \\ A_2 & A_1 & A_o & & \underline{O} & & \\ \cdots & \cdots & \cdots & & & & \\ A_\ell & A_{\ell-1} & \cdots & A_o & & & \\ O & A_\ell & A_{\ell-1} & \cdots & A_o & & \\ O & O & A_\ell & \cdots & & A_o & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Also, we clearly have

$$u = (u_o, u_1, u_2, \ldots\ldots)^T \varepsilon\, \mathcal{U}$$

$$\mathcal{B} = \text{diag}\,\{B, B, \ldots\ldots\} \quad \varepsilon\, \mathcal{L}\left(\bigoplus_{k=0}^{\infty} U, \bigoplus_{k=0}^{\infty} H\right) = \mathcal{L}(\mathcal{U}, \mathcal{H})$$

where $\mathcal{U} = \bigoplus_{k=0}^{\infty} U$.

Finally, we must account for the states $x_{-1}, \ldots, x_{-\ell}$ which will appear in the forcing term f; namely,

$$f = (A_1x_{-1} + \ldots + A_\ell x_{-\ell},\ A_2x_{-1} + \ldots + A_\ell x_{1-\ell},\ \ldots, A_\ell x_{-1}, 0, 0, \ldots)^T.$$

We have therefore replaced the system of equations (3.1) by a system of the form (3.2) and we must now determine conditions under which the operator $\mathcal{A}$ defined on $\mathcal{D}(\mathcal{A})$ generates a semigroup $\mathcal{J}_t$ on $\mathcal{H}$. Consider, then, the unforced system obtained from (3.1) by setting $u_k = 0$; i.e.

(3.4) $$\frac{dx_k(t)}{dt} = A_o x_k(t) + A_1 x_{k-1}(t) + \ldots + A_\ell x_{k-\ell}(t)$$

where we interpret $x_j(t)$ as zero for negative values of j. Continuing with the latter convention and assuming that A_o generates a semigroup T_t we can integrate (formally) the equation (3.4) to obtain

$$x_k(t) = T_t x_k(0) + \sum_{i=1}^{\ell} \int_0^t T_{t-s} A_i x_{k-i}(s)\,ds.$$

It follows easily by induction that

$$(3.5)\quad x_k(t) = T_t x_k(o) + \sum_{i_1=1}^{\ell} K_2^{i_1}(t,o)x_{k-i_1}(0) + \sum_{i_1=1}^{\ell}\sum_{i_2=1}^{\ell} K_3^{i_1 i_2}(t,o)x_{k-i_1-i_2}(o)$$

$$+ \dots + \sum_{i_1=1}^{\ell}\sum_{i_2=1}^{\ell} \dots \sum_{i_k=1}^{\ell} K_{k+1}^{i_1 i_2 \dots i_k}(t,0)x_{k-i_1-i_2-\dots-i_k}(o),$$

where

$$K_1(t,s) = T_{t-s} \; (= K_1^{\phi}(t,s), \text{ with the empty set of suffices})$$

$$K_{k+1}^{i_1 i_2 \dots i_k}(t,s) = \int_s^t K_k^{i_1 \dots i_{k-1}}(t,s_1)A_{i_k} T_{s_1-s} ds_1.$$

Note now that (3.5) was derived purely formally. In order that this relation is well-defined we shall make the following assumption on each operator A_i, $1 \le i \le k$:

either A_i is bounded

or $||A_i T_t|| \le g_i(t)$ for some function g_i

such that $\gamma_i = \int_s^t g_i(s_1 - s)ds_1 < \infty$ for any $t \ge s \ge 0$.

It is then clear that each $K_j^{i_1 \dots i_{j-1}}(t,s)$ is a bounded operator; for

$$||K_1(t,s)|| = ||T_{t-s}|| < \infty$$

and

$$||K_{k+1}^{i_1 i_2 \dots i_k}(t,s)|| \le \sup_{s_1 \le t} ||K_k^{i_1 \dots i_{k-1}}(t,s_1)|| \gamma_{i_k}$$

(for fixed $t \ge 0$). Consider the operator $\mathcal{T}_t$ on $\mathcal{H}$ defined by the matrix

$$\mathcal{J}_t = \begin{array}{r} \\ \\ \\ \\ (k+1)^{th} \text{ row} \rightarrow \\ \\ \end{array} \left(\begin{array}{llll} T_t & 0 & 0 & \cdots\cdots \\ K_2^1(t,0) & T_t & 0 & \cdots\cdots \\ K_2^2(t,0)+K_3^{11}(t,0) & K_2^1(t,0) & T_t & 0\cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sum\limits_{j=2}^{k+1} \sum\limits_{\substack{i_1,\ldots,i_{j-1} \\ (k-i_1-\ldots-i_{j-1}=0)}} K_j^{i_1\ldots i_{j-1}}(t,0) & \sum\limits_{j=2}^{k} \sum\limits_{\substack{i_1,\ldots,i_{j-1} \\ (k-1-i_1-\ldots-i_{j-1}=0)}} K_j^{i_1\ldots i_{j-1}}(t,0) \cdots\cdots & K_2^1(t,0) & T_t \quad 0\ldots. \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array} \right)$$

Note that, in this expression, each i_m is between 1 and ℓ. Hence some of the summations may be empty; e.g. if $\ell = 1$, then $k - i_1 - \ldots - i_{j-1} = k - (j-1) = 0$ only when $j = k+1$. Thus, the first term on the $(k+1)^{th}$ row reduces to

$$K_{k+1}^{\overbrace{1\ldots1}^{k}}(t,0)\;.$$

We now consider the semigroup properties of $\mathcal{T}_t$.

<u>Lemma 3.1</u> Each $K_{k+1}^{i_1\ldots i_k}$ is translation invariant; i.e.

$$K_{k+1}^{i_1\ldots i_k}(t+\tau,s+\tau) = K_{k+1}^{i_1\ldots i_k}(t,s)$$

for each $\tau\geq 0$, $t\geq s\geq 0$ and each $1\leq i_1,\ldots,i_k\leq \ell$, $k\geq 0$.

<u>Proof</u> By induction. For $k=0$, $K_1(t,s) = T_{t-s}$ and the result is trivial. If the result is true for $k-1$, then

$$\begin{aligned}K_{k+1}^{i_1\ldots i_k}(t+\tau,s+\tau) &= \int_{s+\tau}^{t+\tau} K_k^{i_1\ldots i_{k-1}}(t+\tau,s_1)A_{i_k}T_{s_1-s-\tau}ds_1\\ &= \int_s^t K_k^{i_1\ldots i_{k-1}}(t+\tau,s_1+\tau)A_{i_k}T_{s_1-s}ds_1\\ &= \int_s^t K_k^{i_1\ldots i_{k-1}}(t,s_1)A_{i_k}T_{s_1-s}ds_1\\ &= K_{k+1}^{i_1\ldots i_k}(t,s). \quad \square\end{aligned}$$

<u>Lemma 3.2</u> If $s\leq\tau$, we have

$$\begin{aligned}K_{k+1}^{i_1\ldots i_k}(t+\tau,s) = T_tK_{k+1}^{i_1\ldots i_k}(\tau,s) + K_2^{i_1}(t,0)K_k^{i_2\ldots i_k}(\tau,s) + \ldots\\ + K_k^{i_1\ldots i_{k-1}}(t,0)K_2^{i_k}(\tau,s) + K_{k+1}^{i_1\ldots i_k}(t,0)T_{\tau-s},\end{aligned}$$

for $k\geq 1$.

Proof By induction. When k=1, we have

$$K_2^{i_1}(t+\tau,s) = \int_s^{t+\tau} T_{t+\tau-s_1} A_{i_1} T_{s_1-s}\, ds_1$$

$$= \int_s^{\tau} T_{t+\tau-s_1} A_{i_1} T_{s_1-s} ds_1 + \int_\tau^{t+\tau} T_{t+\tau-s_1} A_{i_1} T_{s_1-s} ds_1$$

$$= T_t\, K_2^{i_1}(\tau,s) + K_2^{i_1}(t,0)\, T_{\tau-s}.$$

If the result is true for k–1, then

$$K_{k+1}^{i_1\ldots i_k}(t+\tau,s) = \int_s^{t+\tau} K_k^{i_1\ldots i_{k-1}}(t+\tau,s_1) A_{i_k} T_{s_1-s} ds_1$$

$$= \int_s^{\tau} K_k^{i_1\ldots i_{k-1}}(t+\tau,s_1) A_{i_k} T_{s_1-s} ds_1 + K_{k+1}^{i_1\ldots i_k}(t,0) T_{\tau-s}$$

$$= \int_s^{\tau} \sum_{j=1}^{k} K_j^{i_1\ldots i_{j-1}}(t,0) K_{k-j+1}^{i_j\ldots i_{k-1}}(\tau,s_1) A_{i_k} T_{s_1-s} ds_1$$

$$+ K_{k+1}^{i_1\ldots i_k}(t,0) T_{\tau-s} \quad \text{, by the inductive hypothesis}$$

$$= \sum_{j=1}^{k+1} K_j^{i_1\ldots i_{j-1}}(t,0) K_{k-j+2}^{i_j\ldots i_k}(\tau,s) \;. \quad \square$$

Lemma 3.3 For each $t,\tau \geq 0$ we have $\mathcal{T}_{t+\tau} = \mathcal{T}_t \mathcal{T}_\tau$.

Proof We shall consider the first element of the $(k+1)^{th}$ row of both $\mathcal{T}_{t+\tau}$ and $\mathcal{T}_t \mathcal{T}_\tau$, and show that these are equal. The other elements can be seen to equate in just the same way. Therefore, we have

$$(\mathcal{T}_{t+\tau})_{(k+1),1} = \sum_{j=2}^{k+1} \sum_{\left\{\begin{array}{l} i_1,\ldots,i_{j-1} \\ k-i_1-\ldots-i_{j-1} = 0 \end{array}\right\}} K_j^{i_1\ldots i_{j-1}}(t+\tau,0)$$

$$
(3.6) \qquad = \sum_{j=2}^{k+1} \sum_{\left\{\substack{i_1,\dots,i_{j-1} \\ k-i_1-\dots-i_{j-1}=0}\right\}} \sum_{m=1}^{j} K_m^{i_1\cdots i_{m-1}}(t,0)K_{j-m+1}^{i_m\cdots i_{j-1}}(\tau,0)
$$

by lemma 3.2. Also, by straightforward multiplication of the operator matrices, we see that

$$
(3.7) \quad (\mathcal{T}_t\mathcal{T}_\tau)_{(k+1),1} = \sum_{m=2}^{k}\left(\sum_{j=2}^{k-m+2} \sum_{\substack{i_1,\dots,i_{j-1}=0 \\ k-m+1-i_1-\dots-i_{j-1}=0}} K_j^{i_1\dots i_{j-1}}(t,0)\right) \cdot
$$

$$
\left(\sum_{j'=2}^{m} \sum_{\left\{\substack{i'_1,\dots,i'_{j'-1} \\ m-1-i'_1-\dots-i'_{j'-1}=0}\right\}} K_{j'}^{i'_1\dots i'_{j'-1}}(\tau,0)\right)
$$

$$
+\left(\sum_{j=2}^{k+1} \sum_{\left\{\substack{i_1,\dots,i_{j-1} \\ k-i_1-\dots-i_{j-1}=0}\right\}} K_j^{i_1\dots i_{j-1}}(t,0)\right)T_\tau
$$

$$
+T_t\left(\sum_{j=2}^{k+1} \sum_{\left\{\substack{i_1,\dots,i_{j-1} \\ k-i_1-\dots-i_{j-1}=0}\right\}} K_j^{i_1\dots i_{j-1}}(\tau,0)\right).
$$

We must now show that the expressions on the right hand sides of (3.6) and (3.7) represent the same operator. Each of these expressions is a sum of terms of the form

$$
(3.8) \qquad K_p^{i_1\dots i_{p-1}}(t,0)K_q^{i'_1\dots i'_{q-1}} .
$$

It therefore suffices to show that identical terms for fixed p,q occur in each expression. We shall consider the terms with p=q=2, the general case being similar. In (3.6) therefore, we must have m=2, j-2+1=2, i.e. j=3. Hence (3.6) contains the terms

$$
(3.9) \qquad \sum_{\left\{\substack{i_1,i_2 \\ k-i_1-i_2=0}\right\}} K_2^{i_1}(t,0)K_2^{i_2}(\tau,0).
$$

Now, in (3.7) to obtain the terms of the form (3.8) with p=q=2 we must have j=2, j'=2, and then we obtain the terms

$$\sum_{m=2}^{k} \sum_{\left\{\substack{i_1 \\ k-m+1-i_1=0}\right\}} K_2^{i_1}(t,0) \cdot \sum_{\left\{\substack{i'_1 \\ m-1-i'_1=0}\right\}} K_2^{i'_1}(\tau, 0).$$

$$= \sum_{\left\{\substack{i_1, i'_1 \\ k-i_1-i'_1 = 0}\right\}} K_2^{i_1}(t,0)K_2^{i'_1}(\tau,0)$$

which is the same as (3.9). $\square$

We have now shown that $\mathcal{T}_{t+\tau} = \mathcal{T}_t \mathcal{T}_\tau$ and it is clear that $\mathcal{T}_0 = I$. Note, however, that $\mathcal{T}_t$ is not a bounded operator on $\mathcal{H}$ and therefore does not satisfy the appropriate continuity hypothises. This is easily obviated since we are only interested in a finite number of passes and so we consider the system (3.1) for k up to some fixed value $\bar{m}$. The Hilbert space on which this system exists is now

$$\mathcal{H}_m = \bigoplus_{k=0}^{m} H$$

and with an obvious notation (3.2) becomes

$$\dot{x} = \mathcal{A}_m x + \mathcal{B}_m u + f_m \tag{3.10}$$

when projected on $\mathcal{H}_m$. (To avoid confusion of substripts, we shall use the same symbol x to denote the projection of $x \varepsilon \mathcal{H}$ on $\mathcal{H}_m$. It should be clear from the context to which space x is presumed to belong.) It now follows from the above results that $\mathcal{A}_m$ (with domain $\bigoplus_{k=0}^{m} \{\bigcap_{i=1}^{\ell} \mathcal{D}(A_i)\}$) generates the strongly continuous semigroup

$$\mathcal{T}_t^m = P_m \mathcal{T}_t P_m^*$$

where $P_m : \mathcal{H} \to \mathcal{H}_m$ is the projection. Equation (3.10) can also be written in the integrated (mild)form

$$(3.11)\quad x(t) = \mathcal{T}_t^m\, x(o) + \int_o^t \mathcal{T}_{t-s}^m \mathcal{B}_m u(s)ds + \int_o^t \mathcal{T}_{t-s}^m f_m\, ds\ ,$$

where

$x(o) = (x_0(o), \ldots, x_m(o))^T$ is the given initial states .

4. Controllability

In this section we shall discuss the controllability of the multipass process defined by equation (3.1). Following Collins [3], we introduce the definition

Definition 4.1 The system (3.1) is approximately controllable in m passes if it can be driven from any initial states $x_{-1}, \ldots, x_{-\ell}$ to a dense subspace of $L^2([0,\tau]; H)$ on the $m-1^{th}$ pass. (Recall that the first pass is numbered x_0).

In the finite-dimensional case, where A_o, A_1 and B are matrices, Collins(3) gives the sufficient condition that the matrix

$$(B,\ A_1B, \ldots, A_1^{n-1}B)$$

has full rank, for approximate controllability in n passes. (Here, of course, n is the dimension of the state space of each pass profile $x_k(t)$). In order to derive conditions for approximate controllability in the distributed case, we need the following lemmas:

Lemma 4.2 (Curtain & Pritchard (7), Dolecki, Russell (8)):

If $F\in\mathcal{L}(V,Z)$, $G\in\mathcal{L}(W,Z)$, where V,W,Z are Hilbert spaces, then the following conditions are equivalent.

(a) $\ker(G^*) \subseteq \ker(F^*)$

(b) $\overline{\text{Range}(G)} \supseteq \overline{\text{Range}(F)}$. □

Consider the operator $G: \bigoplus_{i=1}^{m} L^2(0,\tau;U) \to L^2(0,\tau;H)$ defined by

$$G(u) = \int_o^t \sum_{i=0}^{m-1} (\mathcal{T}_{t-s})_{m,i+1}\, Bu_i(s)ds\ , \quad 0\le t\le\tau,$$

where $u = (u_o, u_1, \ldots, u_{m-1}) \in \bigoplus_{i=1}^{m} L^2(0,\tau;U)$ and

$$(\mathcal{J}_t)_{m,i+1} = \sum_{j=2}^{m-i} \sum_{\left\{\begin{array}{l} i_1,\ldots,i_{j-1} \\ k-i_1-\ldots-i_{j-1}=0 \end{array}\right\}} K_j^{i_1\ldots i_{j-1}}(t,0)$$

is the $(m,i+1)^{th}$ element of the operator $\mathcal{J}_t$ defined in section 3.

To find the dual of G, we have

$$\left\langle \int_o^{\cdot} \sum_{i=o}^{m-1} (\mathcal{J}_{\cdot-s})_{m,i+1} Bu_i(s)ds,\ h(\cdot) \right\rangle_{L^2(o,\tau;H),\ L^2(o,\tau;H)}$$

$$= \int_o^{\tau} \left\langle \int_o^t \sum_{i=o}^{m-1} (\mathcal{J}_{t-s})_{m,i+1} Bu_i(s)ds,\ h(t) \right\rangle_H dt$$

$$= \sum_{i=o}^{m-1} \int_o^{\tau} \int_s^{\tau} \left\langle u_i(s)\ ,\ B^*(\mathcal{J}^*_{t-s})_{i+1,m} \quad h(t) \right\rangle_{U,U^*} dtds$$

$$= \sum_{i=o}^{m-1} \left\langle u_i(\cdot)\ ,\ \int_{\cdot}^{\tau} B^*(\mathcal{J}^*_{t-})_{i+1,m} \quad h(t)dt \right\rangle_{L^2(o,\tau;U),L^2(o,\tau\ ;U^*)} ,$$

where $h \epsilon L^2(o,\tau;H)$, Hence,

$$G^* : L^2(o,\tau;H) \to \bigoplus_{i=1}^{m} L^2(0,\tau;U)$$

(identifying the Hilbert spaces H and U with their duals) is given by

$$(G^*h)(s) = \left(\int_s^{\tau} B^*(\mathcal{J}^*_{t-s})_{i+1,m} \quad h(t)dt \right)_{0 \le i \le m}$$

Since $\mathcal{J}_t^{m-1} x(o)$ and $\int_o^t \mathcal{J}^{m-1}_{t-s} f_{m-1} ds$ are fixed functions of t, it follows easily from (3.11) that the system (3.2) is approximately controllable in m passes if and only if

$$\overline{\text{Range (G)}} = L^2(0,\tau;H)$$

and this is true iff

$$\ker G^* = 0 \quad \text{(the zero of } L^2(0,\tau;H)),$$

by Lemma 4.2. Hence we have

Lemma 4.3 The system (3.2) is approximately controllable in m passes iff the relations

$$(4.2) \qquad \int_s^\tau B^*(\mathcal{J}^*_{t-s})_{i+1,m} \, h(t)dt = 0 \quad , \quad h \in L^2(0,\tau;H)$$

for almost all $s \in [0,\tau]$ and $0 \le i \le m-1$ imply that

$$h(t) = 0 \text{ for almost all } t \in [0,\tau] \;. \; \square$$

The following corollary is an easy consequence of lemma 4.3.

Corollary 4.4 The system (3.2) is approximately controllable in m passes iff the relation

$$\int_s^\tau B^*\mathcal{J}^*_{t-s} h_m(t)dt = 0 \;, \tag{4.3}$$

for $h_m = (h,h,\ldots,h) \in \bigoplus_{i=o}^{m-1} L^2(0,\tau;H)$, and for almost all $s \in [0,\tau]$ implies that

$$h \overset{a.e.}{=} 0 \quad \text{on} \quad [0,\tau] \;. \; \square$$

As an example of the application of lemma 4.3, we derive the sufficient condition (4.1) in the finite demensional case. Suppose, therefore, that A_o and A_1 are matrices and that $A_2 = \ldots = A_\ell = 0$. Then, if (4.1) holds, we wish to show that (with $m = n$) the equations

$$\int_s^\tau B^*T^*_{t-s} h(t)dt = 0 \tag{i}$$

$$\int_s^\tau B^*K_2^1(t-s,0)^*h(t)dt = 0 \tag{ii}$$

$$\int_s^\tau B^*K_3^{11}(t-s,0)^*h(t)dt = 0 \tag{iii}$$

$$\vdots$$

$$\int_s^\tau B^*K_h^{1\ldots 1}(t-s,0)^*h(t)dt = 0 \tag{n}$$

imply that $h(t) = 0$ a.e. The left-hand sides of these equations are absolutely continuous and differentiable a.e. , and so, we have, apart from on a set of measure xero,

$$(4.4) \qquad B^*h(s) + \int_s^\tau B^*A_o^*T^*_{t-s}h(t)dt = 0 \qquad \text{from (i)} .$$

Now,

$$K_2^1(t-s,0)^* = \int_o^{t-s} T^*_{s_1} A_1^* T^*_{t-s-s_1} ds_1$$

and so, (ii) implies that

$$\int_s^\tau \int_o^{t-s} B^*T^*_{s_1} A_1^* T^*_{t-s-s_1} h(t)ds_1 dt = 0 .$$

Hence, differentiating twice with respect to s, we have

$$(4.5) \; B^*A_1^*h(s) + \int_s^\tau B^*A_o^*T^*_{t-s}A_1^*h(t)dt + \int_s^\tau \int_o^{t-s} B^*T^*_{s_1}A_1^*A_o^{*2}T^*_{t-s-s_1}h(t)ds_1 dt = 0 .$$

Note that (4.4) and (4.5) can be written in the forms

$$B^* (s) + \int_s^\tau k_1(t,s)h(t)dt = 0$$

$$B^*A_1^*h(s) + \int_s^\tau k_2(t,s)h(t)dt = 0$$

for some bounded operators k_1 and k_2. In exactly the same way, we can use (iii)-(n) to show that

$$B^*A_1^{*2}h(s) + \int_s^\tau k_3(t,s)\, h(t)dt = 0$$

$$B^*A_1^{*n-1}\, h(s) \int_s^\tau k_n(t,s)h(t)dt = 0$$

again, for some bounded kernels $k_3,\ldots,k_n$. Hence, we have

$$(4.6)\qquad \begin{pmatrix} B^* \\ B^*A_1^* \\ \cdot \\ \cdot \\ \cdot \\ B^*A_1^{*n-1} \end{pmatrix} h(s) + \int_s^\tau \begin{pmatrix} k_1(t,s) \\ k_2(t,s) \\ \cdot \\ \cdot \\ \cdot \\ k_n(t,s) \end{pmatrix} h(t)dt = 0$$

for almost all s. However, the matrix $[B\ A_1B \ldots A_1^{n-1}B]$ is of rank n and so by picking out a set of linearly independent columns, an elementary existence and uniqueness argument on (4.6) now implies that

$$h(s) = 0 \quad \text{a.e.} \quad s\varepsilon\ [0,\tau].$$

5. Optimal Control

We now consider the linear-quadratic problem for the system (3.10), which we have written in the 'mild' form of (3.11). We shall consider the regulator problem, since the tracking problem is a simple generalization of the former. Suppose, then, that we wish to minimise the cost functional

$$(5.1)\quad J(u) = \langle x(\tau),\ G_1x(\tau)\rangle_{\mathcal{H}_m} + \int_0^\tau \{\langle x(s), G_2x(s)\rangle_{\mathcal{H}_m} + \langle u(s), Ru(s)\rangle_{\mathcal{U}_m}\}\ ds$$

where $G_1, G_2 \in \mathcal{L}(\mathcal{H}_m)$ and $R\varepsilon\mathcal{L}(\mathcal{U}_m)$. If we just require to control the final pass then we could choose

$$(5.2)\qquad G_1 = \begin{pmatrix} \underline{0} & 0 \\ 0 & \Gamma_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} \underline{0} & 0 \\ 0 & \Gamma_2 \end{pmatrix}$$

where $\Gamma_1, \Gamma_2 \varepsilon \mathcal{L}(H)$, or if we wish to control each pass with identity weighting, then $G_1 = G_2 = I$.

However, the equation (3.11) is inhomogeneous and so it is convenient to put

$$z(t) = x(t) - g(t)$$

where

$$g(t) = \int_o^t \mathcal{T}^m_{t-s} f_m \, ds$$

Then, we have

(5.3) $$\mathbf{z}(t) = \mathcal{T}^m_t z(o) + \int_o^t \mathcal{T}^m_{t-s} \mathcal{B}_m u(s) ds ,$$

where $z(0) = x(0)$, and the cost functional becomes

(5.4) $$J(u) = <\mathbf{z}(\tau)+g(\tau), G_1(\mathbf{z}(\tau)+g(\tau))>$$

$$+ \int_o^\tau \{<z(s)+g(s), G_2(z(s)+g(s))> + <u(s), Ru(s)>\} \, ds$$

and we have reduced the inhomogeneous regulator problem to a tracking problem in a new state $z(t)$. The solution of this problem is well-known (Curtain and Pritchard[7]) and the optimal control is given by

$$u_\infty(t) = -R^{-1}\mathcal{B}^*_m Q_m(t) z(t) - R^{-1}\mathcal{B}^*_m s_\infty(t)$$

where Q_m is the unique solution of the inner product Riccati equation

(5.5) $$\frac{d}{dt}\left\langle Q_m(t)h_m, k_m\right\rangle + \left\langle Q_m(t)h_m, \mathcal{A}_m k_m\right\rangle + \left\langle \mathcal{A}_m h_m, Q_m(t)k_m\right\rangle$$

$$+ \left\langle G_2 h_m, k_m\right\rangle = \left\langle Q_m(t)\mathcal{B}_m R^{-1}\mathcal{B}^*_m Q_m(t)h_m, k_m\right\rangle \qquad \text{for } t \varepsilon [0,\tau]$$

$$Q_m(\tau) = G_1 \quad , \qquad h_m, k_m \,\varepsilon\, \mathcal{D}(\mathcal{A}_m).$$

and s_∞ is the unique solution of the differential equation

(5.6) $$\frac{d}{dt}\left\langle s_\infty(t), h_m\right\rangle = -\left\langle s_\infty(t), (\mathcal{A}_m - \mathcal{B}_m R^{-1}\mathcal{B}^*_m Q_m(t)h_m\right\rangle$$

$$- \left\langle G_2 g(t), h_m\right\rangle$$

$$s_\infty(\tau) = -G_1 g(\tau).$$

Note that, in the equations (5.5) and (5.6), all inner products are with respect to $\mathcal{H}_m$.

If, moreover, we have $\mathcal{J}_t^* G_1$ and $\mathcal{J}_t^* G_2$ map $\mathcal{H}_m$ to $\mathcal{D}(\mathcal{A}_m^*)$ and

$$(5.7)\quad \left.\begin{array}{l} \int_0^\tau \|\mathcal{A}_m^* \mathcal{J}_t^{m*} G_1 h_m\| \, dt < \infty, \\ \int_0^\tau \|\mathcal{A}_m^* \mathcal{J}_t^{m*} G_2 h_m\| \, dt < \infty, \end{array}\right. \quad \forall \; h_m \varepsilon \mathcal{H}_m,$$

then $s_\infty(t)$ is the unique solution of the equation

$$(5.8)\quad \begin{array}{l} \dot{s}_\infty(t) = -(\mathcal{A}_m^* - \mathcal{Q}_m(t)\mathcal{B}_m R^{-1}\mathcal{B}_m^*) s_\infty(t) + G_2 g(t) \\ s_\infty(\tau) = G_1 g(\tau) \end{array}$$

Recall that

$$\mathcal{A}_m = \begin{pmatrix} A_o & & & & \\ A_1 & A_o & & \underline{0} & \\ \vdots & A_1 & A_o & & \\ A_\ell & & & & \\ & A_\ell & & & \\ \underline{0} & & & & \\ & & A_\ell \cdot & \cdots & A_o \end{pmatrix}$$

and since $Q(t)_m \varepsilon \mathcal{L}(\mathcal{H}_m, \mathcal{H}_m)$, we may write

$$Q_m = (Q_{ij})_{0 \le i \le m, \; 0 \le j \le m}$$

where $q_{ij}(t) \; \varepsilon \mathcal{L}(H,H)$. Then, if we let h and k be generic elements in $\bigcap_{i=o}^{\ell} \mathcal{D}(A_i)$, and apply (5.5) with

$$h_m = (0,0,\ldots,\underset{j}{h},0,\ldots,0)^T \ , \ k_m = (0,0,\ldots,\underset{i}{k},0,\ldots 0)^T$$

we obtain the $(m+1)^2$ equations (taking $G_1 = G_2 = I_{\mathcal{H}_m}$) :

$$(5.9)\quad \begin{aligned} &\frac{d}{dt} \langle Q_{ij} h, k\rangle + \sum_{\alpha=o}^{\ell} \langle A_\alpha^* Q_{i+\alpha,j} h,k\rangle + \sum_{\alpha=o}^{\ell} \langle h, A_\alpha^* Q_{j+\alpha,i} k\rangle \\ &+ \langle h,k\rangle \delta_{ij} = \sum_{\alpha=o}^{M} \langle Q_{i\alpha} BB^* Q_{\alpha j} h, k\rangle \end{aligned}$$

$$Q_{ij}(\tau) = I_H \delta_{ij} \ .$$

where, for simplicity, we have assumed that $R = I$ and we interpret $Q_{\beta,j} = 0$ if $\beta > m$.

The equations (5.9) represent a coupled set of nonlinear operator equations which in general would have to be solved numerically and then the solution $Q_m(t)$ would be used to find the evolution operator $U(t,s)$ generated by $(A_m^* - Q_m(\tau-t)\mathcal{B}_m R^{-1}\mathcal{B}_m^*)$ (assuming this exists) and then we could write

$$(5.10)\qquad s_\infty(\tau-t) = U(t,0)G_1 g(\tau) + \int_o^t U(t,s)G_2 g(\tau-s)ds \ .$$

The optimal control can then be written as

$$u_\infty(t) = -R^{-1}\mathcal{B}_m^* Q_m(t)x(t) - R^{-1}\mathcal{B}_m^* Q_m(t)g(t) - R^{-1}\mathcal{B}_m^* s_\infty(t).$$

This control is, however, noncausal since the feedback for the i^{th} pass requires knowledge of future passes $i+1,\ldots,m$. This difficulty can be obviated by using the original equation (3.1), from which we have

$$\begin{aligned} x_k(t) &= T_t x_k(0) + \sum_{i=1}^{\ell} \int_o^t T_{t-s} A_i x_{k-i}(s)ds + \int_o^t T_{t-s} B u_{\infty k}(s)ds \\ &= T_t x_k(0) + \sum_{i=1}^{\ell} \int_o^t T_{t-s} A_i x_{k-i}(s)ds - \sum_{j=o}^{m} \int_o^t T_{t-s} BB^* Q_{kj}(s) x_j(s)ds \\ &\quad - \sum_{j=o}^{m} \int_o^t T_{t-s} BB^* Q_{kj}(s) g(s)ds - \sum_{j=o}^{m} \int_o^t T_{t-s} BB^* s_\infty(s)ds. \end{aligned}$$

where we have set $Q_m = (Q_{kj})_{o\le k,j\le m}$ and we have set R=I for simplicity.

Define the operators η_i, $\zeta_{kj} \in \mathcal{L}(L^2([0,\tau];H))$ and $\xi^m_k \in L^2([o,\tau];H)$ by

$$(\eta_i y)(t) = \int_o^t T_{t-s}A_i y(s)ds \quad , \quad 1\le i\le \ell$$

$$(\zeta_{kj} y)(t) = \int_o^t T_{t-s}\, BB^* Q_{kj}(s)y(s)ds \quad , \quad o\le k,j\le m$$

$$\xi^m_k(t) = T_t x_k(o) - \sum_{j=o}^{m} \int_o^t T_{t-s}BB^* Q_{kj}(s)g(s)ds - \sum_{j=o}^{m} \int_o^t T_{t-s}BB^* s_\infty(s)ds \, ,$$

Then we can write the controlled system in the form:

$$\begin{pmatrix} \mu^m_{00} & \mu^m_{01} & \cdots & \mu^m_{0m} \\ \mu^m_{10} & \mu^m_{11} & \cdots & \mu^m_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \mu^m_{m0} & & \cdots & \mu^m_{mm} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \xi^m_0 \\ \xi^m_1 \\ \vdots \\ \xi^m_m \end{pmatrix}$$

where the matrix of operators (μ^m_{ij}) is defined by

$$\begin{pmatrix} (I+\zeta_{00}) & \zeta_{01} & \cdots & \zeta_{0m} \\ \zeta_{10} & (I+\zeta_{11}) & \cdots & \zeta_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \zeta_{m0} & \cdots & \cdots & (I+\zeta_{mm}) \end{pmatrix} + \begin{pmatrix} 0 & & & & & \\ -\eta_1 & \ddots & & & \underline{0} & \\ -\eta_2 & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ -\eta_\ell & \ddots & \ddots & \ddots & \ddots & \\ \underline{0} & -\eta_\ell & \cdots & -\eta_2 & -\eta_1 & 0 \end{pmatrix}$$

where we have assumed that $m > \ell$.

If we define inductively

$$\left.\begin{aligned} \mu^{k-1}_{ij} &= \mu^k_{ij} - \mu^k_{ik}(\mu^k_{kk})^{-1}\mu^k_{kj} \\ \text{and} \qquad \xi^{k-1}_i &= \xi^k_i - \mu^k_{ik}(\mu^k_{kk})^{-1}\xi^k_k \end{aligned}\right\} \; 0\le i\le k-1, 0\le j\le k, 1\le k\le m$$

(assuming that $(\zeta^k_{kk})^{-1}$ exists for each k) we obtain

$$\begin{pmatrix} \mu^0_{00} & 0 & & & \underline{0} \\ \mu^1_{10} & \mu^1_{11} & 0 & & \\ \cdots & \cdots & \cdots & & \\ \mu^{m-1}_{m-10} & \cdots & \mu^{m-1}_{m-1m-1} & 0 & \\ \mu^m_{mo} & \cdots & \cdots & & \mu^m_{mm} \end{pmatrix} \begin{pmatrix} x_o \\ x_1 \\ \cdot \\ x_{m-1} \\ x_m \end{pmatrix} = \begin{pmatrix} \xi^0_0 \\ \cdot \\ \cdot \\ \vdots \\ \xi^{m-1}_{m-1} \\ \xi^m_m \end{pmatrix} .$$

Now, we have

$$x_m = (\mu^m_{mm})^{-1} (\xi^m_m - \Sigma^{m-1}_{i=o} \; \mu^m_m x_i)$$

$$x_{m-1} = (\mu^{m-1}_{m-1\ m-1})^{-1} (\xi^{m-1}_{m-1} - \Sigma^{m-2}_{i=0} \; \mu^{m-1}_{m-1i} \; x_i)$$

$$\vdots$$

$$x_1 = (\mu^1_{11})^{-1} (\xi^1_1 - \mu^1_{10} \; x_o)$$

Define

$$\mathcal{F}_1(x_o) = (\mu^1_{11})^{-1} (\xi^1_1 - \mu^1_{1o} x_o)$$

$$\mathcal{F}_2(x_o, x_1) = (\mu^2_{22})^{-1} (\xi^2_2 - \mu^2_{2o} x_o - \mu^2_{21} x_1)$$

$$\vdots$$

$$\mathcal{F}_m(x_o, \ldots, x_{m-1}) = (\mu^m_{mm})^{-1} (\xi^m_m - \sum^{m-1}_{i=o} \mu^m_{mi} x_i) .$$

Consider again the optimal control

$$u_\infty(t) = -\mathcal{B}^*_m Q_m(t) x(t) - \mathcal{B}^*_m Q_m(t) \; g(t) - \mathcal{B}^*_m \; s_\infty(t)$$

(with R =I). Then the k^{th} control (for the k^{th} pass) is

$$u_{\infty k}(t) = -B \sum^m_{i=o} Q_{ki} x_k(t) - (\mathcal{B}^*_m Q_m(t) q(t) - \mathcal{B}^*_m \; s_\infty(t))_k \; .$$

In order to make this control causal we must write $x_{k+1}, \ldots, x_m$ in terms of $x_o, \ldots, x_k$. This can be achieved by writing

$$x_{k+1} = \mathcal{F}_{k+1}(x_o, \ldots, x_k)$$

$$x_{k+2} = \mathcal{F}_{k+2}(x_o, \ldots, x_k, x_{k+1})$$

$$(5.11) \qquad = \mathcal{F}_{k+2}(x_o, \ldots, x_k, \mathcal{F}_{k+1}(x_o, \ldots, x_k))$$

$$\vdots$$

$$x_m = \mathcal{F}_m(x_o, \ldots, x_{m-1})$$

$$= \mathcal{F}_m(x_o, \ldots, x_k, \mathcal{F}_{k+1}(x_o, \ldots, x_k), \mathcal{F}_{k+2}(x_o, \ldots, x_k, \mathcal{F}_{k+1}), \ldots) .$$

It should be noted that ζ_{kk} is a Volterra operator and so the right hand sides of these relations require knowledge of $x_o, \ldots, x_k$ only on the interval $[o,t]$.

In order to simplify the solution somewhat, we shall consider the application of receding horizon optimal control in the next section. This type of control has been considered in both the finite-dimensional[9] and the infinite dimensional cases, (Banks[5]), and replaces the linear feedback law in the classical linear-quadratic problem with a nonlinear feedback law which responds more quickly to large disturbances but more slowly to small (possibly noise) perturbations.

6. Receding Horizon Control

In this section we shall assume for simplicity that the initial states $x_{-1}(t), \ldots, x_{-\ell}(t)$ are zero for all t; the case of general initial conditions will be considered in a future paper. From (3.10) it follows that

$$\dot{x} = \mathcal{A}_m x + \mathcal{B}_m u ,$$

since $f \equiv o$. It has been shown[5] that if the pair $(\mathcal{A}_m, \mathcal{B}_m)$ is approximately controllable and we let $G_2 = 0$ and $G_1 = \alpha I$, with $\alpha \to \infty$ then we obtain the open loop control

$$(6.1) \qquad u^* = - R^{-1}\, {}^*\mathcal{J}_{-t}^{m*}\, W^{-1}(T)\, x_o \qquad , \qquad o \leq t \leq T$$

where we have optimized over the subinterval $[0,T] \subseteq [0, \tau,]$ and, of course, if $\mathcal{J}_t$ is a semigroup, then $\mathcal{J}_{-t}$ is defined only on a certain subspace $\mathcal{H}(t)$ of $\mathcal{H}$ (see (5) for details). In (6.1) the invertible operator W(T) is given by

$$(6.2) \qquad W(T) = \int_o^T \mathcal{T}^m_{-s}\mathcal{B}\,R^{-1}\mathcal{B}^*\mathcal{T}^{m*}_{-s}\,ds\ ,$$

which can be shown to be defined on $\mathcal{H}$ for $T\geq 0$. In the receding horizon philosophy, we apply (6.1) as if we were beginning a new optimization over the time interval T at each time t. In other words, we replace (6.1) by the feedback control

$$(6.3) \qquad u^* = -R^{-1}\mathcal{B}^*\,W^{-1}(T)x(t)\ , \qquad \tau\geq t\geq 0\ .$$

It follows from the definition of $\mathcal{T}^m_t$ that, if we take $R = I$, then

$$W(T) = \begin{pmatrix} \int_o^T T_{-s}BB^*T^*_{-s}ds\ , & \int_o^T T_{-s}BB^*K_2^1(-s,o)^*ds\ , & \int_o^T T_{-s}BB^*(K_2^2(-s,0)^*+K_3^{11}(-s,0)^*)ds, \ldots \\ \int_o^T K_2^1(-s,0)BB^*T^*_{-s}ds, & \begin{pmatrix} \int_o^T K_2^1(-s,0)BB^*K_2^1(-s,0)^*ds \\ +\int_o^T T_{-s}BB^*T^*_{-s}ds \end{pmatrix}, & \ldots\ldots\ldots \\ \int_o^T (K_2^2(-s,0)+K_3^{11}(-s,0))BB^*T^*_{-s}ds\ , & \begin{pmatrix} \int_o^T (K_2^2(-s,0)+K_3^{11}(-s,0))BB^*K_2^1(-s,0)^*ds \\ +\int_o^T K_2^1(-s,0)BB^*T^*_{-s}ds \end{pmatrix}, & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}.$$

In order to determine the receding horizon control from (6.3) we must invert the operator matrix W(T). This will be very difficult in general, but we shall give an example in the next section in which this inversion is fairly simple. The inversion of W(T) can be constructed inductively using the following result.

<u>Lemma 6.1</u> Suppose that

$$\mathcal{F} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$$

is a bounded invertible operator defined on $\mathcal{H}_k \; (= \overset{k}{\underset{i=o}{\oplus}} H)$, where

$F_1 \in \mathcal{L}(\mathcal{H}_{k-1}, \mathcal{H}_{k-1})$, $F_2 \in \mathcal{L}(H, \mathcal{H}_{k-1})$, $F_3 \in \mathcal{L}(\mathcal{H}_{k-1}, H)$, $F_4 \in \mathcal{L}(H,H)$,

and assume that F_1 , F_4 , $G_1 \triangleq F_1 - F_2 F_4^{-1} F_3$, $G_2 \triangleq F_4 - F_3 F_1^{-1} F_2$

are invertible operators, on their respective domains. Then, we have

$$\mathcal{F}^{-1} = \begin{pmatrix} G_1^{-1} & -G_1^{-1} F_2 F_4^{-1} \\ -G_2^{-1} F_3 F_1^{-1} & G_2^{-1} \end{pmatrix}$$

The proof of this result is trivial. □

If we write

$$W(T) = (W_{ij}(T))_{o \le i \le m, o \le j \le m}$$

where $W_{ij}(T) \in \mathcal{L}(H,H)$, then the control for two passes is given by

$$(6.4) \qquad u^* = -R^{-1} \begin{pmatrix} B^* & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} W_{oo}(T) & W_{o1}(T) \\ W_{10}(T) & W_{11}(T) \end{pmatrix}^{-1} x(t) .$$

We can find the inverse of the W matrix by using lemma 6.1 (assuming the conditions hold) and then we can apply the lemma again to find the control for three passes by partitioning the W matrix in the form

$$\left(\begin{array}{cc|c} W_{oo}(T) & W_{01}(T) & W_{02}(T) \\ W_{10}(T) & W_{11}(T) & W_{12}(T) \\ \hline W_{20}(T) & W_{21}(T) & W_{22}(T) \end{array}\right).$$

Since we have already inverted the matrix in the top left hand corner the inverse for the case of j passes now follows by induction. This iteration method should be effective when each operator $W_{ij}(T)$ takes a particularly simple form.

It remains only to note that, for receding horizon control, we make the feedback control nonlinear by choosing the interval of optimization T to depend on the current state x(t). However, we again have a noncausal solution as in Section 5, and so we must first put the control in a causal form. This can be done as before by defining η_i as in section 5, and ζ_{kj} and ξ_k^m are now defined by

$$(\zeta_{kj}y)(t) = \int_o^t T_{t-s}\, BB^*(W^{-1})_{kj}(T)\, y(s)ds \ ,$$

$$\xi_k^m(t) = T_t x_k(o) \ ,$$

where we have used the same notation ζ and ξ for convenience (and again R=I). The expression (5.11) therefore allows us to express the values of the states $x_{k+1},\ldots,x_m$ in terms of the states $x_i,\ldots,x_k$ on the k^{th} pass.

The control (6.4) is therefore the receding horizon control when x(t) is expressed in causal from as above. We are now in a position to let T vary so that the control reacts quickly to large rrors and more slowly to small perturbations. For example, a parricular choice of T for the k^{th} pass could be

$$T_k = \min\left\{\frac{1}{||P_k x(t)||_{\mathcal{H}_k}} \ , \ \tau - t\right\}.$$

7. Example

In this simple example, we shall consider a one dimensional heating process. We shall denote the state of the k^{th} pass by $z_k(t,x)$ and let

$$A_o z(t,x) = \frac{\partial^2 z(t,x)}{\partial x^2} \quad , \quad \mathcal{D}(A_o) = \left\{ z \in L^2[0,1] : z_x, z_{xx} \in L^2[0,1], z(t,o) = z(t,1) = 0 \right\}$$

where we assume the process takes place on the unit interval $[0,1]$.

We suppose that $\ell=1$ and $B=I$, in which case the system equation becomes

$$\frac{\partial z}{\partial t}{}_k(t,x) = \frac{\partial^2 z_k(t,x)}{\partial x^2} + A_1 z_{k-1}(t,x) + u_k(t,x) \; .$$

A_1 will frequently be some sort of bounded integral operator. However, in order to obtain an explicit solution we shall take $A_1 = I$. More general operators A_1 could be considered at the expense of more complicated manipulation. The system is clearly controllable in any number of passes and we shall take $m = 1$ (i.e. two passes).

Now A_o has a complete set of orthonormal eigenfunctions

$$\phi_n(x) = \sqrt{2} \sin n\pi x \quad , \quad n \geq 1$$

and it will be convenient to obtain the solution in terms of these functions. Consider first the equations (5.9). If we represent each operator in terms of the basis $\{\phi_n\}$ and write

$$Q_{ij} = (Q_{ij}^{pq})_{1 \leq p,q < \infty} \quad , \quad 0 \leq i,j \leq 1$$

for the matrix of Q_{ij} in this basis, then from (5.9), we obtain the equations

$$\dot{Q}_{oo}^{pq} - \pi^2(p^2+q^2)\, Q_{oo}^{pq} + Q_{10}^{pq} + Q_{10}^{qp} + \delta_{pq} = \sum_{k=1}^{\infty} Q_{oo}^{pk} Q_{oo}^{kq} + \sum_{k=1}^{\infty} Q_{01}^{pk} Q_{10}^{kq}$$

$$\dot{Q}_{01}^{pq} - \pi^2(p^2+q^2) Q_{01}^{pq} + Q_{11}^{pq} = \sum_{k=1}^{\infty} Q_{oo}^{pk} Q_{01}^{kq} + \sum_{k=1}^{\infty} Q_{01}^{pk} Q_{11}^{kq}$$

$$\dot{Q}_{11}^{pq} - \pi^2(p^2+q^2)\, Q_{11}^{pq} + 1 = \sum_{k=1}^{\infty} Q_{10}^{pk} Q_{01}^{kq} + \sum_{k=1}^{\infty} Q_{11}^{pk} Q_{11}^{kq}$$

with $\quad Q_{oo}^{pq}(\tau) = Q_{11}^{pq}(\tau) = \delta_{pq}, \; Q_{01}^{pq}(\tau) = 0 \; , \quad 1 \leq p,q < \infty.$

Note that we have taken $G_1 = G_2 = I$, $R = I$, and, of course, since $Q_1(t)$ is self-adjoint, we have $Q^{pq}_{oo} = Q^{qp}_{01}$. Since the Riccati equation has a unique solution, it follows that

$$Q^{pq}_{ij}(t) \equiv 0 \ , \quad t \in [0,\tau] \ , \quad 0 \leq ij \leq 1 \ , \quad p \neq q \ ,$$

and the diagonal terms satisfy the equations

$$\dot{Q}^{pp}_{oo} - 2\pi^2 p^2 Q^{pp}_{oo} + 2Q^{pp}_{10} + 1 = (Q^{pp}_{oo})^2 + (Q^{pp}_{10})^2$$

$$\dot{Q}^{pp}_{10} - 2\pi^2 p^2 Q^{pp}_{10} + Q^{pp}_{11} = Q^{pp}_{oo} Q^{pp}_{10} + Q^{pp}_{10} Q^{pp}_{11}$$

$$\dot{Q}^{pp}_{11} - 2\pi^2 p^2 Q^{pp}_{11} + 1 = (Q^{pp}_{10})^2 + (Q^{pp}_{11})^2 \ .$$

Hence, if $\Theta_p = \begin{pmatrix} Q^{pp}_{oo} & Q^{pp}_{10} \\ Q^{pp}_{10} & Q^{pp}_{11} \end{pmatrix}$, we have

$$\dot{\Theta}_p - \begin{pmatrix} \pi^2 p & 1 \\ 0 & \pi^2 p \end{pmatrix} \Theta_p - \Theta_p \begin{pmatrix} \pi^2 p & 0 \\ 1 & \pi^2 p \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Theta_p^2 \ , \qquad \Theta_p(\tau) = I$$

and so writing $\Theta_p = Y_p X_p^{-1}$, with $X_p(\tau) = Y_p(\tau) = I$, we have

$$\frac{d}{dt}\begin{pmatrix} X_p \\ Y_p \end{pmatrix} = \begin{pmatrix} -\begin{pmatrix} \pi^2 p & 0 \\ 1 & \pi^2 p \end{pmatrix} & - I \\ - I & \begin{pmatrix} \pi^2 p & 1 \\ 0 & \pi^2 p \end{pmatrix} \end{pmatrix} \begin{pmatrix} X_p \\ Y_p \end{pmatrix} .$$

If Ξ_p denotes the matrix of this equation, then

$$\begin{pmatrix} X_p(t) \\ Y_p(t) \end{pmatrix} = \exp\{\Xi_p(t-\tau)\} \begin{pmatrix} I \\ I \end{pmatrix} .$$

The exponential can be evaluated explicitly by diagonalizing Ξ_p , which has the eigenvalues $\pm\sqrt{\pi^4 p^2+1\pm i}$. We shall not carry this out in detail here, since we intend only to illustrate the theory.

All that remains to do now is to evaluate the causal representation of the feedback control. For simplicity, we shall assume that $x_{-1}(t) \equiv 0$. In this case, we have

$$\xi_1^m(t) = T_t x_1(0)$$

and we require only to write $x_1(t)$ in terms of $x_o(t)$ for use on the first pass. Now,

$$(\zeta_{11} y)(t) = \int_o^t T_{t-s} Q_{11}(s) y(s) ds$$

and so to invert $I + \zeta_{11}$, we must solve the equation

$$(I + \zeta_{11})y = h .$$

In terms of the p^{th} coordinate functions $y_p(t) = \langle y(t),\phi\rangle$, $h_p(t) = \langle h(t),\phi\rangle$, this becomes

$$y_p(t) + \int_o^t e^{-p^2\pi^2 s} \; Q_{11}^{pp}(s)\; y_p(s)\; ds = h_p(t)$$

or

$$y_p(t) + \int_o^t m_p(s)\; y_p(s)\; ds = h_p(t)$$

where

$$m_p(s) = e^{-p^2\pi^2 s} \; Q_{11}^{pp}(s) .$$

This equation has the solution

$$y_p(t) = h_p(t) - \int_o^t h_p(s)\; m_p(s)\; e^{\{-\int_s^t m_p(t_1) dt_1\}}\; ds .$$

We also have

$$(\mu^1_{1o}x_o)(t) = (\zeta_{1o}x_o)(t) - (\eta_1 x_o)(t)$$

$$= \int_o^t T_{t-s}Q_{1o}(s)\ x_o(s)ds - \int_o^t T_{t-s}x_o(s)ds\ .$$

Hence, the p^{th} component of $\xi^m_1(t) - (\mu^1_{1o}x_o)(t)$ is

$$n_p(t) \triangleq e^{-n^2\pi^2 t}\, x^p_1(o) + \int_o^t e^{-n^2\pi^2(t-s)}(\,Q^{pp}_{1o}(s))x^p_o(s)ds$$

where

$$x^p_i(t) = \langle x_i(t), \phi_p \rangle .$$

Since $\quad x_1 = (\mu^{1}_{11}{}^{-1})\,(\xi^1_1 - \mu^1_{1o}x_o)$

we have

$$x^p_1(t) = n_p(t) - \int_o^t n_p(s)m_p(s)\ e^{\{-\int_s^t m_p(t_1)dt_1\}}ds \tag{7.2}$$

and we have written $x_1(t)$ in terms of $x_o(t)$ for use on the first pass.

Let us consider now the receding horizon control for the system (7.1). An elementary computation shows that the W matrix has the form

$$W(T) = \begin{pmatrix} W_{oo}(T) & W_{o1}(T) \\ W_{1o}(T) & W_{11}(T) \end{pmatrix}$$

where, in terms of their basis representations,

$$W_{oo}(T) = \text{diag}\left\{\frac{1}{2p^2\pi^2}(e^{2p^2\pi^2 T} - 1)\right\} \triangleq \text{diag}\ \{u_p(T)\}\quad ,\ \text{say}$$

$$W_{o1}(T) = \text{diag}\left\{\frac{1}{4p^4\pi^4}(e^{2p^2\pi^2 T} - 1) - \frac{T}{2p^2\pi^2}e^{2p^2\pi^2 T}\right\} \triangleq \{\text{diag}\ \ v_p(T)\}$$

$$W_{11}(T) = \text{diag}\left\{\frac{1}{4p^6\pi^6}(e^{2p^2\pi^2 T} - 1) - \frac{T}{2p^4\pi^4}(e^{2p^2\pi^2 T}) + \frac{T^2}{2p^2\pi^2}e^{2p^2\pi^2 T} + u_p(T)\right\}$$

$$\triangleq \text{diag}\ \ \{w_p(T)\}\ .$$

By lemma 6.1 the inverse of W(T) is given by the diagonal matrices

$$(W^{-1})_{oo}(T) = \text{diag}\{\ \bar{u}_p(T) \triangleq (u_p(T) - w^{-1}{}_p(T)v^2_p(T))^{-1}\}$$

$$(W^{-1})_{o1}(T) = \text{diag}\{\ \bar{v}_p(T) \triangleq -\bar{u}_p(T)v_p(T)w^{-1}_p(T)\}$$

$$(W^{-1})_{11}(T) = \mathrm{diag}\{\ \bar{w}_p(T) \triangleq (w_p(T) - v_p^2(T)u_p^{-1}(T))^{-1}\}\quad ,$$

for $T > o$, since we know that $W(T)$ is invertible for $T > o$.

The causal controller for fixed $T > o$ is now given by (7.2) using the same reasoning as before, except that now we have

$$m_p(s) = e^{-p^2\pi^2 s}\,(W_{11}^{-1})^{pp}(T)$$

and

$$n_p(t) = e^{-n^2\pi^2 t}x_1^p(o) + \int_o^t e^{-n^2\pi^2(t-s)}(1-(W_{10}^{-1})^{pp}(s))x_o^p(s)ds.$$

we can then choose, for example

$$T_o = \min\left\{\frac{1}{\|x_o(t)\|_H}\quad ,\quad \tau - t\right\}$$

on the first pass, and

$$T_1 = \min\left\{\frac{1}{(\|x_o(t)\|^2 + \|x_1(t)\|^2)^{\frac{1}{2}}}\quad ,\quad \tau - t\right\}$$

on the second pass.

8. Conclusions

The controllability, optimal control and receding horizon control of a distributed multipass process has been studied in this paper and a general controllability result has been obtained. The optimal control problem, when solved using the semigroup approach is seen to lead to a noncausal solution. However, this solution can be replaced by an equivalent causal one as shown in section 5. The Riccati equation for the general multipass process is a complex set of coupled operator equations which, in general, is likely to require an approximate numerical solution.

If the system matrices $\mathcal{A}_m$, $\mathcal{B}_m$ form an approximately controllable pair, then we have seen that the receding horizon principle may be extended to linear multipass processes. Again a causal (nonlinear) feedback control law can be obtained, which will react more quickly than the linear quadratic solution to large disturbances, but more slowly to small pertubations.

Finally, we have given a very simple example which was chosen to illustrate the theory in such a way that an explicit solution could be obtained. Of course, more general systems would require numerical solutions, which would not bring out the important aspects of the theory in such a clear way. In more realistic systems, it is likely that $\mathcal{B}$ may be unbounded (boundary control) and possibly time dependent - this would be the case, for example, in multipass welding. This type of problem will be considered in a future paper.

(9) References

(1) Edwards, J.B. and Owens, D.H. 1981, 'Analysis and Control of Multipass Processes', Research Studies Press, Wiley.

(2) Owens, D.H. 1977. 'Stability of Linear Multipass Processes'. Proc. IEE 124 , 1079-1082.

(3) Collins, W.D. 1980. IMA Conf. 'Control and its Applications', Sheffield University.

(4) Banks, S.P. 1981. 'An abstract Setting for Non-linear Distributed Multipass Processes', Int. J. Systems Sci., 12 , no 11, 1287-1301.

(5) Banks, S.P. 1982. 'The Receding Horizon Principle for Distributed Systems', University of Sheffield Research Report No. 167, to appear in J. Franklin Institute.

(6) Yosida, K. 1974. 'Functional Analysis, III ed., Springer-Verlag.

(7) Curtain, R.F., and Pritchard, A.J. 1978. 'Infinite-dimensional Linear Systems Theory', Springer-Verlag. Lecture notes on control and information sciences.

(8) Dolecki, S. and Russel, D. 1977. 'A general theory of observation and control, SIAM J. Control, 15 , 185-220.

(9) Shaw, L. 1979. 'Nonlinear control of linear multivariable systems via state-dependent feedback gains', IEEE Trans. Aut. Control, AC-24, No.1. 108-112.