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ROBUST STABILITY OF SMITH PREDICTOR CONTROLLERS FOR TIME-DELAY SYSTEMS

by

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Abstract

The paper considers Smith Predictor control structures for multivariable process plant with separable delays on the output and analyses the robustness of the control scheme with respect to mismatch between the real plant and its model and to simultaneous changes in plant dynamics. The cases of additive and feedback mismatch are considered separately and represented in the form of easily checked graphical stability criteria in the frequency domain.

1. Introduction

Recent work [1-10] has focussed attention on the practical problems of assessing the robustness of feedback control systems with respect to changes in plant dynamic characteristics and of developing design techniques that guarantee a required degree of robustness. The property of robustness is perhaps of particular importance for the case of control systems for plant including significant time-delays in its dynamics [11,12] where, for example, the well-known Smith predictor [11] control scheme incorporates a model of the plant. Off-line analysis of the control configuration may well indicate that the Smith scheme is stable with acceptable transient characteristics if the plant matches the model exactly yet the inevitable mismatch between plant and model dynamics can lead to severe stability and performance problems on implementation [11,12]. It is clearly of vital importance that the off-line design is at least robust enough to cope with the observed plant/model mismatch and preferably robust enough to simultaneously retain stability and acceptable dynamic characteristics in the presence of changes in plant dynamic characteristics. This 'double' robustness problem is the subject of this paper which presents an extension of standard robustness results for feedback systems to cover Smith control schemes.

The general theory is described in section two where an abstract viewpoint is taken, regarding the stability of the Smith scheme as an input/output stability problem in arbitrary Banach spaces [13,14]. Robustness results are then derived for the cases of additive or feedback mismatch between plant and model. Section three then examines the special case of dynamic systems described by convolution operators when it is shown that the rather qualitative structural results of section two take a useful graphical form suitable for computer-aided-design using frequency response loci.

2. Robustness of Smith Control Schemes: General Theory

We consider an ℓ-input/m-output linear system with input vector u and output vector y lying in input and output linear vector spaces U^{ℓ} and Y^{m} respectively. The system (termed the plant) is regarded as a linear operator mapping U^{ℓ} into Y^{m} expressed in the separable form

$$y = Tz \quad , \qquad z = Gu \qquad \ldots(1)$$

The linear operator T maps Y^{m} into itself and is taken to represent delays or other similar dynamic effects at the plant output whereas G, mapping U^{ℓ} into Y^{m}, is linear and represents delay-free dynamics. The delay-free vector $z \in Y^{m}$.

The abstract Smith control scheme is illustrated in Fig.2 where the linear operators G_A and T_A represent models of the plant elements G and T respectively and K is the forward path control element regarded as a mapping of Y^{m} into U^{ℓ}. The signal $r \in Y^{m}$ is the demand or set-point signal. The dynamics of the configuration, for zero-initial conditions, can be represented by the equations

$$y = Tz \qquad \ldots(2)$$

$$z = Gu \qquad \ldots(3)$$

$$u = K(r - G_A u - (Tz - T_A G_A u)) \qquad \ldots(4)$$

In practice, equation (4) has a unique solution u for each choice of r and z and hence we will assume that it can be written in the form

$$u = K^{*}(r - Tz) \qquad \ldots(5)$$

where K^{*} is a uniquely defined linear mapping of Y^{m} into U^{ℓ}. Clearly the Smith scheme therefore has the equivalent feedback structure illustrated in Fig.3.

Let U_o^{ℓ} and Y_o^{m} be linear vector subspaces of U^{ℓ} and Y^{m} respectively (regarded as spaces of 'stable' inputs and outputs resp.) and that they are endowed with norm topologies with respect to which they are Banach spaces. The Smith scheme is then said to be bounded-input/bounded-output (BIBO) stable[14] if the output response y from zero initial conditions lies in Y_o^m whenever the demand signal r lies in Y_o^m. Familiar examples of these ideas can be found in ref.(14) where U^ℓ and Y^m are, for example, cartesian products of extended $L_p(0,+\infty)$ spaces and U_o^ℓ and Y_o^m are cartesian products of $L_p(0,+\infty)$.

In the following subsections, the robustness of the stability of the Smith scheme is investigated in the above general form for two cases of plant/model mismatch using the classical global contraction mapping theorem[13,14]. For notational simplicity the notation $\|\cdot\|$ will be used to denote vector norms and induced operator norms with no reference to the underlying space.

2.1. The Case of Additive Plant Mismatch

It is trivially verified that the delay free plant component G can be represented in terms of the model G_A in the form

$$G = G_A + \Delta G \qquad \ldots(6)$$

where ΔG is an additive[5,6] perturbation of the model G_A to represent the mismatch $G-G_A$ between the delay-free components of plant and model. In a similar manner, it is possible to write

$$T = T_A + \Delta T \qquad \ldots(7)$$

The following theorem characterizes the stability of the Smith scheme in terms of measures of ΔG and ΔT and the stability characteristics of the Smith scheme of Figs. 2 and 3 with G_A and T_A replacing G and T respectively.

Theorem 1: With the above notation, the Smith scheme of Fig.2 is stable in the BIBO sense if

(i) the plant component G and its model G_A map U_o^{ℓ} into Y_o^m and that their restrictions to U_o^{ℓ} have finite induced norms,

(ii) the delay components T, T_A map Y_o^m into itself with restrictions to Y_o^m of finite induced norm,

(iii) the restriction to Y_o^m of the delay-free mapping $r \mapsto u_A \triangleq (I+KG_A)^{-1} Kr$ has range in U_o^{ℓ} and finite induced norm,

(iv)
$$\lambda_1 \triangleq \|(I+KG_A)^{-1} K\Delta TG_A\| < 1 \qquad \ldots(8)$$

and

(v)
$$\lambda_2 = \frac{1}{1-\lambda_1} \|(I+KG_A)^{-1} KT\Delta G\| < 1 \qquad \ldots(9)$$

Proof: As G and T are stable and bounded by assumption it is sufficient to prove that $u \in U_o^{\ell}$ whenever $r \in Y_o^m$. Clearly

$$u = K^*(r - TGu) \qquad \ldots(10)$$

which can be written as

$$u = (I+K^*T_AG_A)^{-1}K^*(r - (TG-T_AG_A)u) \qquad \ldots(11)$$

This is an equation in U^{ℓ} of the form $u = W_r(u)$. Suppose that W_r maps U_o^{ℓ} into itself whenever the demand $r \in Y_o^m$. The BIBO stability requirement can then be replaced by the (sufficient) requirement that W_r is a contraction mapping[13,14] for all r. This is clearly the case if

$$\lambda_o \triangleq \|(I+K^*T_AG_A)^{-1}K^*(TG-T_AG_A)\| < 1 \qquad \ldots(12)$$

We now prove that $(I+K^{*}T_A G_A)^{-1}K^{*} = (I+KG_A)^{-1}K$ by writing $u_A = (I+K^{*}T_A G_A)^{-1}K^{*}r$ in the form

$$u_A = v_1 - v_2 \qquad \ldots(13)$$

with

$$v_1 = K^{*}r \quad , \quad v_2 = K^{*}T_A G_A u_A \qquad \ldots(14)$$

Using the definition of K^{*}, (14) takes the form

$$v_1 = K(r - (I-T_A)G_A v_1) \qquad \ldots(15)$$

$$v_2 = K(T_A G_A u_A - (I-T_A)G_A v_2) \qquad \ldots(16)$$

Subtracting (15) and (16) and using (13) yields, after a little manipulation

$$u_A = K(r - G_A u_A) \qquad \ldots(17)$$

or $u_A = (I+KG_A)^{-1}Kr$ as required. Conditions (i)-(iii) clearly ensure that W_r hence maps U_o^{ℓ} into itself for all $r \in Y_o^m$ and the result follows from (8) and (9) noting that

$$\begin{aligned} \lambda_o &\equiv \|(I+KG_A)^{-1}K(T\Delta G+\Delta TG_A)\| \\ &\leq \|(I+KG_A)^{-1}KT\Delta G\| + \|(I+KG_A)^{-1}K\Delta TG_A\| \\ &< (1-\lambda_1) + \lambda_1 = 1 \end{aligned} \qquad \ldots(18)$$

The interpretation of this result is fairly straightforward despite its abstract form. Conditions (i) and (ii) boil down in practice to the requirement that the plant TG and its approximate model $T_A G_A$ are open-loop stable. Condition (iii) simply states that the input response u_A of the delay-free feedback scheme of Fig.4 should be bounded whenever $\hat{r} = 0$ and the demand r is bounded. That is, the delay free feedback scheme of Fig.4 is stable in the normal practical sense.

Finally, conditions (iv) and (v) provide upper bounds on the additive mismatches ΔG and ΔT that guarantee BIBO stability. Note that if the plant and its model are identical (ie $G = G_A$ and $T = T_A$) we have $\Delta G = 0$, $\Delta T = 0$ and (8) and (9) are trivially satisfied. The result then states the well-known idea that the Smith scheme is stable if the delay-free control scheme of Fig.4 is stable. In the presence of non-zero mismatch, (8) and (9) allow a sequential assessment of the effect of delay mismatch ΔT via equation (8) followed by a consideration of the mismatch ΔG in the delay-free component using equation (9).

An application of the above result is described in section 3.1. It is useful however to make the following general observations on situations when (8) and (9) will be valid:

(i) Using the usual properties of operator norms, it is easily verified that condition (8) will be valid if the mismatch ΔT satisfies

$$\|\Delta T G_A\| < \frac{1}{\|(I+KG_A)^{-1}K\|} \qquad \ldots(19)$$

The model G_A is retained with ΔT on the left-hand-side as infinitessimally small changes in dead-time can lead to large changes in ΔT and hence large values of $\|\Delta T\|$ whereas the low-pass filtering effect of G_A is expected to generate small values of $\|\Delta T G_A\|$. Note that (19) indicates that Theorem 1 is a 'small-gain theorem' as, in general, $\|(I+KG_A)^{-1}K\|$ will tend to become small as the gains in K are reduced to zero. (Note: this argument is rather simplistic as the presence of integral action in the control element generates large d.c. gains but the general principle is expected to carry over in practice).

(ii) Using a similar argument to the above it is clear that (9) is valid for mismatch

$$||\Delta G|| < \frac{1 - \lambda_1}{||(I+KG_A)^{-1}KT||} \qquad \ldots(20)$$

again indicating the 'low-gain' validity of the results. It is more interesting however to note the interplay between the delay and delay-free mismatch terms. If λ_1 is small due to, say, 'small' delay mismatch ΔT, then the permissible delay-free mismatch ΔG could be large. If however λ_1 is close to unity due to, say, poor modelling of the delay term then the permissible delay-free mismatch ΔG could be uncomfortably small.

We now prove the following result indicating the robustness of any design satisfying the conditions of theorem 1:

Theorem 2: Suppose that the controller K is designed so that the conditions of theorem 1 are satisfied and that, over a period of time, the real plant TG changes its dynamic characteristics to those described by the decomposition $\tilde{T}\tilde{G}$. If both $\tilde{T}$ and $\tilde{G}$ are BIBO stable, then the Smith scheme will retain its BIBO stability if

$$||\tilde{T}(\tilde{G}-G)|| + ||(\tilde{T}-T)G|| < \frac{(1-\lambda_1)(1-\lambda_2)}{||(I+KG_A)^{-1}K||} \qquad \ldots(21)$$

Proof: Following the argument of the proof of theorem 1, the perturbed scheme is stable if

$$\tilde{\lambda}_o \triangleq ||(I+KG_A)^{-1}K(\tilde{T}\tilde{G} - T_AG_A)|| < 1 \qquad \ldots(22)$$

Noting that

$$\tilde{\lambda}_o \leq ||(I+KG_A)^{-1}K|| \cdot ||(\tilde{T}\tilde{G} - TG)|| + \lambda_o \qquad \ldots(23)$$

and that $\tilde{T}\tilde{G}-TG = \tilde{T}(\tilde{G}-G)+(\tilde{T}-T)G$, it is clear that (22) is satisfied if

$$\|(I+KG_A)^{-1}K\| \cdot \{ \|\tilde{T}(\tilde{G}-G)\| + \|(\tilde{T}-T)G\| \} < 1-\lambda_o \qquad \ldots(24)$$

The result follows from (8) and (9) noting that (c.f. equation (18))

$$\begin{aligned} \lambda_o &\leq \|(I+KG_A)^{-1}KT\Delta G\| + \|(I+KG_A)^{-1}K\Delta TG_A\| \\ &= \lambda_2(1-\lambda_1) + \lambda_1 \end{aligned} \qquad \ldots(25)$$

and hence that $1-\lambda_o \geq (1-\lambda_1)(1-\lambda_2)$.

Condition (21) estimates the magnitude of the permissible future changes $\tilde{G}-G$ and $\tilde{T}-T$ in plant dynamics in terms of the design parameters G_A and K and the computed parameters λ_1 and λ_2 describing mismatch characteristics with the initial plant data G and T. Note again that, if λ_1 and λ_2 are small and small gains are used, the right-hand-side of (21) is large, hence allowing large changes in plant dynamics before instability results. Conversely, if high-gains are used and λ_1 and λ_2 are close to unity, small changes in plant characteristics will violate (21) indicating the possibility of instability.

Finally, we note that, in general, delay operators have unit norm in both frequency domain and time-domain spaces. It is very often the case therefore that $\|\tilde{T}(\tilde{G}-G)\| = \|\tilde{G}-G\|$ providing a partial simplification of (21).

2.2. The Case of Feedback Plant Mismatch

Despite the simplicity of the analysis of additive mismatch the results do suffer from the practical restriction that they require (condition (i) of theorem 1) both the plant G and its model G_A to be

BIBO stable whereas there are obvious practical situations where either or both could be unstable. A useful general analysis of this problem has not yet been achieved. A partial solution to the problem has been suggested in ref.4 using the notion of a feedback perturbation. In the context of the Smith scheme discussed in this paper, the mismatch between G and G_A has a feedback perturbation structure if there exists a linear mapping H of Y^m into U^ℓ such that

$$G = (I + G_A H)^{-1} G_A \qquad \ldots(26)$$

when G is as illustrated in Fig.5. If both G and G_A have an inverse then (26) indicates that

$$G^{-1} = G_A^{-1} + H \qquad \ldots(27)$$

and hence that a feedback perturbation H of G_A is an additive perturbation of its inverse. Such perturbations have been used to great advantage in first-order multivariable process control theory (7,9) and approximation of large scale systems (15) and can be expected to have at least comparable advantages for time-delay systems analysis.

The following result describes the stability of the Smith scheme in terms of additive mismatch $\Delta T = T - T_A$ between the plant and model delay components and feedback mismatch H between plant and model delay free components.

Theorem 3: With the above notation, suppose that:

(i) the delay components T and T_A map Y_o^m into itself with restriction to Y_o^m of finite induced norm,

(ii) the model G_A maps U_o^ℓ into Y_o^m with restriction to U_o^ℓ of finite induced norm,

(iii) the feedback mismatch H maps Y_o^m into U_o^ℓ with restriction to Y_o^m of finite induced norm,

(iv) the delay free mappings $r \mapsto z_A \triangleq (I+G_AK)^{-1}G_AKr$ and $\hat{r} \mapsto \hat{z}_A \triangleq (I+G_AK)^{-1}G_A\hat{r}$ map Y_o^m into itself and U_o^ℓ into Y_o^m respectively with finite induced norm,

(v) $$\lambda_1' \triangleq \|(I+G_AK)^{-1}G_AK\Delta T\| < 1 \quad , \qquad \ldots(28)$$

and

(vi) $$\lambda_2' \triangleq \frac{1}{1-\lambda'_1} \|(I+G_AK)^{-1}G_A(I+K(I-T_A)G_A)H\| < 1 \qquad \ldots(29)$$

Then the Smith scheme of Fig.2 is stable in the BIBO sense.

Proof: As T is stable it is sufficient to prove that $z \in Y_o^m$ whenever $r \in Y_o^m$. Clearly, from Fig.3,

$$\begin{aligned} z &= GK^*(r - Tz) \\ &= GK^*(r - T_Az - (T-T_A)z) \end{aligned} \qquad \ldots(30)$$

or, substituting for G from (26) and rearranging,

$$\begin{aligned} z = {} & (I+G_AK^*T_A)^{-1}G_AK^*(r - (T-T_A)z) \\ & - (I+G_AK^*T_A)^{-1}G_AHz \end{aligned} \qquad \ldots(31)$$

which can be regarded as an equation $z = W_rz$ in Y^m. Suppose now that W_r maps Y_o^m into itself whenever $r \in Y_o^m$. As in the proof of theorem 2, the stability requirement can then be replaced by the requirement that W_r is a contraction on Y_o^m for all $r \in Y_o^m$. That is, we need

$$\lambda'_o \triangleq \|(I+G_AK^*T_A)^{-1}G_A(K^*(T-T_A) + H)\| < 1 \qquad \ldots(32)$$

We now prove that

$$(I+G_AK^*T_A)^{-1}G_A = (I+G_AK)^{-1}G_A(I+K(I-T_A)G_A) \qquad \ldots(33)$$

and hence (using the definition of K^*) that $(I+G_AK^*T_A)^{-1}G_AK^* = (I+G_AK)^{-1}G_AK$ by writing the equation $\hat{z}_A = (I+G_AK^*T_A)^{-1}G_A\hat{r}$ in the form

$$\hat{z}_A + G_A\xi = G_A\hat{r} \qquad \ldots(34)$$

where $\xi = K^*T_A\hat{z}_A$ or, equivalently,

$$\xi = K(T_A\hat{z}_A - (I-T_A)G_A\xi) \qquad \ldots(35)$$

Multiplying (35) by G_A and eliminating $G_A\xi$ using (34) yields, after a little manipulation, the relation

$$\hat{z}_A = (I+G_AK)^{-1}G_A(I+K(I-T_A)G_A)\hat{r} \qquad \ldots(36)$$

which implies (33) by comparing with $\hat{z}_A = (I+G_AK^*T_A)^{-1}G_A$. Clearly these results combined with conditions (i)-(iv) ensure that W_r maps Y_o^m into itself whenever $r \in Y_o^m$, the remainder of the theorem following from (28) and (29) noting that

$$\lambda'_o \leq ||(I+G_AK)^{-1}G_AK\Delta T|| + ||(I+G_AK)^{-1}G_A(I+K(I-T_A)G_A)H||$$
$$< \lambda'_1 + 1 - \lambda'_1 = 1 \qquad \ldots(37)$$

The overall structure of the result is similar to that of Theorem 1 and will not, for brevity, be discussed in detail except to note that,

(a) although the result requires that the model G_A and the feedback mismatch H be stable this does not imply that the plant component G is stable as is illustrated by taking the transfer function descriptions $G_A(s) = 1/(s+1)$ and $H(s) = -2/(s+1)$ when $G(s) = (s+1)/(s^2+2s-1)$,

(b) if the feedback mismatch $H = 0$, condition (29) is trivially verified and (28) will be satisfied for small control gains but, in the case of $H \neq 0$, (29) may not be satisfied at low gains if the mismatch contribution $||G_A H|| > 1$ as it is easily seen that

$$\limsup_{||K|| \to 0} \lambda'_2 = ||G_A H|| \qquad \ldots(38)$$

We conclude this section with the following result characterizing the robustness of the Smith scheme designed to satisfy the conditions of theorem 3:

Theorem 4: If the controller K is designed to ensure the validity of the conditions of theorem 3 and if the real plant TG changes its dynamic characteristics to those described by $\tilde{T}\tilde{G}$, where the delay component $\tilde{T}$ is BIBO stable and $\tilde{G} = (I+G(\tilde{H}-H))^{-1}G$ is generated by the stable perturbation $\tilde{H}-H$ of G, then the Smith scheme will retain its stability if

$$||(I+G_A K)^{-1} G_A K(\tilde{T}-T)|| + ||(I+G_A K)^{-1} G_A (I+K(I-T_A)G_A)(\tilde{H}-H)|| < (1-\lambda'_1)(1-\lambda'_2) \qquad \ldots(39)$$

Proof: Note initially that a feedback perturbation $\tilde{H}-H$ of G is just a feedback perturbation $\tilde{H}$ of G_A. Stability is hence retained (c.f. (32)) if

$$\tilde{\lambda}'_o \triangleq ||(I+G_A K^* T_A)^{-1} G_A (K^*(\tilde{T}-T_A)+\tilde{H})|| < 1 \qquad \ldots(40)$$

Clearly

$$\tilde{\lambda}'_o \leq ||(I+G_A K^* T_A)^{-1} G_A K^*(\tilde{T}-T)|| + ||(I+G_A K^* T_A)^{-1} G_A (\tilde{H}-H)|| + \lambda_o \qquad \ldots(41)$$

indicating that $\tilde{\lambda}'_o < 1$ if

$$\| (I+G_A K^* T_A)^{-1} G_A K^* (\tilde{T}-T) \| \;+$$
$$\| (I+G_A K^* T_A)^{-1} G_A (\tilde{H}-H) \| \;<\; (1-\lambda'_1)(1-\lambda'_2) \qquad \ldots(42)$$

as $(1-\lambda'_1)(1-\lambda'_2) \leqslant 1-\lambda'_o$ from (28), (29) and (37). The result follows by eliminating K^* from (42) using (33).

Note that poor robustness margins are obtained if either or both of the mismatch parameters λ'_1 and λ'_2 is close to unity and that robustness is optimized by making λ'_1 and λ'_2 as small as possible either by use of low gains (with consequent loss in performance) or good models G_A and T_A of G and T.

3. Graphical Stability Criteria

The analyses of section 2 have great generality allowing some distributed, non-rational and even non-causal dynamics in G and non-delay elements in T. The underlying spaces U^ℓ and Y^m can be freely chosen as can the Banach subspaces U_o^ℓ and Y_o^m characterizing bounded-inputs and bounded-outputs. In general stability theory(13,14), typical examples for continuous systems could be $U_o^\ell = L_p^\ell(0,+\infty)$ and $Y_o^m = L_q^m(0,+\infty)$ with U^ℓ and Y^m defined as their extended spaces whereas, for sampled-data systems $U_o^\ell = \ell_p^\ell$ and $Y_o^m = \ell_q^m$ with U^ℓ and Y^m as their extended spaces may be suitable choices. There are clearly an infinity of stability criteria derivable from the results of section 2. For simplicity however we will use the framework provided by Freeman(16) by choosing $U_o = Y_o$ to be the vector space of functions of a complex variable s that are holomorphic and bounded in the open, connected set

$$\Omega = \{s : \mathrm{Re}\, s > 0 \;,\quad |s| < R\} \qquad \ldots(43)$$

with R 'large enough' to interpret Ω as the 'unstable region' of the complex plane. We will denote the boundary (the Nyquist contour) of Ω by $\partial\Omega$ and the norm of a qx1 vector x(s) in the product space Y_o^q as

$$\|x\| \triangleq \max_{1\leq i\leq q} \sup_{s\in\partial\Omega} |x_i(s)| \qquad \ldots(44)$$

All mappings T, G, T_A, G_A, K are represented by transfer function matrices of appropriate dimensions and a operator M mapping U_o^q into Y_o^p (say) is bounded iff its pxq transfer function matrix has elements that are holomorphic and bounded in Ω. The induced operator norm is [16]

$$\|M\| = \max_{1\leq i\leq p} \sup_{s\in\partial\Omega} \sum_{j=1}^{q} |M_{ij}(s)| \qquad \ldots(45)$$

(Note: In the time domain M represents a convolution operator with exponentially bounded with negative exponent kernel).

In the following sections, the abstract mismatch stability criteria of theorems 1 and 3 are converted into graphical stability criteria based upon frequency response data. For simplicity of presentation we will suppose that G and G_A are rational and strictly proper transfer function matrices, that K is rational and proper and that both $T = \mathrm{diag}\{e^{-sT_j}\}_{1\leq j\leq m}$ and $T_A = \mathrm{diag}\{e^{-sT_{Aj}}\}_{1\leq j\leq m}$ are mxm diagonal matrices of pure delays. The case of single-input/single-output systems clearly corresponds to the case of $m = \ell = 1$.

3.1. Graphical Assessment of Additive Mismatch

With the above definitions, theorem 1 has the following simple form describing stability in the presence of additive mismatch:

Theorem 5: If the plant component G and its model G_A are asymptotically stable and the delay free feedback system of Fig.4 is input-output stable then the Smith scheme of Fig.2 is BIBO stable if

$$\lambda_1 \triangleq \max_{1\leq i\leq \ell} \sup_{s\in\partial\Omega} \sum_{j=1}^{\ell} |((I+K(s)G_A(s))^{-1}K(s)(T(s)-T_A(s))G_A(s))_{ij}| < 1 \qquad \ldots(46)$$

and

$$\lambda_2 \triangleq \frac{1}{1-\lambda_1} \max_{1\leq i\leq \ell} \sup_{s\in\partial\Omega} \sum_{j=1}^{\ell} |((I+K(s)G_A(s))^{-1}K(s)T(s)(G(s)-G_A(s)))_{ij}| < 1 \qquad \ldots(47)$$

Proof: The stability assumptions are equivalent to conditions (i)-(iii) of theorem 1 whilst (46) and (47) are identical to (8) and (9) respectively.

The result has a clear frequency domain flavour and can be checked by numerical evaluation of λ_1 and λ_2. It does not however have a Nyquist-like structure unless G_A and K are diagonal. This includes the important single-input/single-output case but it also includes square multivariable cases where interaction is either regarded from physical considerations as being small enough to be neglected in the model or it is neglected to simplify the structure of the model and hence the design of K to stabilize the delay-free feedback scheme (Fig.4). The details are described in the following result:

Theorem 6: With the assumptions of theorem 5 and the condition that $m = \ell$ and both G_A and K are diagonal, the Smith scheme is stable if, for $1\leq k\leq m$,

(i) the band $B_k^{(1)}(T-T_A)$ generated by plotting the <u>inverse</u> Nyquist locus Γ_k of $(G_A(s))_{kk}K_{kk}(s)$ with $s = i\omega$ and $\omega \geq 0$ and superimposing at each frequency point circles of radius

$$r_k^{(1)}(\omega) \triangleq \sqrt{2}\,(1 - \cos\,\omega(T_k - T_{Ak}))^{\frac{1}{2}} \qquad \ldots(48)$$

and centre $((G_A(i\omega))_{kk}K_{kk}(i\omega))^{-1}$ does not contain the (-1,0) point of the complex plane, and

(ii) noting that $\lambda_1 = \max\limits_{1\leq k\leq m}\ \max\limits_{\omega\geq 0}\{r_k^{(1)}(\omega)/|1+((G_A(i\omega))_{kk}K_{kk}(i\omega))^{-1}|\}$,

then the band $B_k^{(2)}(G-G_A)$ generated from Γ_k as described in (i) but with $r_k^{(1)}(\omega)$ replaced by

$$r_k^{(2)}(\omega) \triangleq \frac{1}{(1-\lambda_1)} \sum_{j=1}^{m} |\delta_{kj} - (G_A(i\omega))_{kk}^{-1} G_{kj}(i\omega)| \qquad \ldots(49)$$

does not contain the (-1,0) point of the complex plane.

<u>Proof</u>: As G and G_A are strictly proper, λ_1 and λ_2 can be evaluated from (46) and (47) by replacing $\partial\Omega$ by the positive imaginary axis $\{s : s = i\omega,\ \omega\geq 0\}$. The condition $\lambda_1<1$ is then equivalent to, $1\leq k\leq m$,

$$\left| \frac{(G_A(i\omega))_{kk}K_{kk}(i\omega)}{1 + (G_A(i\omega))_{kk}K_{kk}(i\omega)} \left(e^{-i\omega T} - e^{-i\omega T_A}\right) \right| < 1 \quad , \qquad \omega\geq 0 \qquad \ldots(50)$$

or,

$$|1 + ((G_A(i\omega))_{kk}K_{kk}(i\omega))^{-1}| > |e^{-i\omega T} - e^{-i\omega T_A}|$$

$$= |1 - e^{i\omega(T-T_A)}| = r_k^{(1)}(\omega) \quad , \qquad \omega\geq 0 \qquad \ldots(51)$$

which can be represented graphically by condition (i). Condition (ii) follows from the requirement that $\lambda_2<1$ after similar manipulations to the above noting that $|e^{-i\omega T_k}| = 1$, $\omega\geq 0$, $1\leq k\leq m$.

The result has a similar structure to the well-known inverse Nyquist array design method(17,18) with the two sets of controller-independent bands $B_k^{(1)}(T-T_A)$ and $B_k^{(2)}(G-G_A)$, $1 \leq k \leq m$, replacing the single set of Gershgorin bands. In practice the controller K is designed to stabilize the delay-free scheme shown in Fig.4. The inverse Nyquist loci Γ_k, $1 \leq k \leq m$, could, in fact be used to check this preliminary stability requirement and subsequently the bands $B_k^{(1)}(T-T_A)$ representing time-delay mismatch superimposed to check conditon (i). If this approach is successful, λ_1 can be calculated and the bands $B_k^{(1)}(T-T_A)$, $1 \leq k \leq m$, replaced by $B_k^{(2)}(G-G_A)$ to check the validity of condition (ii). Note that the time-delay mismatch parameter λ_1 has a significant effect on the width of $B_k^{(2)}(G-G_A)$ via the multiplicative factor $(1-\lambda_1)^{-1}$ in $r_k^{(2)}(\omega)$.

3.2. Graphical Assessment of Feedback Mismatch

Using the frequency domain spaces described above, theorem 3 has the following graphical form describing the effect of feedback mismatch on stability.

Theorem 7: If the model G_A and the feedback mismatch H are asymptotically stable and the delay free feedback system of Fig.4 is input-output stable then the Smith scheme of Fig.2 is BIBO stable if

$$\lambda'_1 \triangleq \max_{1 \leq i \leq \ell} \sup_{s \in \partial\Omega} \sum_{j=1}^{\ell} \left| \left((I+G_A(s)K(s))^{-1} G_A(s)K(s)(T(s)-T_A(s)) \right)_{ij} \right| < 1 \qquad \ldots(52)$$

and

$$\lambda'_2 \triangleq \frac{1}{1-\lambda'_1} \max_{1\le i\le \ell} \sup_{s\in\partial\Omega} \sum_{j=1}^{\ell} |((I+G_A(s)K(s))^{-1}G_A(s)(I+K(s)(I-T_A(s))G_A(s))H(s))_{ij}|$$

$$< 1 \qquad \ldots(53)$$

Proof: Conditions (i)-(iv) of theorem 3 hold from the assumptions and (52) and (53) are simply (28) and (29).

Despite its frequency-domain flavour, a result of more classical structure paralleling theorem 6 is stated as follows. The graphical interpretation is obvious.

Theorem 8: With the assumptions of theorem 7 and the condition $m = \ell$, suppose that both G_A and K are diagonal and K is invertible. Then $\lambda_1 = \lambda_1'$ and the Smith scheme is stable if conditions (i) and (ii) of theorem 6 hold with $r_k^{(2)}(\omega)$ replaced by

$$r_k^{(2)}(\omega) \triangleq \frac{1}{1-\lambda_1'} |K_{kk}^{-1}(i\omega)+(1-e^{-i\omega T_{Ak}})(G_A(i\omega))_{kk}| \sum_{j=1}^{m} |H_{kj}(i\omega)| \qquad \ldots(54)$$

Proof: It is clear that $\lambda_1 = \lambda_1'$ as G_A, T, T_A and K are diagonal. Condition (52) therefore reduces to condition (i) of theorem 6. The remainder of the result follows from (53) and its graphical interpretation in a similar manner to the proof of theorem 6.

It is interesting to note that, as the delays T_k and T_{Ak} tend to zero, we clearly have $\lambda_1' = 0$ and

$$r_k^{(2)}(\omega) = |K_{kk}^{-1}(i\omega) \sum_{j=1}^{m} H_{kj}(i\omega)| \qquad \ldots(55)$$

If the approximate model G_A is chosen to have diagonal elements with inverses equal to the diagonal elements of the inverse of the plant element G, then a moments reflection (using (27)) indicates that the bands $B_k^{(2)}(G-G_A)$ are simply the Gershgorin bands of Q = GK. In this sense theorem 8 is a generalization of the inverse Nyquist array technique to Smith control schemes.

4. Conclusions

The paper has considered the dual robustness problem of stability in the presence of plant/model mismatch and stability in the presence of plant variations for the classical Smith control scheme for multi-input/multi-output control schemes. The abstract functional analysis viewpoint taken illustrates that the problem is very similar in structure to the robustness problem for conventional feedback schemes with the added complication that (i) we are interested in variations in two plant components (the delay and delay-free elements) and (ii) the effective forward path controller K^* is a nonlinear function of the delay-free control K. Despite these added complexities, it has been shown that the double robustness problem is capable of being analysed for both additive and feedback mismatch using the contraction mapping fixed point theorem providing results that can be applied to a wide class of continuous or discrete systems in the time or frequency domains. This fact may be of particular importance if the plant is partially unknown when time domain information should be brought into the design procedure in a similar manner to the delay free case discussed in reference (10).

The results clearly lead to a large number of distinct design techniques covering different areas of application. Their potential has been illustrated by a simple frequency domain design of multivariable Smith schemes using diagonal plant models (a special case that includes the classical single-input/single-output case). Stability analysis is undertaken using inverse Nyquist loci of the diagonal terms of the plant model and the use of bands around these loci that represent the destabilizing effect of mismatch in a manner remeniscent of the inverse Nyquist array design technique. In fact the frequency domain result for feedback mismatch reduces to the INA criterion as plant and model delays tend to zero indicating that the results described here are a generalization of the INA to Smith control schemes for plants with significant time-delay.

The frequency domain results described in the paper were chosen to reflect the similarity of the abstract results to well-known multivariable control concepts and robustness notions and do not necessarily take the best form for design work. The refinement of these results for practical computer-aided-design is under study and will be reported in future papers.

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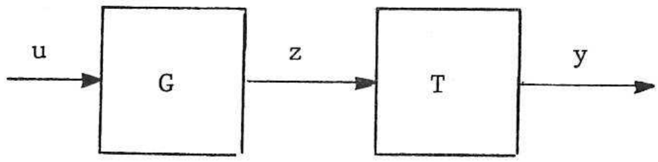

Fig.1. Plant decomposition

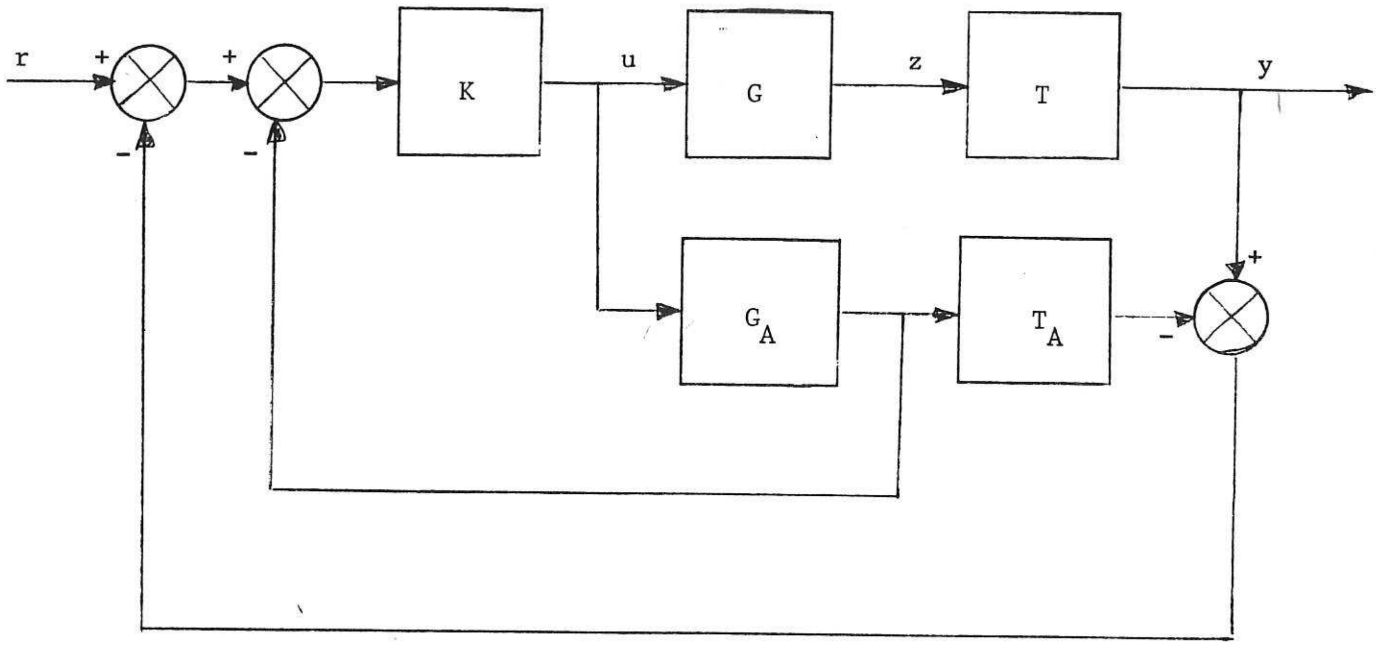

Fig.2. Smith Control Scheme

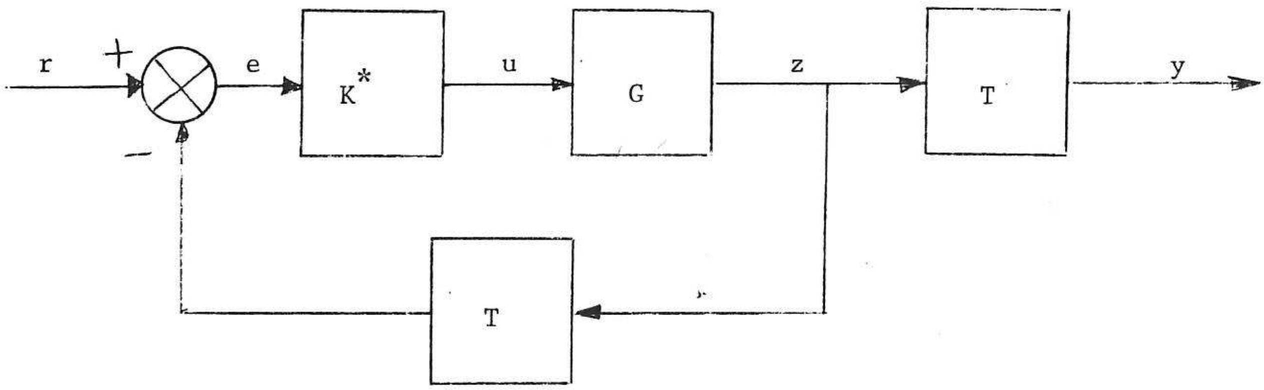

Fig.3. Equivalent Smith Scheme

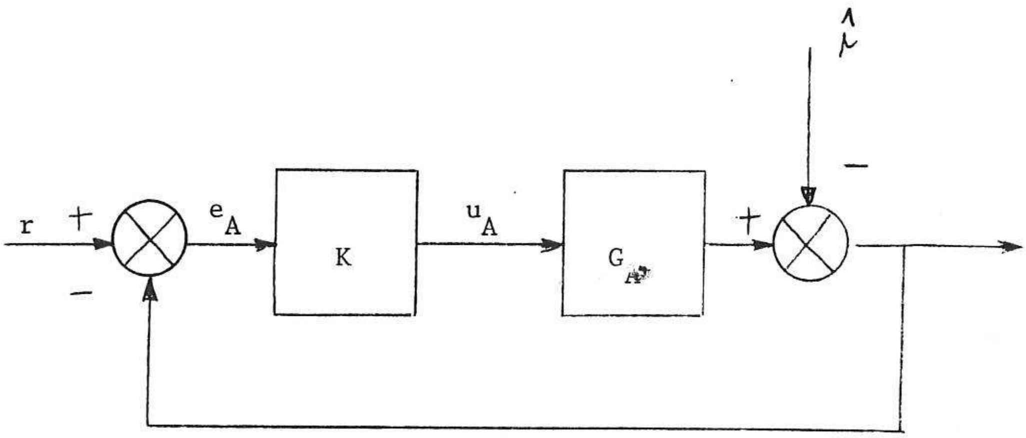

Fig.4. Delay-Free Control Scheme

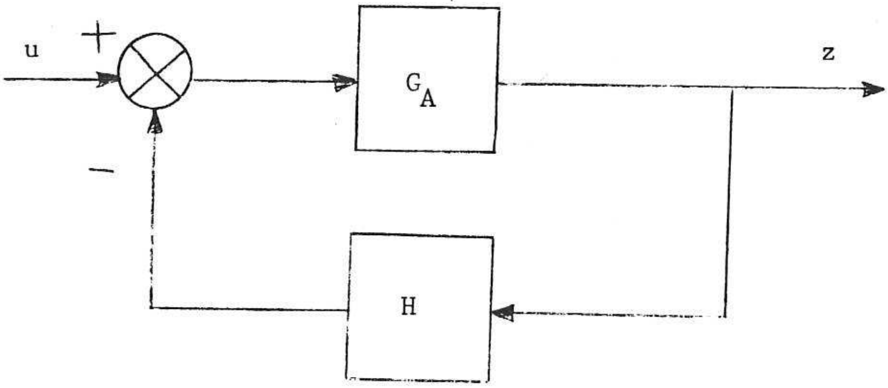

Fig.5. Plant regarded as a feedback perturbation of the model