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OPTIMAL CONTROL OF LINEAR DIFFERENTIAL MULTIPASS PROCESSES

by

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1. INTRODUCTION

MULTIPASS PROCESSES [1] are a relatively new class of control system first introduced by Edwards in 1974 [2] to model processes which may be considered as a repeated sequence of actions by a processing tool on a material or workpiece. Examples of such systems include the longwall cutting of coal, metal rolling, agricultural ploughing and image processing.

The stability of such systems when governed by linear equations has been considered by Owens in 1977 [3], whilst the controllability of linear multipass systems has been considered by Collins in 1980 [4].

This report is concerned with the optimal control of multipass systems described by linear ordinary differential equations with constant coefficients. We denote the state and control vectors on pass k by $x_k(t)$ and $u_k(t)$ respectively, where $k = 0, 1, 2, \ldots$. Here $x_k(t)$ is assumed to be an n-vector and $u_k(t)$ an r-vector. The variable t can be taken as the distance along the pass or, if each pass is described with constant velocity, time. The passes are assumed to have constant length T, so that $t \in [0, T]$. The rate of change of the state vector x_k at position t is assumed to be dependent only on the state and control vectors on the current pass k at position t, i.e. $x_k(t)$ and $u_k(t)$, and also on the state vector from the pass previous to the current one at the same position t, i.e. $x_{k-1}(t)$. We then have the following system of ordinary differential equations:

$$\frac{dx_k(t)}{dt} = A_1 x_k(t) + A_2 x_{k-1}(t) + Bu_k(t), \qquad (1.1)$$

$0 \leq t \leq T$, $k = 1, 2, 3, \ldots$, where A_1 and A_2 are constant $n \times n$ matrices, B a constant $n \times r$ matrix and T the fixed length of the pass.

To determine the system we require an initial pass profile, i.e. the state on the zeroth pass:

$$x_o(t) = f(t), \qquad 0 \leq t \leq T \qquad (1.2)$$

and boundary conditions at the beginning of each pass:

$$x_k(0) = c_k, \qquad k = 1, 2, 3, \ldots . \tag{1.3}$$

In the following discussion we shall limit our attention to processes involving a fixed finite number of passes only. Thus k will assume the values $k = 1, 2, \ldots, M$ where M is the total number of passes.

Our problem is to synthesize, if possible, a feedback controller of the above system by suitable choice of quadratic performance criteria.

In general there are two main types of controller required in quadratic optimization problems, namely terminal controllers and regulators. A terminal controller is designed to bring a system close to a desired terminal condition (usually taken to be zero) while exhibiting "acceptable" behaviour on the way. A regulator is designed to keep a stationary system within "acceptable" deviation from a reference condition using "acceptable" amounts of control. This report will be concerned with finding terminal controllers for the system governed by (1.1), (1.2) and (1.3).

Thus, given a system of the form of (1.1) we desire to bring it from an initial state $c_k (=x_k(0))$ to a terminal state $x_k(T) \simeq 0$ on each pass k, $1 \leq k \leq m$, using "acceptable" levels of control $u_k(t)$ and not exceeding "acceptable" levels of the state on the way, given also initial profile f(t).

The method we shall consider to do this is to minimize a quadratic performance index made up of terms involving a quadratic form in the terminal state added to an integral of quadratic forms in the state and control over 0 to T for each pass k and then summed over all the passes $k = 1, 2, \ldots, M$:

$$J = \frac{1}{2} \sum_{k=1}^{M} \left\{ x_k'(T) G x_k(T) + \int_0^T \left[x_k'(\tau) Q x_k(\tau) + u_k'(\tau) R u_k(\tau) \right] d\tau \right\} \tag{1.4}$$

Here G and Q are positive semidefinite matrices and R a positive definite matrix, all constant and of conformable dimensions. Primes will be used throughout this report to denote the transpose of a vector or matrix. Of course, an appropriate choice of these matrices is now required in order to obtain the desired "acceptable" levels of $x_k(T)$, $x_k(t)$ and $u_k(t)$, $k = 1, 2, \ldots, M$. When dealing with a real process one might, for example, choose them to be diagonal with

$$1/(G)_{ii} \;=\; \text{maximum acceptable value of } \left[(x_k(T))_i\right]$$

$$1/(Q)_{ii} \;=\; T \times \text{maximum acceptable value of } \left[(x_k(t))_i\right]$$

$$1/(R)_{ii} \;=\; T \times \text{maximum acceptable value of } \left[(u_k(t))_i\right]$$

for $k = 1, 2, \ldots, M$. Clearly (1.4) could be generalized to contain 'cross' terms in the control and state, for example, or by allowing Q and R to vary with τ and all three matrices to be dependent on k. For simplicity, however, we shall deal only with performance criteria of the form (1.4).

The choice of G and Q to be positive semidefinite and R positive definite is motivated on the grounds that we would like the performance index to have a non-negative value for all $x_k(t)$ and $u_k(t)$, $1 \le k \le m$. In particular we choose R positive definite so that any use of the control is penalized. The weaker condition is applied to G and Q since we may wish to neglect some of the system states.

More information on the use of quadratic performance criteria in optimal regulator and terminal controller problems with regard to other types of linear systems is available in the standard texts on optimal control (see e.g. [5]).

We now wish to investigate conditions to be satisfied by the control $u_k(t)$, $0 \le t \le T$, $1 \le k \le M$, for the performance criterion (1.4) to be minimized.

It is worth noting here that this type of system bears considerable resemblance to the quadratic optimization problem for differential delay systems where the delay occurs explicitly in the system equation as in

$$\min J = \frac{1}{2} x'(T_f)Gx(T_f) + \frac{1}{2}\int_0^{T_f}\left[x'(t)Qx(t) + u'(t)Ru(t)\right]dt$$

subject to

$$\frac{dx(t)}{dt} = A_1x(t) + A_2x(t-\tau) + Bu(t), \qquad 0 \le t \le T_f$$

$$x(\theta) = \phi(\theta) \quad -\tau \le \theta \le 0 .$$

Here A_1, A_2, B, G, Q and R are constant matrices similar to those in (1.1) and (1.4), T_f is the fixed terminal time of the process and τ the fixed length of the delay. The multipass problem we are considering reduces to the above form if we set $c_k = x_{k-1}(T)$, $1 \le k \le M$. In this case our delay length τ becomes T and the final time T_f becomes $M \times T$.

A solution to the above problem was determined by Chang and Lee [7].

Section 2 of this report will deal with basic concepts and preliminaries and will consider relevant aspects of causality in these processes. A "non-causal" controller for the system will be derived in section 3 of linear feedback form. The existence of causal linear feedback controllers will be demonstrated in section 4 and the main results concerning the conditions satisfied by two such controllers stated and discussed in sections 5 and 8. Section 5 also contains the conditions satisfied by another form of non-causal controller. Sections 6 and 7 contain computational and implementation considerations for the controllers derived.

2. BASIC DEFINITIONS, ADMISSIBLE CONTROLS AND CAUSALITY CONCEPTS

Let us first introduce some notation:

$C_n = C([0, T], \mathbb{R}^n)$ will denote the Banach space of continuous functions with domain [0, T] and range $\mathbb{R}^n$. The norm of this space is defined as $\|f\| = \sup_{0\le t\le T} |f(t)|$, where $|.|$ denotes the usual Euclidean norm of $\mathbb{R}^n$.

The Euclidean norm on $\mathbb{R}^n$ gives rise to the spectral matrix norm: $\|A\|_n = (\rho(A^TA))^{\frac{1}{2}}$ where A is an $n \times n$ real valued square matrix and $\rho(A^TA)$ denotes the maximum eigenvalue of A^TA.

Let $\hat{C}_F = \hat{C}_F([0, T] \times [0, T], A^n)$ denote the Banach space of real valued $n \times n$ matrix functions of two variables with domain $[0, T] \times [0, T]$. These functions are piecewise continuous with finite discontinuity along the diagonal of the domain such that

$$A(t, t+) - A(t, t-) = F(t), \quad 0 \le t \le T \quad \text{for all } A \in \hat{C}_F$$

where $F(t)$ is a real valued continuous $n \times n$ matrix function defined on $[0, T]$. The norm of this space is defined as:

$$\|A\|_F = \sup_{\substack{t,\theta \in [0,T] \\ t \neq \theta}} \{ \|A(t, \theta)\|_n \}, \qquad A \in \hat{C}_F$$

where $\|\cdot\|_n$ denotes the spectral matrix norm defined above.

Finally, we introduce the Hilbert spaces $H_1 = H([0, T], \underset{\sim}{\mathbb{R}}^n)$ and $H_2 = H([0, T], \underset{\sim}{\mathbb{R}}^r)$ of Lebesgue integrable vector functions defined on $[0, T]$ with ranges in $\underset{\sim}{\mathbb{R}}^n$ and $\underset{\sim}{\mathbb{R}}^r$ respectively. The inner products for these spaces are

$$\langle x, y \rangle_1 = \int_0^T x'(t)Qy(t)dt \qquad \text{for } x, y \in H_1$$

and

$$\langle x, y \rangle_2 = \int_0^T x'(t)Ry(t)dt \qquad \text{for } x, y \in H_2$$

where Q is an $n \times n$ positive definite constant matrix and R an $r \times r$ positive definite matrix.

These inner products give rise to the following norms:

$$\|x\|_1 = \left(\int_0^T x'(t)Qx(t)dt \right)^{\frac{1}{2}}$$

and

$$\|y\|_2 = \left(\int_0^T y'(t)Ry(t)dt \right)^{\frac{1}{2}}$$

We are now in a position to define an admissible control for system (1.1).

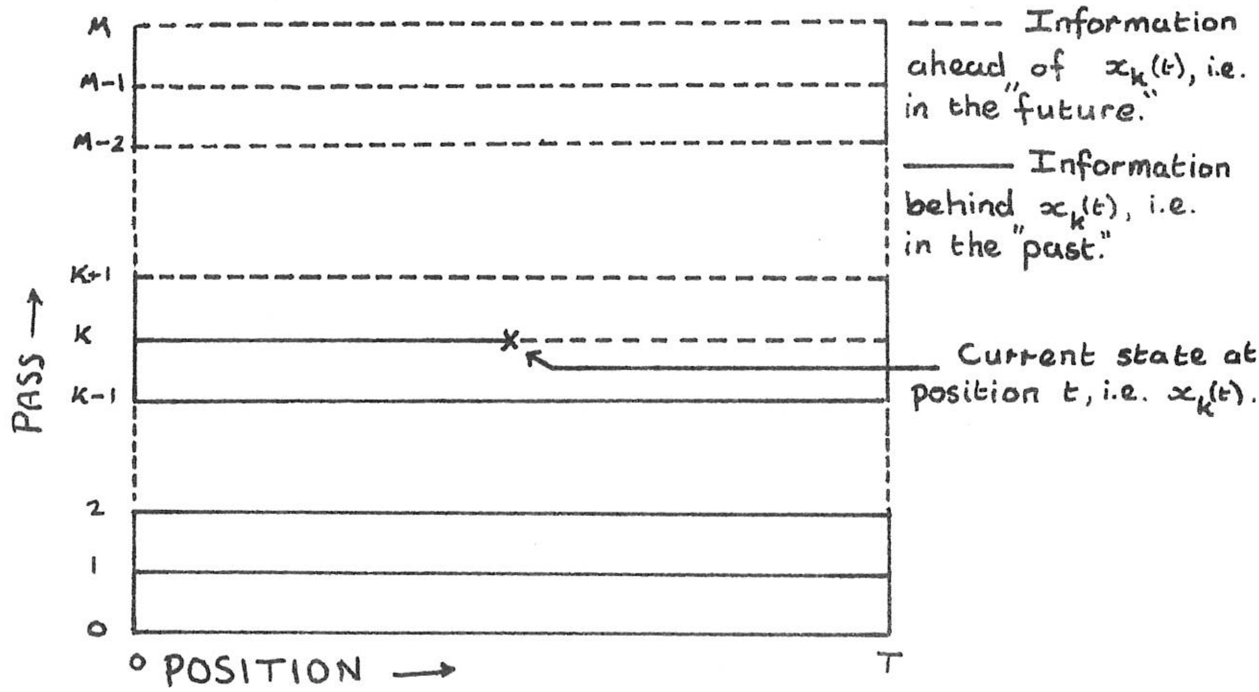
DEFINITION 2.1. An admissible control for system (1.1) is to satisfy the requirement that it is a continuous time-dependent functional of the system states, i.e. $u(t) = u(t, x)$ where

$$x = [x_0(\theta)\, x_1(\theta) \;\ldots\; x_M(\theta)]^T, \qquad 0 \le \theta \le T.$$

The control is otherwise unconstrained.

It is useful here to introduce the concept of causality with regard to multipass systems.

In the multipass systems considered here the passes are regarded as being described in sequence: $x_1(t)$ first, $x_2(t)$ second and so on. Thus if we are on pass k at position t all passes $x_\ell(t)$ for which $\ell > k$, $0 \le t \le T$ are considered as being in the "future", i.e. they have yet to be described. Similarly $x_k(\tau)$ for $t \le \tau \le T$ is considered as "future" information. The following diagram may help to illustrate the position.



This idea leads to the following definitions:

<u>DEFINITION 2.2.</u> A controller $u_k(t)$, $1 \le k \le M$, $0 \le t \le T$ is said to be causal if it feeds back the state only from the following set: $\{x_k(\tau): 0 \le \tau \le t\} \cup \{x_\ell(t): 0 \le t \le T, 0 \le \ell < k\} = C_k$. It need not, of course, feed back the whole of this set to be causal as a part of it may suffice, but it may not feed back any state information not contained in C_k. C_k will be known as the causal data set for controller $u_k(t)$.

We thus see that a causal controller may feedback only "past" information in the sense described above.

DEFINITION 2.3. A controller $u_k(t)$, $\ell \leq k \leq M$, $0 \leq t \leq T$ is said to be non-causal if at least some part of its feedback is from the set $N_k = \{x_k(\tau): t < \tau \leq T\} \cup \{x_\ell(t): 0 \leq t \leq T, k < \ell \leq M\}$. It may also feedback information from C_k. N_k will be known as the non-causal data set for controller $u_k(t)$.

Finally we remark that in the following discussion we shall be searching for linear controllers of (1.1). A controller is said to be linear if:

$$u_k(t, \alpha y + \beta z) = \alpha u_k(t, y) + \beta u_k(t, z) \qquad \begin{matrix} 1 \leq k \leq M, \\ 0 \leq t \leq T. \end{matrix}$$

for all

$$y, z \in C_k \cup N_k \text{ for all scalars } \alpha, \beta.$$

3. A NON-CAUSAL SOLUTION

We shall now obtain a solution to this problem of non-causal type. Write:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_M(t) \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_2 & A_1 & 0 & \cdots & 0 \\ 0 & A_2 & A_1 & \cdots & 0 \\ & & \ddots & \ddots & \\ 0 & & 0 & A_2 & A_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & B \end{bmatrix},$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_M(t) \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & Q \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & R \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} A_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & G \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}.$$

Here $x(t)$ will be an n.m vector, $u(t)$ an m.r vector, c an n.m vector, $\tilde{A}$ an n.m × n.m matrix, $\tilde{B}$ an m.r × m.r matrix, $\tilde{D}$ an n.m × n matrix, $\tilde{G}$ and $\tilde{Q}$ $n^2 \times n^2$ positive semidefinite matrices and $\tilde{R}$ an $r^2 \times r^2$ positive definite matrix.

In terms of these new quantities our problem now becomes:

$$\min J = \frac{1}{2} x'(T)\tilde{G}x(T) + \frac{1}{2}\int_0^T [x'(\tau)\tilde{Q}x(\tau) + u'(\tau)\tilde{R}u(\tau)]\, d\tau \tag{3.1}$$

subject to

$$\frac{dx(t)}{dt} = \tilde{A}x(t) + \tilde{B}u(t) + \tilde{D}f(t),\ 0 \le t \le T,\ x(0) = c. \tag{3.2}$$

From the calculus of variations we know that this problem has solution

$$u(t) = -\tilde{R}^{-1}\tilde{B}^{-1}p(t) \tag{3.3}$$

where $p(t)$ is a solution of the two point boundary value problem :

$$\left.\begin{aligned} \frac{dp(t)}{dt} &= -\tilde{A}p(t) - \tilde{Q}x(t), \quad 0 \le t \le T,\ p(T) = \tilde{G}x(T) \\ \frac{dx(t)}{dt} &= \tilde{A}x(t) + \tilde{B}u(t) + \tilde{D}f(t), \quad 0 \le t \le T,\ x(0) = C \end{aligned}\right\} \tag{3.4}$$

Here $p(t)$ is a differentiable, time-dependent m.n vector. This result is obtained by augmenting the functional (3.1) with the differential equation (3.2) in the usual way. The derivation can be found in the standard texts on optimal control, see, for example [5].

In the absence of the term $\tilde{D}f(t)$ in (3.4) this two point boundary value problem has the well known solution

$$p(t) = K(t)x(t) \tag{3.5}$$

where the n.m × m.n differentiable time-dependent matrix $K(t)$ satisfies the equation

$$\frac{dK(t)}{dt} + K(t)\tilde{A} + \tilde{A}'K(t) + \tilde{Q} - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^{-1}K(t) = 0, \quad K(T) = \tilde{G} \tag{3.6}$$

This type of equation is known as a Riccati since it contains a quadratic term in K. For derivation see again [5].

As can be seen, this yields a linear feedback control law given by:

$$u(t) = -\tilde{R}^{-1}\tilde{B}^{-1}K(t)x(t) . \tag{3.7}$$

As we also seek a linear feedback control for (3.2) in the presence of the $\tilde{D}f(t)$ term we shall try a solution of (3.4) of the form of (3.5) but with an added "tracking" vector in order to take into account the $\tilde{D}f(t)$ term.

We thus make a trial solution of (3.4) of the form

$$p(t) = K(t)x(t) + g(t) \tag{3.8}$$

where K(t) is an n.m × n.m differentiable time-dependent matrix and g(t) a differentiable time-dependent n.m vector.

Then

$$\frac{dp(t)}{dt} = \frac{d}{dt}(K(t)x(t) + g(t)) = \frac{dK(t)}{dt}x(t) + K(t)\frac{dx(t)}{dt} + \frac{dg(t)}{dt}$$

$$= \frac{dK(t)}{dt}x(t) + K(t)[\tilde{A}x(t) + \tilde{B}u(t) + \tilde{D}f(t)] + \frac{dg(t)}{dt} \quad \text{(using (3.2))}$$

$$= \frac{dK(t)}{dt}x(t) + K(t)\tilde{A}x(t) - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^{-1}(K(t)x(t) + g(t)) + K(t)\tilde{D}f(t) + \frac{dg(t)}{dt}$$

(using (3.3) & (3.8))

$$= \left\{\frac{dK(t)}{dt} + K(t)\tilde{A} - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^{-1}K(t)\right\}x(t) + \frac{dg(t)}{dt} - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^{-1}g(t) + K(t)\tilde{D}f(t) \tag{3.9}$$

From (3.4) we also have

$$\frac{dp(t)}{dt} = -\tilde{A}p(t) - \tilde{Q}x(t) = -\tilde{A}'K(t)x(t) - \tilde{A}'g(t) - \tilde{Q}x(t) \tag{3.10}$$

Equating the R.H.S's of (3.9) and (3.10) and collecting terms yields

$$\left[\frac{dK(t)}{dt} + K(t)\tilde{A} + \tilde{A}'K(t) - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^{-1}K(t) + \tilde{Q}\right]x(t)$$

$$+ \left[\frac{dg(t)}{dt} + (\tilde{A}' - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^{-1})g(t) + K(t)\tilde{D}f(t)\right] = 0, \quad 0 \le t \le T. \tag{3.11}$$

We can thus satisfy (3.11) identically if we set both brackets to zero. We also require that $p(T) = \tilde{G}x(T)$

i.e. $\tilde{G}x(T) = K(T)x(T) + g(T)$

Hence we choose $K(T) = \tilde{G}$, $g(T) = 0$.

To summarize, for a solution of the form (3.8) the matrix K and vector g must satisfy

$$\left.\begin{aligned} &\frac{dK(t)}{dt} + K(t)\tilde{A} + \tilde{A}'K(t) - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}'K(t) + \tilde{Q} = 0, \quad 0 \le t \le T \\ &K(T) = \tilde{G} \\ &\frac{dg(t)}{dt} + (\tilde{A}' - K(t)\tilde{B}\tilde{R}^{-1}\tilde{B}')g(t) + K(t)\tilde{D}f(t) = 0, \quad 0 \le t \le T \\ &g(T) = 0 \end{aligned}\right\} \qquad (3.12)$$

This is the standard Riccati solution of (3.1), (3.2) with tracking vector g.

Now, partition K(t), g(t) and p(t) so that

$$K(t) = \begin{bmatrix} K_{11}(t) & K_{12}(t) & \cdots & K_{1M}(t) \\ K_{21}(t) & \ddots & & \\ \vdots & & \ddots & \\ K_{M1}(t) & \cdots & & K_{MM}(t) \end{bmatrix}, \quad g(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_M(t) \end{bmatrix}, \quad p(t) = \begin{bmatrix} p_1(t) \\ \vdots \\ p_M(t) \end{bmatrix}$$

where each $K_{ij}(t)$ is an $n \times n$ matrix and $g_i(t)$, $p_i(t)$ are n-vectors.

Then we have

$$p_i(t) = \sum_{j=1}^{M} K_{ij}(t)x_j(t) + g_i(t), \quad 1 \le i \le M, \ 0 \le t \le T \qquad (3.13)$$

and hence from (3.3)

$$u_i(t) = -R^{-1}B'\left[\sum_{j=1}^{M} K_{ij}(t)x_j(t) + g_i(t)\right], \quad 1 \le i \le M, \ 0 \le t \le T. \qquad (3.14)$$

Since in general $K_{ij}(t) \not\equiv 0$, $1 \le i, j \le M$, $0 \le t \le T$, our controller at position t on pass i feedsback the state vector at the corresponding position t from all passes, 1, 2, ..., M. It is thus a non-causal controller in the sense of (2.3). It is, however, linear and thus, raises the following question:

Is there a causal linear feedback solution of (1.1) - (1.4)?
We answer this in the next section.

4. EXISTENCE OF A CAUSAL LINEAR FEEDBACK SOLUTION

In this section we reformulate the problem in an abstract Hilbert space setting in order to demonstrate the existence of a linear feedback solution which is causal. We shall also indicate the existence of a further non-causal solution of a different form from (3.14).

Let H_1 snd H_2 be the Hilbert spaces as defined in section 2. In this setting our problem now becomes

$$\left.\begin{aligned} &\min J = \frac{1}{2}\sum_{k=1}^{M}\left\{\|x_k\|_1^2 + \|u_k\|_2^2\right\} \\ &\text{subject to} \\ &x_{k+1} = Ax_k + Bu_k, \quad 1 \le k \le M, \\ &x_k \in H_1, \; u_k \in H_2, \quad 1 \le k \le M \end{aligned}\right\} . \qquad (4.1)$$

Here A and B are specified bounded linear operators.

We now quote the following theorem:

THEOREM 4.2 There exist bounded linear operators L_k and $\tilde{L}_k$ for which the choices

$$u_k = L_k x_{k-1} \quad \text{or} \quad u_k = \tilde{L}_k x_k \qquad 1 \le k \le M$$

both minimize J of (4.1).

The proof relies on the ideas of dynamic programming and is contained in Appendix 1.

The control law $u_k = L_k x_{k-1}$ of the above theorem is clearly causal since it only feeds back information from the pass previous to the current one. The second controller $u_k = \tilde{L}_k x_k$ may not, however, be causal since it could feedback information from the current pass "ahead" of the current position (i.e. $u_k(t)$ might depend on $x_k(\tau)$ for $\tau > t$).

We have, thus, answered the question of the existence of a causal linear feedback solution which was posed in the previous section.

It is also worth noting at this point that we have determined the existence in the form of $u_k = \tilde{L}_k x_k$ a (possibly) non-causal solution of a different type to (3.14), i.e. non-causal solutions to this problem are not necessarily unique. This suggests that a causal linear feedback solution to this problem may also be non-unique, as indeed is the case as will be seen later in this report.

While the theorem answers the question of existence of a linear causal feedback controller for (1.1) it does not indicate the conditions to be satisfied by such a solution. It is to the end of finding such conditions that we now turn.

5. MAIN RESULTS PART 1

The theorem to follow is a summary of the first main result of this report; it is a statement of sufficient conditions for a causal linear feedback control law of the form of theorem (4.2) (i.e. $u_k = L_k x_{k-1}$) to be optimal for criterion (1.4).

THEOREM 5.1 The linear control law

$$u_k(t) = -R^{-1}B'\left\{\int_0^T\left[\Pi_o(t)X(k,t,\theta)+\Pi_1(k,t,\theta)\right]x_{k-1}(\theta)d\theta+\Pi_o(t)p_k(t)+c_k(t)\right\},$$

$$0 \le t \le T, \quad 1 \le k \le M$$

provides the absolute minimum of performance criterion (1.4) for the dynamic system (1.1), (1.2) and (1.3) provided the $n \times n$ symmetric matrix, $\Pi_o(t)$, of functions defined on $[0, T]$ together with the $n \times n$ matrices, $\Pi_1(k,t,\theta)$, $X(k,t,\theta)$, of functions of two variables having domain $[0, T] \times [0, T]$ and k-vectors, $p_k(t)$ and $c_k(t)$ defined on $[0, T]$ satisfy the relations:

$$\frac{d\Pi_o}{dt}(t) + \Pi_o(t)A_1 + A_1'\Pi_o(t) + Q - \Pi_o(t)BR^{-1}B'\Pi_o(t) = 0, \quad 0 \le t \le T, \qquad (5.1a)$$

$$\Pi_o(T) = G$$

$$\left.\begin{aligned}
&\frac{\partial \Pi_1}{\partial t}(k,t,\theta)+(A_1'-\Pi_o(t)BR^{-1}B')\Pi_1(k,t,\theta)+A_2'\int_0^T\Big[\Pi_1(k+1,t,s)+\Pi_o(t)X(k+1,t,s)\Big] \\
&\qquad\qquad X(k,s,\theta)\,ds = 0, \\
&\qquad\qquad \theta \neq t,\ 0 \le t,\ \theta \le T,\ 1 \le k \le M \\
&\Pi_1(k,t,t+) - \Pi_1(k,t,t-) = \Pi_o(t)A_2, \quad 0 \le t \le T,\ 1 \le k \le M \\
&\Pi_1(k,T,\theta) = 0, \quad 0 \le \theta \le T,\ 1 \le k \le M \\
&\Pi_1(M+1,t,\theta) \equiv 0, \quad 0 \le \theta,\ t \le T \\
&\frac{\partial X}{\partial t}(k,t,\theta) = (A_1-BR^{-1}B'\Pi_o(t))X(k,t,\theta)-BR^{-1}B'\Pi_1(k,t,\theta), \quad \theta \neq t,\ 0\le t,\theta \le T, \\
&\qquad\qquad 1 \le k \le M \\
&X(k,t,t-) - X(k,t,t+) = A_2, \quad 0 \le t \le T,\ 1 \le k \le M \\
&X(k,0,\theta) = 0, \quad 0 \le \theta \le T,\ 1 \le k \le M \\
&X(M+1,t,\theta) \equiv 0,\ 0 \le t,\ \theta \le T
\end{aligned}\right\} \quad (5.1b)$$

$$\left.\begin{aligned}
&\frac{d\ell_k}{dt}(t)+(A_1'-\Pi_o(t)BR^{-1}B')\ell_k(t)+A_2'\Big[\ell_{k+1}(t)+\Pi_o(t)p_{k+1}(t)+\int_0^T\Big\{\Pi_1(k+1,t,s)+ \\
&\qquad\qquad \Pi_o(t)X(k+1,t,s)\Big\}p_k(s)\,ds\Big], \\
&\qquad\qquad 0 \le t \le T,\ 1 \le k \le M \\
&\ell_k(T) = 0, \quad 1 \le k \le M \\
&\ell_{M+1}(t) \equiv 0, \quad 0 \le t \le T \\
&\frac{dp_k}{dt}(t) = (A_1-BR^{-1}B'\Pi_o(t))p_k(t) - BR^{-1}B'\ell_k(t), \quad 0 \le t \le T,\ 1 \le k \le M \\
&p_k(0) = c_k,\ 1 \le k \le M \\
&p_{M+1}(t) \equiv 0, \quad 0 \le t \le T
\end{aligned}\right\} \quad (5.1c)$$

Here $\Pi_1(k,t,t+)$ denotes $\lim_{\theta\to t+} \Pi_1(k,t,\theta)$. Similarly $\Pi_1(k,t,t-)$ denotes $\lim_{\theta\to t-} \Pi_1(k,t,\theta)$. The proof of the above theorem is contained on $x_{k-1}^{(\theta)}$, $0 \le \theta \le T$. This control thus feeds back the whole of the pass previous to the current one and is thus causal and of the form $u_k = L_k x_{k-1}$ from theorem 4.2.

2) $\Pi_1(k,t,\theta)$ and $X(k,t,\theta)$ are piecewise continuous functions with discontinuity along the $\theta = t$ diagonal of their domain of definition.

3) $\Pi_o(t)$, which is independent of pass number, is just the standard time dependent Riccati matrix indicated in section 3.

4) We may write (see Appendix 2)

$$x_k(t) = \int_0^T X(k,t,\theta)x_{k-1}(\theta)d\theta + p_k(t), \quad 0 \le t \le T, \ 1 \le k \le M \ . \tag{5.2}$$

Using this we may eliminate $X(k,t,\theta)$ and $p_k(t)$ from (5.1) to obtain

$$u_k(t) = -R^{-1}B'\left\{\Pi_o(t)x_k(t) + \int_0^T \Pi_1(k,t,\theta)x_{k-1}(\theta)d\theta + \ell_k(t)\right\} ,\begin{matrix} 0 \le t \le T \\ 1 \le k \le M \end{matrix} \tag{5.3}$$

5) Putting $A_2 \equiv 0$ results in $\Pi_1(k,t,\theta)$ and $\ell_k(t)$ vanishing for $0 \le t_1\theta \le T$ and $1 \le k \le M$. (5.3) then becomes

$$u_k(t) = - R^{-1}B'\Pi_o(t)x_k(t) \quad . \tag{5.4}$$

This is just the standard Riccati feedback on each pass which is the expected result of setting $A_2 \equiv 0$ since this removes the "interpass" coupling. Note also under these conditions (5.2) ceases to be valid.

6) Setting $c_k = 0$, $1 \le k \le M$ results in (5.1c) collapsing to zero for $1 \le k \le M$ which is a considerable simplification. We shall make use of this fact for computation purposes.

We see, then, that to characterize the feedback kernels in (5.1) requires the solution of the standard time-dependent Riccati matrix equation (5.1a) together with the pair of two point boundary value problems (5.1b) and (5.1c).

We now indicate a solution of the type $u_k = \tilde{L}_k x_k$ which, as has already been stated, may not be causal. We thus derive a less general solution than (5.1) for ease of derivation since such a solution is of less practical value.

The extra restrictions we require are that $G \equiv 0$ in (1.4) and that $x_k(0) = c_k$ 0, $1 \le k \le M$ in (1.3). We thus have

$$J = \frac{1}{2} \sum_{k=1}^{M} \int_0^T \left[x_k'(\tau) Q x_k(\tau) + u_k'(\tau) R u_k(\tau) \right] d\tau \qquad (5.5)$$

with dynamical system

$$\left.\begin{aligned} &\frac{dx_k}{dt}(t) = A_1 x_k(t) + A_2 x_{k-1}(t) + Bu_k(t), \quad 0 \le t \le T, \\ &\qquad\qquad\qquad\qquad\qquad\qquad 1 \le k \le M \\ &x_k(0) = 0, \qquad 1 \le k \le M \\ &x_o(t) = f(t), \; 0 \le t \le T \end{aligned}\right\} \qquad (5.6)$$

<u>THEOREM 5.7</u> The linear control law

$$u_k(t) = - R^{-1} B' \int_0^T F_k(t,\theta) x_k(\theta) d\theta, \qquad 0 \le t \le T, \; 1 \le k \le M$$

provides the absolute minimum of performance criterion (5.5) for the dynamic system (5.6) provided the n × n matrix function of two variables, $F_k(t,\theta)$, with domain [0, T] × [0, T] together with the n × n matrix function of two variables, $Y_k(t,\theta)$, also with domain [0, T] × [0, T] satisfy the relations:

$$\left.\begin{aligned} &\frac{\partial F_k}{\partial t}(t,\theta) + A_1' F_k(t,\theta) + A_2 \int_0^T F_{k+1}(t,s) Y_{k+1}(s,\theta) ds = 0, \quad 0 \le t, \theta \le T, \\ &F_k(t,t+) - F_k(t,t-) = Q, \quad 0 \le t \le T, \; 1 \le k \le M \qquad 1 \le k \le M, \; \theta \ne t \\ &F_k(T,\theta) = 0, \quad 0 \le \theta \le T, \; 1 \le k \le M \\ &F_{M+1}(t,\theta) = 0, \quad 0 \le t, \theta \le T \\ &\frac{\partial Y_k}{\partial t}(t,\theta) = A_1 Y_k(t,\theta) - BR^{-1} B' \int_0^T F_k(t,s) Y_k(s,\theta) ds, \quad 0 \le t, \theta \le T, \\ &\qquad\qquad\qquad\qquad\qquad\qquad 1 \le k \le M, \quad \theta \ne t \\ &Y_k(t,t-) - Y_k(t,t+) = A_2, \quad 0 \le t \le T, \; 1 \le k \le M \\ &Y_k(0,\theta) = 0, \quad 0 \le \theta \le T, \; 1 \le k \le M \\ &Y_{M+1}(t,\theta) \equiv 0, \quad 0 \le t, \theta \le T \end{aligned}\right\} \qquad (5.7a)$$

The proof of this theorem is contained in Appendix 2.

<u>Remarks:</u>

1) The control law (5.7) may be non-causal since it is an integral control over $x_k(\theta)$, $0 \le \theta \le T$ on pass k, i.e. it may feed back information from "ahead" of the current position on the current pass.

2) As with Theorem 5.1, the feedback kernels $F_k(t,\theta)$ and $Y_k(t,\theta)$ are discontinuous along the $\theta = t$ diagonal of their domain.

3) We may write (see Appendix 2)

$$x_k(t) = \int_0^T Y_k(t, \theta)x_{k-1}(\theta)d\theta, \quad 0 \le t \le T, \ 1 \le k \le M. \qquad (5.8)$$

We now state an existence theorem giving sufficient conditions for a unique solution to the two point boundary value problems (5.1b) and (5.1c). As existence and uniqueness of solutions of the Riccati equation (5.1a) are already well known this theorem will show a control of the form (5.1) exists and is unique under appropriate conditions.

THEOREM 5.9 The two point boundary value problems (5.1b) and (5.1c) have unique solution for $0 \le t, \theta \le T$, $1 \le k \le M$ provided $\Pi_1(k+1, t, s)$ and $X(k+1, t, s)$ exist for each k and that

$$T^3 \cdot (\sup_{\tau,t\in[0,T]} \| \phi(t,\tau) \|_n)^2 \cdot \|BR^{-1}B'\|_n \cdot \| A_2 \|_n \cdot \sup_{\substack{t,s\in[0,T] \\ t\neq s}} \| \Pi_1(k+1,t,s) + \Pi_o(t) X(k+1,t,s) \|_n < 1 \qquad (5.9a)$$

where $\|\cdot\|_n$ is a matrix norm as described in section 2 and $\phi(t,\tau)$ is the transition matrix associated with the solution of the linear system

$$\frac{\partial C}{\partial t}(t, \theta) = (A_1 - BR^{-1}B'\Pi_o(t))C(t,\theta) + D(t,\theta), \quad 0 \le t, \ \theta \le T$$

where C and D are $n \times n$ matrix functions of two variables having domain $[0, T] \times [0, T]$.

REMARK This theorem gives a condition for the existence of a unique solution to (5.1b) and (5.1c) for pass k provided (5.1b) has solution on pass k+1. Now for $k = M$, $X(M+1, t, s)$ and $\Pi_1(M+1, t, s)$ are identically zero and thus (5.1b) and (5.1c) are guaranteed to have unique solutions. Once $\Pi_1(M, t, s)$ and $X(M, t, s)$ are determined we may then test the condition for $k = M-1$ and so on solving the system in reverse order.

Remarks:

1) While the theorem is able to give sufficient conditions for unique solutions to (5.1b) and (5.1c) the expression obtained in (5.9a) is difficult to evaluate since the transition matrix, $\phi(t, \tau)$, is, in practice, very difficult to compute. For this reason no attempt is made to evaluate it.

2) The theorem indicates certain qualitative conditions for a solution, for example if T and A_2 are "small" or R is "large" then (5.9a) will be satisfied.

6. NUMERICAL SOLUTIONS

In this section we consider the numerical solution of systems (5.1b) and (5.1c), the solution of (5.1a) being already known and well documented. The ideas discussed are then applied to scalar examples.

The computation of these control kernels for implementation in a real process may require considerable effort. Since they are independent of the system state, however, this may be achieved off line prior to the commencement of the process and the kernels then stored in the system's control unit. The control itself is then calculated on line by the control unit using the system state from the previous pass, which must also be stored.

The proof of theorem 5.9 relies on the contraction mapping principle, (5.9a) being the condition under which an integrated version of (5.1b) is a contraction. Hence, if (5.9a) holds we may take any function satisfying the boundary and discontinuity conditions and iterate using the contraction and obtain convergence to the unique solution of (5.1b).

This idea suggests the following successive approximation method for the numerical solution of (5.1b).

Since $\Pi_1(M+1,t,\theta) = X(M+1,t,\theta) \equiv 0$ we have, for $k = M$

$$\frac{\partial \Pi_1}{\partial t}(M,t,\theta) + (A_1' - \Pi_o(t)BR^{-1}B')\Pi_1(M,t,\theta) = 0, \quad \theta \neq t, 0 \le t, \theta \le T \qquad (6.1)$$

$$\Pi_1(M,t,t+) - \Pi_1(M,t,t-) = \Pi_o(t)A_2, \quad 0 \le t \le T \qquad (6.2)$$

$$\Pi_1(M,T,\theta) = 0, \quad 0 \le \theta \le T \qquad (6.3)$$

$$\frac{\partial X}{\partial t}(M,t,\theta) = (A_1 - BR^{-1}B'\Pi_o(t))X(M,t,\theta) - BR^{-1}B'\Pi_1(M,t,\theta), \quad 0 \le t, \theta \le T, \; \theta \neq t \qquad (6.4)$$

$$X(M,t,t-) - X(M,t,t+) = A_2, \quad 0 \le t \le T \qquad (6.5)$$

$$X(M,0,\theta) = 0, \quad 0 \le \theta \le T \qquad (6.6)$$

We notice that in this instance equation (6.1) is no longer coupled to equation (6.4). We may, thus, integrate (6.1) (with appropriate boundary conditions (6.1) and (6.2)) numerically without difficulty (it is assumed here, of course, that $\Pi_o(t)$ has already been found from equation (5.1a)) and obtain $\Pi_1(M,t,\theta)$. Once $\Pi_1(M,t,\theta)$ is known we may then integrate (6.4) with boundary conditions (6.5) and (6.6) to give $X(M,t,\theta)$.

Now for $k = M-1$ we have

$$\frac{\partial \Pi_1}{\partial t}(M-1,t,\theta) + (A_1' - \Pi_o(t)BR^{-1}B')\Pi_1(M-1,t,\theta) + A_2'\int_0^T \left[\Pi_1(M,t,s) + \Pi_o(t)X(M,t,s)\right] X(M-1,t,s)\,ds = 0, \quad 0 \le t \le T, \; \theta \neq t \qquad (6.7)$$

$$\Pi_1(M-1,t,t+) - \Pi_1(M-1,t,t-) = \Pi_o(t)A_2, \quad 0 \le t \le T \qquad (6.8)$$

$$\Pi_1(M-1,T,\theta) = 0, \quad 0 \le \theta \le T \qquad (6.9)$$

$$\frac{\partial X}{\partial t}(M-1,t,\theta) = (A_1 - BR^{-1}B'\Pi_o(t))X(M-1,t,\theta) - BR^{-1}B'\Pi_1(M-1,t,\theta), \quad \theta \neq t, \; 0 \le \theta, t \le T \qquad (6.10)$$

$$X(M-1,t,t-) - X(M-1,t,t+) = A_2, \quad 0 \le t \le T \qquad (6.11)$$

$$X(M-1,0,\theta) = 0, \quad 0 \le \theta \le T \qquad (6.12)$$

Since $\Pi_1(M,t,\theta)$ and $X(M,t,\theta)$ are known we make an initial guess at $X(M-1,t,\theta)$ which satisfies the boundary conditions. A suitable choice is say

$$X^{(1)}_{(M-1,t,\theta)} = \begin{cases} A_2 & \theta < t \\ 0 & \theta > t \end{cases} \quad , \qquad 0 \leq t,\ \theta \leq T\ .$$

We then substitute $X^{(1)}_{(M-1,t,\theta)}$ into equation (6.7) and integrate it numerically to obtain the corresponding function Π_1, say $\Pi^{(1)}_{1(M-1,t,\theta)}$. We now substitute $\Pi^{(1)}_{1(M-1,t,\theta)}$ into (6.10) and integrate to obtain $X^{(2)}_{(M-1,t,\theta)}$, which is in turn substituted into (6.7) and so on until the iterations converge.

It is noted that a similar method can be used to obtain a solution of (5.1c) for $1 \leq k \leq M$ once the X's and Π's have been found.

We now look at some scalar examples of the computation of the Π_1 and X matrices and the corresponding optimal control and state trajectories. The dynamical systems for which we shall choose to do this will all have zero initial conditions since, as was remarked in section 5, this results in (5.1c) collapsing to zero thus reducing the amount of computation.

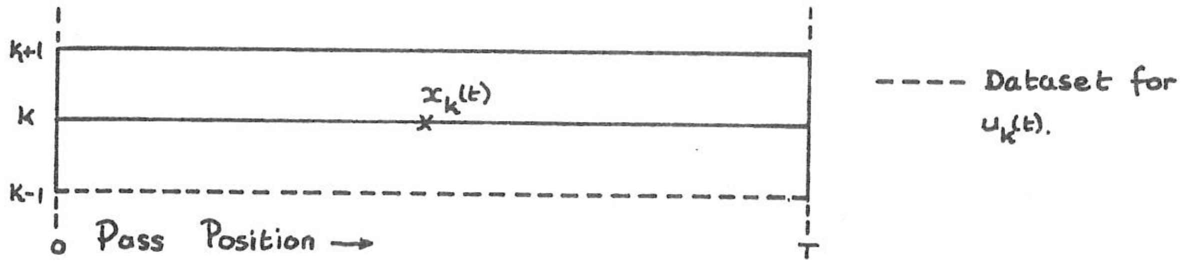
For comparison, the trajectories obtained using a numerical optimization technique developed by Jones and Owens [6] will be shown although in this case the control is not in feedback form. The corresponding open loop trajectories will also be displayed.

7. IMPLEMENTATION CONSIDERATIONS

We now consider the storage problems involved in implementing the control law (5.1) and develop the idea of a minimal data storage controller.

Once Π_o, Π_1, X etc. have been calculated for all passes off line we then implement the following integral feedback control or $p_k(t)$.

Our dataset for such a feedback is thus the current state plus the whole of the pass previous to the current one, as illustrated in the diagram.

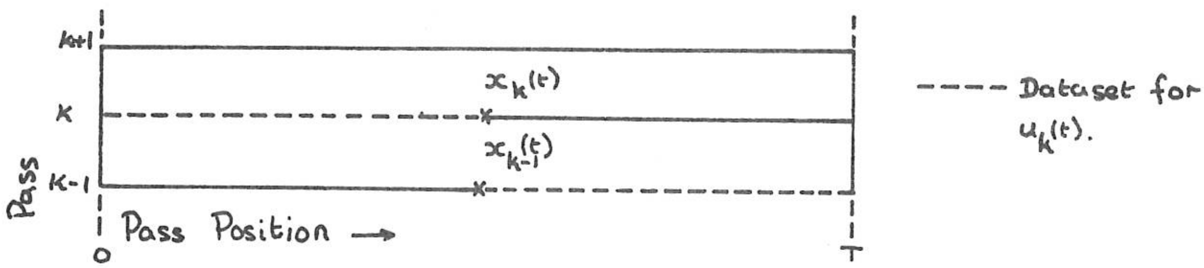


Hence, as we traverse pass k we need to store all of pass k-1 to determine u_k and also pass k as we go along since we shall need it to determine u_{k+1}. We thus need storage space for two whole passes, whilst our feedback dataset is only one pass in length.

This gives rise to the following question: can we find a causal feedback control for our system which has less storage requirements?

In view of the analogy between our system and the differential delay system it seems likely that our feedback dataset for a causal controller will be not less than one pass long (this is not true for a non-causal controller of course, e.g. (3.14)).

Suppose we could find a controller for our system whose dataset was $\left\{x_{k-1}^{(\tau)} \;:\; t \le \tau \le T\right\} \cup \left\{x_k(\tau) \;:\; 0 \le \tau \le t\right\}$ for $u_k(t)$.
This clearly has a dataset of one passlength as illustrated below.



We see, however, that this type of controller would require only one passlength of data storage since as we traverse pass k we overwrite the storage area containing $x_{k-1}(\tau)$ for $0 \leq \tau \leq t$, where t is the current position, with $x_k(\tau)$, $0 \leq \tau \leq t$.

The desirability of reducing storage leads us to find a control of this form in the next section.

8. MAIN RESULTS PART 2

We begin by recapping the problem:

$$\min J = \frac{1}{2} \sum_{k=1}^{M} \left\{ x_k'(T) G x_k(T) + \int_0^T \left[u_k'(\tau) R u_k(\tau) + x_k'(\tau) Q x_k(\tau) \right] d\tau \right\} \qquad (8.1)$$

subject to

$$\frac{dx_k}{dt}(t) = A_1 x_k(t) + A_2 x_{k-1}(t) + B u_k(t), \quad 0 \leq t \leq T, \; k = 1, 2, \ldots, M \qquad (8.2)$$

with

$$x_k(0) = c_k, \quad 1 \leq k \leq M \qquad (8.3)$$

and

$$x_o(t) = f(t), \quad 0 \leq t \leq T \quad . \qquad (8.4)$$

Henceforth in this section we shall consider only system with zero initial conditions on each pass for reasons of simplicity, so that we have

$$x_k(0) = 0, \quad 1 \leq k \leq M \quad . \qquad (8.5)$$

A more general version of the following result is contained in Appendix 3.

The following theorem is a summary of the second main result of this report; namely it is a statement of sufficient conditions to be satisfied by a causal linear feedback controller requiring minimum data storage to be optimal for criterion (8.1).

THEOREM 8.6 The linear control law

$$u_k(t) = -R^{-1}B'\left[K_o(k,t) x_k(t) + \int_t^T K_1(k,t,\theta) x_{k-1}(\theta) d\theta + \int_0^t K_2(k,t,\theta) x_k(\theta) d\theta \right],$$

$$0 \leq t \leq T, \; 1 \leq k \leq M$$

provides the absolute minimum of performance criterion (8.1) for the dynamical system (8.2), (8.4) and (8.5) provided the $n \times n$ symmetric matrix, $K_o(k,t)$, of function defined on $[0, T]$ together with the $n \times n$ matrices, $K_1(k,t,\theta)$ and $K_2(k,t,\theta)$, of functions of two variables defined in the regions $0 \le t \le T$, $t \le \theta \le T$ and $0 \le t \le T$, $0 \le \theta \le t$ respectively and the three $n \times n$ matrices, $K_3(k,t,\theta,\sigma)$, $K_4(k,t,\theta,\sigma)$ and $K_5(k,t,\theta,\sigma)$, of functions in three variables with domains of definition $0 \le t \le T$, $t \le \theta, \sigma \le T$; $0 \le t \le T$, $0 \le \theta \le t \le \sigma \le T$ and $0 \le t \le T$, $0 \le \theta,\sigma \le t$ satisfy the relations:

$$\frac{dK_o}{dt}(k,t)+Q+A_1'K_o(k,t)+K_o(k,t)A_1-K_o(k,t)BR^{-1}B'K_o(k,t)+K_2(k,t,t)+K_2'(k,t,t) = 0, \quad 1 \le k \le M,\ 0 \le t \le T \qquad (8.6a)$$

$$K_o(k,T) = G, \qquad 1 \le k \le M \qquad (8.6b)$$

$$\frac{\partial K_1}{\partial t}(k,t,\theta)+\left\{A_1'-K_o(k,t)BR^{-1}B'\right\}K_1(k,t,\theta)+K_4(k,t,t,\theta) = 0, \quad 1 \le k \le M,\ 0 \le t \le \theta \le T \qquad (8.6c)$$

$$K_1(k,t,t) = K_o(k,t)A_2, \qquad 1 \le k \le M \qquad (8.6d)$$

$$\frac{\partial K_2}{\partial t}(k,t,\theta)+\left\{A_1'-K_o(k,t)BR^{-1}B'\right\}K_2(k,t,\theta)+K_5(k,t,t,\theta) = 0, \quad 1 \le k \le M,\ 0 \le \theta \le t \le T \qquad (8.6e)$$

$$K_2(k,T,\theta) = 0, \qquad 0 \le \theta \le T,\ 1 \le k \le M \qquad (8.6f)$$

$$\frac{\partial K_3}{\partial t}(k,t,\theta,\sigma) = K_1'(k,t,\theta)BR^{-1}B'K_1(k,t,\sigma), \qquad 0 \le t \le \theta,\ \sigma \le T,\ 1 \le k \le M, \qquad (8.6g)$$

$$K_3(k,t,t,\theta) = A_2'K_1(k,t,\theta), \qquad 1 \le k \le M,\ 0 \le t \le \theta \le T \qquad (8.6h)$$

$$K_3(k,t,\theta,\sigma) = K_3'(k,t,\theta,\sigma), \qquad 1 \le k \le M,\ 0 \le t \le \theta,\ \sigma \le T \qquad (8.6i)$$

$$\frac{\partial K_4}{\partial t}(k,t,\theta,\sigma) = K_2'(k,t,\theta)BR^{-1}B'K_1(k,t,\sigma), \qquad 1 \le k \le M,\ 0 \le \theta \le t \le \sigma \le T \qquad (8.6j)$$

$$K_4(k,t,\theta,t) = K_2'(k,t,\theta)A_2, \qquad 1 \le k \le M,\ 0 \le \theta \le t \le T \qquad (8.6k)$$

$$\frac{\partial K_5}{\partial t}(k,t,\theta,\sigma) = K_2'(k,t,\theta)BR^{-1}B'K_2(k,t,\sigma), \quad 1 \le k \le M,\ 0 \le \theta,\ \sigma \le t \le t \qquad (8.61)$$

$$K_5(k,T,\theta,\sigma) = K_3(k+1,0,\theta,\sigma), \quad 1 \le k \le M-1,\ 0 \le \theta,\ \sigma \le T \qquad (8.6m)$$

$$K_5(M,T,\theta,\sigma) = 0, \quad 0 \le \theta,\ \sigma \le T \qquad (8.6n)$$

The proof of this theorem is contained in Appendix 3.

Remarks:

1) The linear control law stated in (8.6) is an integral operation on $x_{k-1}^{(\theta)}$, $t \le \theta \le T$ and $x_k(\theta)$, $0 \le \theta \le t$ at position t and is thus of the form desired for minimal data storage, as discussed in the previous section.

2) Unlike the control of theorem (5.1), all the feedback kernels in this instance are continuous.

3) The system of equations obtained in (5.1) for the control kernels bears considerable resemblance to the system obtained for the corresponding optimal solution for the differential delay system with quadratic performance index. See for example [7].

4) Putting $A_2 = 0$ results in all the kernels except u_o vanishing. k_o itself then becomes pass independent and reduces to the standard Riccati matrix. (8.6) then becomes

$$u_k(t) = -R^{-1}B'K_o(t)x_k(t) \qquad (8.7)$$

where K_o satisfies the standard Riccati equation and boundary condition as in (5.1a). This is the expected result, c.f. remark 5 to (5.1).

APPENDIX 1

In this appendix we provide the proof of theorem 4.2. The proof will rely on the principle of optimality as used in dynamic programming.

THE PRINCIPLE OF OPTIMALITY states that, if an optimal path from the initial stage t_o to the final stage t_N of a process exists and at an intermediate stage t_k it passes through the state x_α, then the optimal path from x_α at stage t_k to stage t_N coincides with the overall optimal path.

The proof of (4.2) is as follows:

In the Hilbert space setting our problem becomes

$$\min J = \frac{1}{2} \sum_{k=1}^{M} \left\{ \|x_k\|_1^2 + \|u_k\|_2^2 \right\} \tag{A1.1a}$$

subject to

$$x_{k+1} = Ax_k + Bu_{k+1}, \quad 0 \le k \le M-1 \tag{A1.1b}$$

where $x_k \in H_1$ and $u_R \in H_2$ for $1 \le k \le M$, A and B are specified bounded linear operators. H_1 and H_2 are the Hilbert spaces defined in Section 2.

We define J_k by

$$J_k = \frac{1}{2} \sum_{j=M-k+1}^{M} \left\{ \|x_j\|_1^2 + \|u_j\|_2^2 \right\} \tag{A1.2}$$

i.e. J_k is the value of the performance index for the process (A1.1b) restricted to the last k-1 passes.

We now make the following induction hypothesis:

$$\min J_k = \langle x_{M-k}, K_k x_{M-k} \rangle_1 \tag{A1.3}$$

where K_k is a self-adjoint positive definite linear operator for $k = 1, 2, 3, \ldots, M$.

Since our process has only M passes

$$K_o \equiv 0 \quad . \tag{A1.4}$$

Now assume (A1.3) holds for some value of $k > 0$. Then, using the principle of optimality:

$$\min J_{k+1} = \min\left\{\frac{1}{2}\|x_{M-k}\|_1^2 + \frac{1}{2}\|u_{M-k}\|_2^2 + \frac{1}{2}\langle x_{M-k}, K_k x_{M-k}\rangle_1\right\}.$$

Using (A1.1b) this becomes

$$\min J_{k+1} = \min\left\{\frac{1}{2}\|Ax_{M-k-1}+Bu_{M-k}\|_1^2 + \frac{1}{2}\|u_{M-k}\|_2^2 + \frac{1}{2}\langle Ax_{M-k-1}+Bu_{M-k}, K_k(Ax_{M-k-1}+Bu_{M-k})\rangle_1\right\}$$

$$= \min\left\{\frac{1}{2}\langle Ax_{M-k-1}+Bu_{M-k}, Ax_{M-k-1}+Bu_{M-k}\rangle_1 + \frac{1}{2}\langle u_{M-k}, u_{M-k}\rangle_2 + \frac{1}{2}\langle Ax_{M-k-1}+Bu_{M-k}, K_k(Ax_{M-k-1}+Bu_{M-k})\rangle_1\right\}.$$

Now, for simplicity, write $x_{M-k-1} = x_o$, $u_{M-k} = u$, $K_k = K$.

Then

$$\min J_{k+1} = \min\left\{\frac{1}{2}\langle Ax_o+Bu, (I+K)(Ax_o+Bu)\rangle_1 + \frac{1}{2}\langle u, u\rangle_2\right\}$$

Now $\frac{1}{2}\langle Ax_o+Bu, (I+K)(Ax_o+Bu)\rangle_1 + \frac{1}{2}\langle u, u\rangle_2 = \frac{1}{2}\langle x_o, A^*(I+K)Ax_o\rangle_1$

$$+ \frac{1}{2}\langle Ax_o, (I+K)Bu\rangle_1 + \frac{1}{2}\langle Bu, (I+K)Ax_o\rangle_1 + \frac{1}{2}\langle u, (I+B^*(I+K)B)u\rangle_2$$

$$= \frac{1}{2}\langle x_o, A^*(I+K)Ax_o\rangle_1 + \langle B^*(I+K)Ax_o, u\rangle_1 + \frac{1}{2}\langle u, (I+B^*(I+K)B)u\rangle_2. \quad \text{(A1.5)}$$

Now, since x_o is not determined by u, $\frac{1}{2}\langle x_o, A^*(I+K)Ax_o\rangle_1$ is fixed.

Writing $u = u_o + \delta u$ it is easily verified that a necessary condition for a minimum of $\min\left\{\frac{1}{2}\langle u, Lu\rangle_2 + \langle b, u\rangle_2\right\}$ is that $Lu + b = 0$.

Thus, for a minimum of (A1.5) we require

$$(I+B^*(I+K)B)u + B^*(I+K)Ax_o = 0 \quad \text{(A1.6)}$$

i.e.

$$u = -(I+B^*(I+K)B)^{-1}B^*(I+K)Ax_o \quad \text{(A1.7a)}$$

or

$$u_{M-k} = L_{M-k}x_{M-k-1} \quad \text{(A1.7b)}$$

where

$$L_{M-k} = -(I+B^*(I+K)B)^{-1}B^*(I+K_k)A. \quad \text{(A1.7c)}$$

Notice: $B^*(I+K)(I+BB^*(I+K)) \equiv (I+B^*(I+K)B)B^*(I+K)$

Thus, premultiplying each side of the above by $(I+B^*(I+K)B)^{-1}$ and post multiplying by $(I+BB^*(I+K))^{-1}$ we obtain

$$B^*(I+K)(I+BB^*(I+K))^{-1} = (I+B^*(I+K)B)^{-1}B^*(I+K)$$

Thus

$$u = -B^*(I+K)(I+BB^*(I+K))^{-1}Ax_o \quad .$$

Substituting into (A1.1b) gives

$$\begin{aligned} x &= Ax_o + Bu \\ &= \left[I - BB^*(I+K)(I+BB^*(I+K))^{-1}\right]Ax_o \\ &= (I+BB^*(I+K) - BB^*(I+K))(I+BB^*(I+K))^{-1}Ax_o \\ &= (I+BB^*(I+K))^{-1}Ax_o \ . \end{aligned} \qquad (A1.8)$$

We thus have

$$u = -B^*(I+K)x \quad . \qquad (A1.9)$$

Now

$$\min J_{k+1} = \min\left\{\frac{1}{2}\langle x, x\rangle_1 + \frac{1}{2}\langle u, u\rangle_2 + \frac{1}{2}\langle x, Kx\rangle_1\right\}$$

Substituting for u using (A1.9) gives

$$\begin{aligned} \min J_{k+1} &= \frac{1}{2}\langle x, (I+K)x\rangle_1 + \frac{1}{2}\langle B^*(I+K)x, B^*(I+K)x\rangle_2 \\ &= \frac{1}{2}\langle x, (I+K)(I+BB^*(I+K))x\rangle_1 \end{aligned}$$

Now substituting (A1.8) gives

$$\begin{aligned} \min J_{k+1} &= \frac{1}{2}\langle (I+BB^*(I+K))^{-1}Ax_o, (I+K)(I+BB^*(I+K))(I+BB^*(I+K))^{-1}Ax_o\rangle_1 \\ &= \frac{1}{2}\langle (I+BB^*(I+K))^{-1}Ax_o, (I+K)Ax_o\rangle_1 \\ &= \frac{1}{2}\langle x_o, A^*(I+BB^*(I+K))^{-1}(I+K)Ax_o\rangle_1 \quad . \end{aligned}$$

Thus

$$K_{k+1} = A^*(I+BB^*(I+K_k))^{-1}(I+K_k)A \qquad (A1.10)$$

and the proof is complete by induction.

Notice

$$L_n = -(I+B^*(I+K_{M-k})B)^{-1}B^*(I+K_{M-k})A \qquad (A1.11)$$

and

$$\tilde{L}_n = -B^*(I+K_{M-k}) \qquad (A1.12)$$

in the statement of theorem 4.2.

The K's satisfy the recurrence relation

$$K_{k+1} = A^*(I+BB^*(I+K_k))^{-1}(I+K_k)A, \quad 0 \le k \le M \tag{A1.13}$$

with

$$K_o \equiv 0 \ . \tag{A1.14}$$

APPENDIX 2

In this appendix we give the proofs of theorems (5.1), (5.7) and (5.9). We begin with theorem (5.1).

Proof of theorem 5.1

By writing (3.4) and (3.3) in terms of $x_k(t)$ and $\lambda_k(t)$, $1 \le k \le M$ we obtain

$$\left.\begin{aligned} \frac{d\lambda_k}{dt}(t) = -A_1'\lambda_k(t) - A_2'\lambda_{k+1}(t) - Qx_k(t), \quad & \lambda_k(T) = Gx_k(T) \\ & \lambda_{M+1}(t) \equiv 0 \end{aligned}\right\} \begin{aligned} & 0 \le t \le T \\ & 1 \le k \le M \end{aligned} \tag{A2.1}$$

$$\left.\begin{aligned} \frac{dx_k}{dt}(t) = A_1x_k(t) + A_2x_{k-1}(t) - BR^{-1}B'\lambda_k(t), \quad & x_k(0) = c_k \\ & x_o(t) = f(t) \end{aligned}\right\} \begin{aligned} & 0 \le t \le T \\ & 1 \le k \le M \end{aligned} \ . \tag{A2.2}$$

From (A1.8) we know that there exists a linear relationship between x_k and x_{k-1}. In our trial solution we shall thus write

$$x_k(t) = \int_0^T X(k,t,\theta)x_{k-1}(\theta)d\theta + p_k(t), \quad 1 \le k \le M, \ 0 \le t \le T \tag{A2.3}$$

and attempt to find equations for X and p.

We also know there exists a linear relationship between u_k (and hence λ_k) and x_{k-1} from theorem 4.2. We make the trial solution of

$$\lambda_k(t) = \Pi_o(t)x_k(t) + \int_0^T \Pi_1(k,t,\theta)x_{k-1}(\theta)d\theta + \ell_k(t), \quad 1 \le k \le m \tag{A2.4}$$
$$0 \le t \le T \ .$$

In view of (A2.3) this is clearly a linear feedback of x_{k-1}. This form of control is chosen since it is a perturbation of the standard terminal controller with added tracking vector and integral feedback to account for the interpass coupling of our system.

From (A2.3) we have

$$\frac{dx_k}{dt}(t) = \{X(k,t,t-)-X(k,t,t+)\}x_{k-1}(t)+\int_0^T \frac{\partial X}{\partial t}(k,t,\theta)x_{k-1}(\theta)d\theta + \frac{dp_k}{dt}(t) \quad . \quad (A2.5)$$

Now, substituting (A2.3) into (A2.2) yields

$$\frac{dx_k}{dt}(t) = A_1\int_0^T X(k,t,\theta)x_{k-1}(\theta)d\theta + A_2x_{k-1}(t) + A_1p_k(t)$$
$$-BR^{-1}B'\left\{\int_0^T\left[\Pi_1(k,t,\theta)+\Pi_o(t)X(k,t,\theta)\right]x_{k-1}(\theta)d\theta+\Pi_o(t)p_k(t)+\ell_k(t)\right\} \quad (A2.6)$$

Equating (A2.5) and (A2.6) gives

$$\int_0^T\left[\frac{\partial X}{\partial t}(k,t,\theta)-\left\{A_1-BR^{-1}B'\Pi_o(t)\right\}X(k,t,\theta)+BR^{-1}B'\Pi_1(k,t,\ )\right]x_{k-1}(\theta)d\theta$$
$$+\frac{dp_k}{dt}(t) - \left[A_1-BR^{-1}B'\Pi_o(t)\right]p_k(t) + BR^{-1}B'\ell_k(t)$$
$$+\left\{X(t,t-) - X(k,t,t+) - A_2\right\}x_{k-1}(t) = 0 \quad . \quad (A2.7)$$

This equation is satisfied identically if we choose X and p to satisfy $(5.1b)_2$ and $(5.1c)_2$.

Similarly, by differentiating (A2.4) and equating it with (A2.1) we obtain

$$\left\{\frac{d\Pi_o}{dt}(t)+\Pi_o(t)A_1+A_1'\Pi_o(t)+Q-\Pi_o(t)BR^{-1}B'\Pi_o(t)\right\}x_k$$
$$+\left\{\Pi_1(k,t,t-)-\Pi_1(k,t,t+)+\Pi_o(t)A_2\right\}x_{k-1}(t)$$
$$+\frac{d\ell_k}{dt}(t)+(A_1'-\Pi_o(t)BR^{-1}B')\ell_k(t)+A_2'\ell_{k+1}(t)+A_2'\Pi_o(t)p_{k+1}(t)$$
$$+A_2'\int_0^T\left[\Pi_1(k+1,t,s)+\Pi_o(t)X(k+1,t,s)\right]p_k(s)ds \quad (A2.8)$$
$$+\int_0^T\left\{\frac{\partial\Pi_1}{\partial t}(k,t,\theta)+\left[A_1'-\Pi_o(t)BR^{-1}B'\right]\Pi_1(k,t,\theta)\right.$$
$$\left.+A_2'\int_0^T\left[\Pi_1(k+1,t,s)+\Pi_o(t)X(k+1,t,s)\right]X_k(s,\theta)ds\right\}x_{k-1}(\theta)d\theta = 0.$$

This equation is satisfied identically by choosing Π_o, Π_1 and ℓ to satisfy equations (5.1a), $(5.1b)_1$ and $(5.1c)_1$.

This completes the proof.

Theorem (5.7) is obtained in similar fashion using the trial solution

$$\left.\begin{aligned} x_k(t) &= \int_0^T Y_k(t,\theta)x_{k-1}(\theta)d\theta, \quad 0 \le t \le T,\ 1 \le k \le M \\ \lambda_k(t) &= \int_0^T F_k(t,\theta)x_{k-1}(\theta)d\theta, \quad 0 \le t \le T,\ 1 \le k \le M \end{aligned}\right\} \quad (A2.9)$$

Finally in this section, we indicate a proof of theorem (5.9).

Write

$$A_1 - BR^{-1}B'\Pi_o(t) = A_o(t),$$

$$BR^{-1}B' = B_o,$$

$$\Pi_1(k+1,t,s)+\Pi_o(t)X(k+1,t,s) = g(k,t,s),$$

$$\Pi_1(k,t,t+)-\Pi_1(k,t,t-) = f(t) \ (=\Pi_o(t)A_2).$$

Then we have, for $\theta \neq t$ and some k with $1 \le k \le M-1$

$$\left.\begin{aligned} &\frac{\partial X}{\partial t}(k,t,\theta) = A_o(t)X(k,t,\theta) - B_o\Pi_1(k,t,\theta), \\ &X(k,t,t-) - X(k,t,t+) = A_2,\ X(k,0,\theta) = 0 \\ &\frac{\partial \Pi_1}{\partial t}(k,t,\theta) = -A_o'(t)\Pi_1(k,t,\theta) - A_2'\int_0^T g(k,t,s)X(k,s,\theta)ds, \\ &\Pi_1(k,t,t+) - \Pi_1(k,t,t-) = f(t),\ \Pi_1(k,T,\theta) = 0. \end{aligned}\right\} \quad (A2.10)$$

Now $X(k,t,\theta) \in \hat{C}_{A_2}$ and $\Pi_1(k,t,\theta) \in \hat{C}_{f(t)}$.

Our proof relies on the contraction mapping

CONTRACTION MAPPING PRINCIPLE: Let V be a complete normed vector space (i.e. a Banach space) and let $f: V \to V$ be such that for all $x, y \in V$

$$\|f(x)-f(y)\| \le \lambda \|x-y\|,$$

where $\lambda < 1$. Then the mapping f has a unique fixed point, i.e. there exists a unique $x' \in V$ such that $f(x) = x$.

The proof of this theorem tells us how to find this fixed point. Simply take any element of V and reapply f to it where upon convergence to the fixed point will be obtained.

This is how we shall demonstrate the existence of a unique solution of (A2.10), i.e. by showing that under condition (5.9a), (A2.10) is a contraction. The unique fixed point of the theorem will then be the solution we seek.

Now, integrating (A2.10) we have

$$\left.\begin{aligned} X(k,t,\theta) &= \int_0^t \Phi(t,\tau)B_o\Pi_1(k,\tau,\theta)d\tau + \alpha(t,\theta) \\ \Pi_1(k,t,\theta) &= \int_t^T \Phi'(\tau,t)A_2' \int_0^T g(k,\tau,s)X(k,s;\theta)dsd\tau + \beta(t,\theta) \end{aligned}\right\} \quad (A2.11)$$

where

$$\alpha(t,\theta) = \begin{cases} 0 & t > \theta \\ A_2 & t < \theta \end{cases}$$

$$\beta(t,\theta) = \begin{cases} 0 & t < \theta \\ f(t) & t > \theta \end{cases}$$

and $\Phi(t,\tau)$ is the transition matrix associated with $A_o(t)$.

Now, take any $X^{(1)} \in \hat{C}_{A_2}$. Then we have, on application of (A2.11)

$$X^{(2)}(k,t,\theta) = \int_0^t \Phi(t,\tau)B_o\Pi_1^{(1)}(k,\tau,\theta)d\tau + k(t,\theta) \quad (A2.12a)$$

$$\Pi_1^{(1)}(k,t,\theta) = \int_t^T \Phi'(\tau,t)A_2' \int_0^T g(k,\tau,s)X^{(1)}(k,s,\theta)dsd\tau + \beta(t,\theta) \;. \quad (A2.12b)$$

Substituting (A2.12b) into (A2.12a) gives

$$X^{(2)}(k,t,\theta) = \int_0^t \Phi(t,\tau)B_o\left\{\int_t^T \Phi'(u,\tau)A_2' \int_0^T g(k,u,s)X^{(1)}(k,s,\theta)dsdu + \beta(\tau,\theta)\right\}d\tau + \alpha(t,\varepsilon)$$

$$= W(X^{(1)}(k,t,\theta)) \text{ say.} \quad (A2.13)$$

We shall now find a sufficient condition for which w is a contraction and the proof will be complete.

Let $X, X' \in \hat{C}_{A_2}$ and let $Y = w(X)$, $Y' = w(X')$. Then

$$\|Y-Y'\|_{A_2} = \sup_{\substack{t,\theta\in[0,T]\\ t\neq\theta}} \left\| \int_0^t \Phi(t,\tau)B_o \int^T \Phi'(u,\tau)A_2' \int_0^T g(k,u,s)\{X(s,\theta)-X'(s,\theta)\}dsdud\tau \right\|_n$$

$$\leq T^3\left(\sup_{t,\tau\in[0,T]} \|\Phi(t,\tau)\|_n\right)^2 \|B_o\|_n \|A_2'\|_n \sup_{\substack{t,\theta\in[0,T]\\ \theta\neq t}} \|g(k,t,\theta)\|_n$$

$$\sup_{\substack{t,\theta\in[0,T]\\ t\neq\theta}} \|X(t,\theta)-X'(t,\theta)\|_n$$

(using $\int_a^b f(x)dx \leq \sup_{x\in[a,b]} \|f(x)\| \cdot (b-a)$)

$$= T^3\left(\sup_{t,\tau\ 0,T} \|\Phi(t,\tau)\|_n\right)^2 \|B_o\|_n \|A_2'\|_n \sup_{\substack{t,\theta\in[0,T]\\ \theta\neq t}} \|g(k,t,\theta)\|_m \cdot \|X-X'\|_{A_2}$$

$$= \lambda\|X-X'\|_{A_2}$$

where $\lambda = T^3\left(\sup_{t,\tau\in[0,T]} \|\Phi(t,\tau)\|_2\right)^2 \cdot \|B_o\|_n \cdot \|A_2'\|_n \cdot \sup_{\substack{t,\theta\in[0,T]\\ t\neq\theta}} \|g(k,t,\theta)\|_n$.

Thus, if $\lambda < 1$, w is a contraction and this completes the proof.

The proof that (5.9a) is also a sufficient condition for a unique solution to (5.1c) follows similar lines.

APPENDIX 3

In this final appendix we supply the proof of theorem (8.6). Consider the following

$$\frac{d}{dt}\Big\{x_k'(t)K_o(k,t)x_k(t)+x_k'(t)\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+x_k'(t)\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta$$

$$+\int_t^T x_{k-1}'(\theta)K_1'(k,t,\theta)d\theta x_k(t)+\int_0^t x_k'(\theta)K_2'(k,t,\theta)d\theta x_k(t)$$

$$+\int_t^T\int_t^T x_{k-1}'(\theta)K_3(k,t,\theta,\sigma)x_{k-1}(\sigma)d\theta d\sigma + \int_0^t\int_t^T X_k'(\theta)K_4(k,t,\theta,\sigma)x_{k-1}(\sigma)d\sigma d\theta$$

$$+\int_t^T\int_0^t x_{k-1}'(\theta)K_4'(k,t,\sigma,\theta)x_k(\sigma)d\sigma d\theta+\int_0^t\int_0^t x_k'(\theta)K_s(k,t,\theta,\sigma)x_k(\sigma)d\theta d\sigma\Big\}$$

$$+ x_k'(t)Qx_k(t) + u_k'(t)Ru_k(t)$$

$$= \frac{dx_k'}{dt}(t)K_o(k,t)x_k(t)+x_n'(t)\frac{dK_o}{dt}(k,t)x_k(t)+x_k'(t)K_o(k,t)\frac{dx_k}{dt}(t)$$

$$+ \frac{dx_k'}{dt}(t)\left[\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]$$

$$- x_k'(t)K_1(k,t,t)x_{k-1}(t)+x_k'(t)\int_t^T \frac{\partial K_1}{\partial t}(k,t,\theta)x_{k-1}(\theta)d\theta+x_k'(t)K_2(k,t,t)x_k(t)$$

$$+ \int_0^t \frac{\partial K_2}{\partial t}(k,t,\theta)x_k(\theta)d\theta+\left[\int_t^T x_{k-1}'(\theta)K_1'(k,t,\theta)d\theta+\int_0^t x_k'(\theta)K_2'(k,t,\theta)d\theta\right]\frac{dx_k}{dt}(t)$$

$$- x_{k-1}'(t)K_1'(k,t,t)x_k(t)+\int_t^T x_{k-1}'(\theta)\frac{\partial K_1'}{\partial t}(k,t,\theta)d\theta+x_k'(t)K_2'(k,t,t)x_k(t)$$

$$+ \int_0^t x_k'(\theta)\frac{\partial K_2'}{\partial t}(k,t,\theta)d\theta x_k(t)-x_{k-1}'(t)\int_t^T K_3(k,t,t,\sigma)x_{k-1}(\sigma)d\sigma-\int_t^T x_{k-1}'(\theta)K_3(k,t,\theta,t)d\theta x_{k-1}(t)$$

$$+ \int_t^T\int_t^T x_{k-1}'(\theta)\frac{\partial K_3}{\partial t}(k,t,\theta,\sigma)x_{k-1}(\sigma)d\sigma+x_k(t)\int_t^T K_4(k,t,t,\sigma)x_{k-1}(\sigma)d\sigma$$

$$- \int_0^t x_k'(\theta)K_4(k,t,\theta,t)d\theta x_{k-1}(t)$$

$$+ \int_0^t\int_t^T x_k'(\theta)\frac{\partial K_4}{\partial t}(k,t,\theta,\sigma)x_{k-1}(\sigma)d\sigma d\theta - x_{k-1}'(t)\int_0^t K_4'(k,t,\sigma, t)x_k(\sigma)d\sigma$$

$$+ \int_t^T x_{k-1}'(\theta)K_4'(k,t,t,\theta)d\theta x_k(t)$$

$$+ \int_t^T\int_0^t x_{k-1}'(\theta)\frac{\partial K_4'}{\partial t}(k,t,\theta,r)x_k(r)d\sigma d\theta+x_k'(t)\int_0^t K_5(k,t,t,\sigma)x_k(\sigma)d\sigma$$

$$+ \int_0^t x_k'(\theta)K_5(k,t,\theta,t)d\theta x_k(t)$$

(Substituting in (1.1) and rearranging gives)

$$= \left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]\right)' R$$

$$\left(u_k(t)+R^{-1}B'\left[K_o(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]\right)$$

$$+ x_k'(t)\left\{\frac{dK_0}{dt}(k,t)+Q+A_1'K_0(k,t)+K_0(k,t)A_1-K_0(k,t)BR^{-1}B'K_1(k,t)\right.$$

$$\left. + K_2(k,t,t)+K_2'(k,t,t)\right\}x_k(t)$$

$$+ x_k'(t)\int_t^T\left\{\frac{\partial K_1}{\partial t}(k,t,\theta)+\left\{A_1'-K_0(k,t)BR^{-1}B'\right\}K_1(k,t,\theta)+K_4(k,t,t,\theta)\right\}x_{k-1}(\theta)d\theta$$

$$+ \int_t^T x_{k-1}'(\theta)\left\{\frac{\partial K_1'}{\partial t}(k,t,\theta)+K_1'(k,t,\theta)[A_1-BR^{-1}B'K_o(k,t)]+K_4'(k,t,t,\theta)\right\}d\theta x_k(t)$$

$$+ x_k'(t)\int_0^t\left\{\frac{\partial K_2}{\partial t}(k,t,\theta)+[A_1'-K_0(k,t)BR^{-1}B']K_2(k,t,\theta)+K_5(k,t,t,\theta)\right\}x_k(\theta)d\theta$$

$$+ \int_0^t x_n'(\theta)\left\{\frac{\partial K_2'}{\partial t}(k,t,\theta)+K_2'(k,t,\theta)[A_1-BR^{-1}B'K_0(k,t)]+K_5(k,t,\theta,t)\right\}d\theta x_k(t)$$

$$+ \int_t^T\int_t^T x_{k-1}'(\theta)\left\{\frac{\partial K_2}{\partial t}(k,t,\theta,\sigma)-K_1'(k,t,\theta)BR^{-1}B'K_1(k,t,\sigma)\right\}x_{k-1}(\sigma)d\theta d\sigma$$

$$+ \int_0^t\int_t^T x_k'(\theta)\left\{\frac{\partial K_4}{\partial t}(k,t,\theta,\sigma)-K_2'(k,t,\theta)BR^{-1}B'K_1(k,t,\sigma)\right\}x_{k-1}(\sigma)d\sigma d\theta$$

$$+ \int_t^T\int_0^t x_{k-1}'(\theta)\left\{\frac{\partial K_4'}{\partial t}(k,t,\sigma,\theta)-K_1'(k,t,\theta)BR^{-1}B'K_2(k,t,\sigma)\right\}x_k(\sigma)d\sigma d\theta$$

$$+ \int_0^t\int_0^t x_k'(\theta)\left\{\frac{\partial K_5}{\partial t}(k,t,\theta,\sigma)-K_2'(k,t,\theta)BR^{-1}B'K_2(k,t,\sigma)\right\}x_k(\sigma)d\theta d\sigma$$

$$+ x_k'\left\{K_0(k,t)A_2-K_1(k,t,t)\right\}x_{k-1}(t) + x_{k-1}'(t)\left\{A_2'K_0(k,t)-K_1'(k,t,t)\right\}x_k(t)$$

$$+ x_{k-1}'(t)\int_t^T\left\{A_2'K_1(k,t,\theta)-K_3(k,t,t,\theta)\right\}x_{k-1}(\theta)d\theta+\int_t^T x_{k-1}'(\theta)\left\{K_1'(k,t,\theta)A_2 - K_3(k,t,\theta,t)\right\}d\theta x_{k-1}(t)$$

$$+ x_{k-1}'(t)\int_0^t\left\{A_2'K_2(k,t,\theta)-K_4'(k,t,\theta,t)\right\}x_k(\theta)d\theta+\int_0^t x_k'(\theta)\left\{K_2'(k,t,\theta)A_2 - K_4(k,t,\theta,t)\right\}d\theta x_{k-1}(t) .$$

Choose K_0, K_1, K_2, K_3, K_4 and K_5 to satisfy 8.6a,c,d,e,g,h,i,j,k, and l.

Then we have

$$\frac{d}{dt}\Big\{x_k'(t)K_0(k,t)x_k(t)+x_k'(t)\Big[\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\Big]$$

$$+\Big[\int_t^T x_{k-1}'(\theta)K_1'(k,t,\theta)d\theta+\int_0^t x_k'(\theta)K_2'(k,t,\theta)d\theta\Big]x_k(t)$$

$$+\int_t^T\int_t^T x_{k-1}'(\theta)K_4(k,t,\theta,\sigma)x_{k-1}(\sigma)d\theta d\sigma+\int_0^t\int_t^T x_k'(\theta)K_4(k,t,\theta,\sigma)x_{k-1}(\sigma)d\sigma d\theta$$

$$+\int_t^T\int_0^t x_{k-1}'(\theta)K_4'(k,t,\sigma,\theta)x_n(\sigma)d\sigma d\theta+\int_0^t\int_0^t x_k'(\theta)K_5(k,t,\theta,\sigma)x_k(\sigma)d\sigma d\theta\Big\}$$

$$+\ x_k'(t)Qx_k(t)\ +\ u_k'(t)Ru_k(t)$$

$$=\ \Big(u_k(t)+R^{-1}B'\Big[K_0(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\Big]\Big)'R$$

$$\Big(u_k(T)+R^{-1}B'\Big[K_0(k,t)x_k(t)+\int_t^T k_1(k,t,\theta)x_{k-1}(\theta)d\theta$$

$$+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\Big]\Big)\ .$$

Integrating with respect to t over 0 to T and summing over k from 1 to M we get

$$\sum_{k=1}^{M}\Big\{x_k'(T)K_0(k,T)x_k(T)+x_k'(T)\int_0^T K_2(k,T,\theta)x_n(\theta)d\theta+\int_0^T x_k'(\theta)K_2'(k,T,\theta)d\theta\, x_k(T)$$

$$+\int_0^T\int_0^T x_k'(\theta)K_5(k,T,\theta,\sigma)x_k(\sigma)d\theta d\sigma-\int_0^T\int_0^T x'_{k-1}(\theta)K_4(k,0,\theta,\sigma)x_{k-1}(\sigma)d\theta d\sigma\Big\}$$

$$+\sum_{k=1}^{M}\int_0^T (x_k'(t)Qx_k(t)\ +\ u_k'(t)Ru_k(t))dt$$

$$=\sum_{k=1}^{M}\int_0^T\Big(u_k(t)+R^{-1}B'\Big[K_0(k,t)x_k(t)+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta\Big]\Big)'R$$

$$\Big(u_k(t)+R^{-1}B'\Big[K_0(k,t)x_k(t)+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta\Big]\Big)dt\ .$$

Choose 8.6 b, f, m and n to be satisfied. Then we have

$$J[u] = \frac{1}{2}\sum_{k=1}^{M}\left\{x_k'(T)Gx_k(T)+\int_0^T\left[x_k'(\tau)Qx_k(\tau)+u_k(\tau)Ru_k(\tau)\right]d\tau\right\}$$

$$= \frac{1}{2}\sum_{k=1}^{M}\int_0^T\left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]\right)'R$$

$$\left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]\right)dt$$

$$+\frac{1}{2}\int_0^T\int_0^T f'(\theta)K_3(1,0,\theta,\sigma)f(\sigma)d\theta d\sigma \quad .$$

Choose $\tilde{x}_k(t)$ and $\tilde{u}_k(t)$ so that

$$\tilde{u}_k(t) = -R^{-1}B'\left[K_0(k,t)x_k(t)+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta\right] . \quad (A3.1)$$

Then

$$J[\tilde{u}] = \frac{1}{2}\int_0^T\int_0^T f'(\theta)K_3(1,0,\theta,\sigma)f(\sigma)d\theta d\sigma \quad .$$

For any other $x_k(t)$ and $u_k(t)$

$$J[u] = \frac{1}{2}\int_0^T\int_0^T f'(\theta)K_3(1,0,\theta,\sigma)f(\sigma)d\theta d\sigma$$

$$+\frac{1}{2}\sum_{k=1}^{M}\int_0^T\left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta\right.\right.$$

$$\left.\left.+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]\right)'R$$

$$\left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta\right]\right)dt .$$

Thus

$$J[u]-J[\tilde{u}]$$

$$= \frac{1}{2}\sum_{k=1}^{m}\int_0^T\left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta\right]\right)'R$$

$$\left(u_k(t)+R^{-1}B'\left[K_0(k,t)x_k(t)+\int_0^t K_2(k,t,\theta)x_k(\theta)d\theta+\int_t^T K_1(k,t,\theta)x_{k-1}(\theta)d\theta\right]\right)dt$$

≥ 0 Since R is positive definite.

We thus have that $J[\tilde{u}] \leq J[u]$ for any choice of u_k $(1 \leq k \leq M)$ and hence $\tilde{u}$ as chosen in (A3.1) provides a minimum of J, i.e. 8.6 a - n are sufficient conditions for $\tilde{u}_k$ $(1 \leq k \leq m)$ to be an optimal control for system (1.1), (1.2), (1.3) and (1.4) and the proof is complete.

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