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MODE ANALYSIS OF A TUBULAR STRUCTURE OF COUPLED NON-LINEAR OSCILLATORS FOR SMALL-INTESTINAL MODELLING

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ABSTRACT

A commonly accepted mathematical model for the slow-wave electrical activity of the gastro-intestinal tract of humans and animals comprises a set of interconnected non-linear oscillators. Using a van der Pol oscillator with third-power conductance characteristics as the unit oscillator a number of structures have been analysed using a matrix Krylov-Bogolioubov method linearisation. Thus mode analysis of one dimensional chains and two-dimensional arrays have been reported. In this paper the method is extended to consider a tubular structure which is relevant to modelling small-intestinal rhythms. It is shown that this structure is capable of producing stable single modes, nonresonant double modes and degenerate modes. General expressions are obtained for an mxn structure and examples given of two special conditions of 3 x 4 (i.e. odd numbers of oscillators in a ring) and 4 x 3 cases. The analytical results obtained for these two cases have been verified experimentally using an electronic implementation of coupled van der Pol oscillators.

1. Introduction

Nelsen and Becker in 1968 first made the hypothesis that slow-wave electrical activity in the mammalian gastro-intestinal tract should be modelled as a set of interconnected non-linear oscillators. Since then the data available on these electrical rhythms have steadily improved and the number of modelling simulations and analyses have increased. The common theme in all these models has been that of mutual interaction (i.e. bidirectional coupling), while structure and unit oscillator dynamics have varied.

In terms of simulation studies human small-intestine has been simulated as a one-dimensional chain of coupled van der Pol oscillators (Sarna et al, 1971), and the human stomach by a two-dimensional array (Sarna et al, 1972). For the large-intestine, a fifth-power van der Pol chain has been considered to account for apparent periods of absence of electrical rhythms (Linkens et al, 1976). Using a unit oscillator dynamic based on the Hodgkin-Huxley equations, the established gastro-intestinal phenomenon of entrainment has been demonstrated both for small numbers (Linkens and Datardina, 1977) and large numbers (Patton and Linkens, 1977) of coupled oscillators. These computer simulations of Hodgkin-Huxley based models have been complemented by electronic simulations based on simplified dynamics. Such fast models have enable synchronisation phenomena to be quantified rapidly.

Mode analysis of intercoupled van der Pol oscillators has also been keeping pace with simulation studies. Harmonic balance techniques have established the multimode behaviour for general RLC coupling of two oscillators (Linkens 1977), while one-dimensional chains with intrinsic frequency gradients have yielded analytical entrainment conditions (Linkens,1974). The same methods have also been applied successfully to the condition of 'almost-entrainment', or modulation, where multiple spectral components occur (Linkens 1979). In parallel with this a matrix Krylov-Bogolioubov method has been developed and applied to one-dimensional ladder (Endo and Mori 1976a), two-dimensional arrays (Endo and Mori, 1976b), and a ring of oscillators (Endo and Mori, 1978). In this paper the matrix linearisation method is extended to analyse the case of a tubular structure of coupled third-power van der Pol oscillators. Such a tubular structure is clearly of relevance to the small-intestine with its tube-like nature. It is shown that such a structure has a particularly rich mode behaviour. The types of mode considered are single modes, nonresonant non degenerate modes (i.e. having non-equal frequencies), and degenerate modes (i.e. having equal frequencies) which may be either regular (i.e. with exact phase) or irregular (i.e. non-exact phase).

In section 2 the basic equations under consideration are derived, while in section 3 the diagonalisation of the system matrices via a suitable linear transformation and equivalent linearisation are described. Stationary amplitude conditions are determined in section 4 and their stability found in section 5. Two particular examples are dealt with in section 6 comprising odd and even numbers of oscillators around the periphery of the tube, and these examples are followed through via electronic experimentation in section 7.

2. Derivation of the Fundamental Mode Equation

The unit oscillator of the tube structure under investigation is a parallel resonant circuit, and consists of a capacitance C, an inductance L, and an active element characterised by a cubic nonlinearity

$$I_{ij}(V_{ij}) = -g_1 V_{ij} + g_3 V_{ij}^3 \quad (g_1, g_3 > 0) \qquad (1)$$

where i represents the location of the oscillator in each ring and j the location of the ring in the structure

$i = 1, 2, \ldots, m$

$j = 1, 2, \ldots, n$

where m is the number of oscillators in each ring and n the number of rings in the whole structure (see Fig. 1).

Now the tubular structure is described by the integral difference - differential equations:

$$I_{T(i-1)j} = \int \frac{1}{L_c} (V_{(i-1)j} - V_{ij})\,dt \tag{2a}$$

where L_c is the coupling inductance,

$$I_{Tij} = \int \frac{1}{L_c} (V_{ij} - V_{(i+1)j})\,dt \tag{2b}$$

(Subscript T means peripheral direction of the structure while L means its longitudinal direction).

$$I_{Li(j+1)} = \int \frac{1}{L_c} (V_{i(j+1)} - V_{ij})\,dt \tag{2c}$$

$$I_{Lij} = \int \frac{1}{L_c} (V_{ij} - V_{i(j-1)})\,dt \tag{2d}$$

where I_{Tij} is the current passing through the transverse coupling branch. I_{Lij} is the current passing through the longitudinal coupling branch V_{ij} is the output voltage of the oscillator of location i,j. Applying Kirchoff's law at the node i,j, leads to

$$I_{ij} + C\frac{dV_{ij}}{dt} + \frac{1}{L}\int V_{ij}\,dt = I_{T(i-1)j} + I_{Li(j+1)} - I_{Tij} - I_{Lij} \tag{3}$$

Substituting the various currents with their corresponding values into equation (3) and then differentiation w.r.t. time t, yields

$$\frac{d^2V_{ij}}{dt^2} - \frac{g_1}{C}\left(1 - \frac{3g_3}{g_1} V_{ij}^2\right)\frac{dV_{ij}}{dt} - \frac{1}{CL_c} V_{i(j-1)} + \left(\frac{1}{2CL} + \frac{2}{CL_c}\right) V_{ij}$$

$$- \frac{1}{CL_c} V_{i(j+1)} - \frac{1}{CL_c} V_{(i-1)j} + \frac{2}{CL_c}) V_{ij} - \frac{1}{CL_c} V_{(i+1)j} = 0 \tag{4}$$

Substituting

$$V_{ij} = \sqrt{\frac{g_1}{3g_3}}\, x_{ij} = k_v\, x_{ij} \tag{5a}$$

$$\tau = \sqrt{\frac{1}{2CL} + \frac{1}{CL_c}} \quad t \tag{5b}$$

Then, (4) can be written as

$$\frac{d^2 x_{ij}}{d\tau^2} - \frac{1}{C\sqrt{\frac{1}{2CL} + \frac{1}{CL_c}}} (1-x_{ij}^2)\frac{dx_{ij}}{d\tau}$$

$$- \frac{2L}{2L+L_c} x_{(i-1)j} + (1+ \frac{2L}{2L+L_c}) x_{ij} - \frac{2L}{2L+L_c} x_{(i+1)j}$$

$$- \frac{2L}{2L+L_c} x_{i(j-1)} + (1+ \frac{2L}{2L+L_c}) x_{ij} - \frac{2L}{2L+L_c} x_{i(j+1)} = 0 \tag{6}$$

Defining

$$\alpha \equiv \frac{2L}{2L+L_c} \tag{7a}$$

which represents the inductive coupling factor ($0 < \alpha < 1$)

$$\xi \equiv \frac{g_1}{\sqrt{\frac{C}{2L} + \frac{C}{L_c}}} \tag{7b}$$

Then, (6) becomes

$$\frac{d^2 x_{ij}}{d\tau^2} -\alpha x_{(i-1)j} + (1+\alpha) x_{ij} - \alpha x_{(i+1)j} - \alpha x_{i(j-1)} + (1+\alpha) x_{i\,j} - \alpha x_{i(j+1)}$$

$$= \xi \left(\frac{dx_{ij}}{d\tau} - \frac{1}{3}\frac{d(x_{ij}^3)}{d\tau} \right) \tag{8}$$

where $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$

The above equation (8) can be expressed in matrix form by defining two matrices X and X_c

$$X = \left(x_{ij} \right)$$

$$X_c = \left(x_{ij}^3 \right)$$

Both of them are of row m and column n, i.e. each matrix is of the order mxn.

Now (8) can be expressed in the matrix differential equation

$$X^{\cdot\cdot} + B\,X + X\,D = \xi X' - \frac{1}{3}\xi X'_c \qquad (9)$$

where B is determined to be square and symmetric and by using the boundary conditions of ring connection

$$x_{oj} = x_{mj} \;;\; x_{(m+1)j} = x_{ij} \qquad (10)$$

it can be written as

$$B = \begin{pmatrix} 1+\alpha & -\alpha & & & -\alpha \\ -\alpha & 1+\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 1+\alpha & -\alpha \\ -\alpha & & & -\alpha & 1+\alpha \end{pmatrix} \qquad (11)$$

and D is determined to be square symmetric and of the order equal to the number of rings in the structure (n)

$$D = \begin{pmatrix} 1 & -\alpha & & & \\ -\alpha & 1+\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 1+\alpha & -\alpha \\ & & & -\alpha & 1 \end{pmatrix} \qquad (12)$$

Matrix D is tridiagonal when assuming that the two ends of the tube are open, which means the boundary conditions are

$$x_{i1} = x_{io} \; ; \; x_{in} = x_{i(n+1)} \tag{13}$$

The matrix differential equation (9) is defined as the fundamental mode equation of the structure.

Applying an orthogonal transformation

$$X = P\, YQ^T \tag{14}$$

where P and Q are two orthogonal square matrices of the order m and n respectively gives

$$P^T P = I_m \quad (P\, P^T = I_m)$$

and

$$Q^T Q = I_n \quad (Q\, Q^T = I_n)$$

where I_m is the unit matrix of order m.

Substituting the orthogonal transformation (14) into the fundamental equation (9)

$$P\, Y^{\cdot\cdot}\, Q^T + B\, PY\, Q^T + P\, YQ^T\, D = \xi PY^{\cdot} Q^T - \frac{1}{3}\, \xi\, X_c^{\cdot} \tag{15}$$

and then multiplying by P_T from the left hand side and by Q from the right hand side, we have

$$Y^{\cdot\cdot} + (P^T\, B\, P)Y + Y\, (Q^T\, D\, Q) = \xi Y^{\cdot} - \frac{1}{3}\, \xi P^T\, X_c^{\cdot}\, Q \tag{16}$$

which is the fundamental equation.

To solve it we have firstly to solve the unperturbed differential equation and then linearise the nonlinear term by using the Kryloff and Bogoliuboff linearization technique.

3.1 Solution of the Unperturbed Matrix Differential Equation

The unperturbed equation of (16) (which is the fundamental equation with $\xi = 0$) is

$$Y^{\cdot\cdot} + (P^T B P)Y + Y(Q^T D Q) = 0 \qquad (17)$$

The elements of the matrix P, to diagonalise the matrix B, can be determined as an eigenvalue problem of matrices by solving the corresponding difference equation (Gantmacher, 1960) which is taken from (11) as

$$- \alpha p_{(i-1)j} + (1+\alpha)p_{ij} - \alpha p_{(i+1)j} = \lambda_j p_{ij} \qquad (18a)$$

where p_{ij} is the element of the matrix P and λ_j is the eigenvalue of B with the boundary conditions

$$p_{oj} = p_{mj} \; ; \; p_{1j} = p_{(m+1)j} \qquad (18b)$$

From the above two equations and the orthogonal condition, the elements of the matrix P and the eigenvalues of B can be determined uniquely (Endo and Mori, 1978) by

$$p_{i1} = \sqrt{\frac{1}{m}} \qquad \text{for } i = 1,2,\ldots m$$

$$p_{i\frac{(m+1)}{2}} = (-1)^i \sqrt{\frac{1}{m}} \qquad \text{for } i = 1,2,\ \ldots,\ m$$

only for m is even

$$p_{ij} = \sqrt{\frac{2}{m}} \cos \frac{2\pi i(j-1)}{m} \text{ for } j = 2,3,\ \ldots\ (\frac{m+1}{2})$$

of m is odd but for m even

$$j = 2,3,\ \ldots,\ \frac{m}{2} - 1,\ \frac{m}{2} + 1$$

$$p_{ij} = \sqrt{\frac{2}{m}} \sin \frac{2\pi(i(j-1)}{m} \quad \text{for } j = (\frac{m+1}{2}),\ldots,\ m-1,\ m$$

if m is odd but for m even

$$j = (\frac{m}{2}+2),\ \ldots,\ m-1,\ m \qquad (19)$$

$$\lambda_j = 1 + \alpha - 2\,\alpha \cos \frac{2\pi(j-1)}{m}, \quad j = 1,2,\ \ldots,\ m \qquad (20)$$

Thus, the matrix combination $(P^T BP)$ in (17) can be replaced by the diagonal matrix

$$P^T BP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} \qquad (21)$$

Similarly, in order to diagonalise the matrix D (12) the elements of the orthogonal matrix Q must be determined first. This can be done by solving the difference equation

$$-\alpha q_{i(j-1)} + (1+\alpha)q_{ij} - \alpha q_{i(j+1)} = \gamma_i q_{ij} \qquad (22a)$$

where q_{ij} is the element of the matrix Q

γ_i is the eignevalue of the matrix D.

Equation (22a) has the boundary conditions

$$q_{io} = q_{i1} \quad ; \quad q_{in} = q_{i(n+1)} \qquad (22b)$$

From the above two equations and the orthogonal condition, the elements of the matrix Q can be determined uniquely as

$$q_{i1} = \frac{1}{\sqrt{n}}$$

$$q_{ij} = \sqrt{\frac{2}{n}} \cos \frac{(2i-1)(j-1)\pi}{2n} \text{ , } j = 2,3, \ldots, n \qquad (23)$$

$$\gamma_i = 1 + \alpha - 2\alpha \cos \frac{(i-1)\pi}{n} \qquad (24)$$

Thus, the matrix combination $(Q^T DQ)$ in (17) can also be replaced by the diagonal matrix

$$Q^T DQ = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \end{pmatrix} \quad (25)$$

Substituting (21) and (25) into (17), the general equation for the element y_{ij} is

$$\ddot{y}_{ij} + (\lambda_i + \gamma_j) y_{ij} = 0 \quad (26)$$

from which y_{ij} can be solved as

$$y_{ij} = A_{ij} \sin (\omega_{ij} \quad + \phi_{ij}) \quad (27a)$$

$$\omega_{ij} = \sqrt{\lambda_i + \gamma_j} \quad (27b)$$

where λ_i and λ_j can be determined from (20) and (24) respectively, so

$$\omega_{ij} = \sqrt{2 + 2\alpha - 2\alpha \cos \frac{2(i-1)\pi}{m} - 2\alpha \cos \frac{(j-1)\pi}{n}}$$

$$= \sqrt{2 \{1 + \alpha \left[1 - \cos \frac{2(i-1)\pi}{m} - \cos \frac{(j-1)\pi}{n}\right]\}} \quad (28)$$

where $i = 1,2, \ldots, m$; $j = 1,2, \ldots, n$

Thus, the different modes y_{ij} of the unperturbed equation are given by (27a) with amplitudes A_{ij} and angular frequencies ω_{ij} given by (28).

Here, it is clear that the angular velocity ω_{ij} of the mode y_{ij} is equal to the square root of the summation of the eigenvalue λ_i of the matrix B and the eigenvalue γ_j of the matrix D. As an example of the calculation of the mode frequencies as functions of the inductive coupling α where the number of oscillators in each ring of the structure (m) is 3 and the number of rings in the tube structure (n) is 4. For the tube structure with m=4, n=3, the mode frequencies are given by Table 5.2.

3.2 Equivalent Linearization of Nonlinear Terms

To linearise the nonlinear term in the fundamental equation (16), the whole equation should firstly be in the Y-space. So the elements of the nonlinear term $P^T X_c Q$ must be written in the form of the nonlinear combination of y_{ij} , by linearising the cubic nonlinear term x_{ab}^3. Now, defining

$$H \equiv P^T X_c Q = [h_{ij}] \tag{29}$$

where H is a matrix of the order mxn and its elements h_{ij} given by

$$h_{ij} = \sum_{a=1}^{m} \sum_{b=1}^{n} p_{ai} q_{bj} x_{ab}^3 \tag{30}$$

From the transformation (14)

$$x_{ab} = \sum_{k=1}^{m} \sum_{l=1}^{n} p_{ak} q_{bl} y_{kl} \tag{31}$$

x_{ab}^3 becomes

$$x_{ab}^3 = \sum_{k=1}^{m} \sum_{l=1}^{n} p_{ak}^3 q_{bl}^3 y_{kl}^3$$

$$+3 \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} p_{ak}^2 q_{bl}^2 p_{ar} q_{bs} y_{kl}^2 y_{rs}$$

$$+ (y_{kl} y_{rs} y_{tu} \text{ terms}) \tag{32}$$

so that $(k,l) \neq (r,s) \neq (t,u)$.

From (28) no two of all the mode frequencies ω_{ij} have a rational ratio (i.e. all modes are assumed nonresonant), and ignoring resonant interaction between modes (Scott, 1970) the $y_{kl}\, y_{rs}\, y_{tu}$ terms can be ignored, butthe $y_{kl}^2\, y_{rs}$ terms and the y_{kl}^3 terms cannot be ignored.

In quasiharmonic analysis, the higher harmonics can be ignored, and therefore y_{kl}^3 can be written as

$$y_{kl}^3 = A_{kl}^3 \sin^3(\omega_{kl} + \phi_{kl}) \tag{33a}$$

$$= \tfrac{3}{4} A_{kl}^2 \; y_k$$

Similarly,

$$y_{kl}^2 \, y_{rs} = A_{kl}^2 \, A_{rs} \sin^2 \; (\omega_{kl} + \phi_{kl}).\sin\;(\omega_{rs} + \phi_{rs})$$

$$= \tfrac{1}{2} A_{kl}^2 \, y_{rs} \tag{33b}$$

Substituting (33) into (32) x_{ab}^3 becomes

$$x_{ab}^3 = \tfrac{3}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} p_{ak}^3 \, q_{bl}^3 \, A_{kl}^2 \, y_{kl}$$

$$+ \frac{3}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} p_{ak}^2 \, q_{bl}^2 \, p_{ar} \, q_{bs} \, A_{kl}^2 \, y_{rs}$$

where $(k,l) \neq (r,s)$ (34)

Substituting (34) into (30) produces h_{ij} in the form of a linear combination of y_{kl}, i.e.

$$h_{ij} = \sum_{k=1}^{m} \sum_{l=1}^{n} \eta_{ij} \; (k,l) \; y_{kl} \tag{35}$$

where η_{ij} (k,l) is written as

$$\eta_{ij}(i,j) = \frac{3}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \Psi_{mn}(i,j,k,l) A_{kl}^2 - \frac{3}{4} \Psi_{mn}(i,j,i,j) A_{ij}^2 \tag{36}$$

where

$$\Psi_{mn}(i,j,k,l) = \sum_{a=1}^{m} \sum_{b=1}^{n} p_{ai}^2 \, q_{bj}^2 \, p_{ak}^2 \, q_{bl}^2$$

$$i,k = 1,2, \ldots, \; m; \; j,l = 1.2. \;,,,.\; n \tag{37}$$

Hence, the parameter $\Psi_{mn}(i,j,k,l)$ can be determined from the elements of the two orthogonal matrices P and Q which are given by (19) and (20) respectively.

Thus, the elements (h_{ij}) of the nonlinear matrix $P^T X_c Q$ have been linearised, i.e. the equivalent linearised equation of the fundamental equation (16) has been obtained as

$$\ddot{y}_{ij} + (\lambda_i + \gamma_j) y_{ij} = \xi \dot{y}_{ij} - \frac{1}{3} \xi \sum_{k=1}^{m} \sum_{l=1}^{n} \eta_{ij}(k,l) \dot{y}_{kl} \tag{38}$$

Supposing that the left hand side of the above equation has a resonance centred around $\sqrt{\lambda_i + \gamma_j}$, the modes of the frequencies which are not equal to ω_{ij} have little effect upon the solution provided that each mode frequency is separated enough, or that the Q-value of the resonance is fairly high. So, we can ignore all the y_{kl} terms of the right hand side except y_{ij}.

Therefore, $\eta_{ij}(k,l)$ can be replaced by $\eta_{ij}(i,j)$ in the linearised equation (38) to give the equation in simple form

$$\ddot{y}_{ij} + \omega_{ij}^2 y_{ij} = \xi \dot{y}_{ij} - \frac{1}{3} \xi \eta_{ij}(i,j) \dot{y}_{ij} \tag{39}$$

where $\eta_{ij}(i,j)$ is given by (36).

Now, equation (39) is the equivalent linearised equation of the mode y_{ij}.

4. Evaluation of the Stationary Amplitude Values by Averaging the Equivalent Linearised Equation

In order to determine which of the modes are stable, it is necessary to evaluate first the stationary values of the amplitudes A_{ij}. Using the quasiharmonic approximation, the amplitudes and phases are assumed to be slowly varying functions of time. Therefore, substituting the unperturbed solution y_{ij} and its first and second derivatives into the equivalent linearised equation (39), the following averaged equations are obtained

$$2\dot{A}_{ij} = \xi\left[1 - \frac{1}{3}\ \eta_{ij}(i,j)\right] A_{ij} \tag{40}$$

Multiplying both sides by A_{ij}, and assuming $A_{ij}^2 = U_{ij}$, then (40) becomes

$$\dot{U}_{ij} = \frac{1}{3}\,\xi\, U_{ij}\left[3 - \eta_{ij}(i,j)\right] \tag{41}$$

Substituting (36) and (41) gives

$$\dot{U}_{ij} = \frac{1}{3}\,\xi\, U_{ij}\Big[3 - \frac{3}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Psi_{mn}(i,j,k,l)U_{kl}$$

$$+ \tfrac{3}{4}\,\Psi_{mn}(i,j,i,j)U_{ij}\Big] \tag{42}$$

where $i = 1,2,\ \ldots,\ m$; $j = 1,2,\ \ldots,\ n$.

The statinary values of amplitudes can be taken by putting all the first order time derivatives in the averaged equations (42) to zero, i.e.

$$\dot{U}_{ij} = 0 \qquad\qquad \text{for all } i,j \tag{43}$$

Consequently, equation (40) leads to

$$\eta_{ij}(i,j) = 3 \tag{44}$$

which is the condition for stationary amplitudes.

But from (36), η_{ij} is a function of the parameter Ψ_{mn} and amplitudes of modes. So, the values of Ψ_{mn} should be clarified before the calculation of the stationary amplitudes.

From (37) Ψ_{mn} can be written in the form

$$\Psi_{mn}(i,j,k,l) = \sum_{a=1}^{m} p_{ai}^2\ p_{ak}^2 \cdot \sum_{b=1}^{n} q_{bj}^2\ q_{bl}^2 \tag{45}$$

$$= \Psi_m(i,k)\ .\ \Psi_n(j,l)$$

where

$$\Psi_m(i,k) = \sum_{a=1}^{m} p_{ai}^2\, p_{ak}^2 \quad ; \; i,k = 1,2, \ldots, m \tag{46a}$$

$$\Psi_n(j,l) = \sum_{b=1}^{n} q_{nj}^2\, q_{bl}^2 \quad ; j,l = 1,2, \ldots, n \tag{46b}$$

Using Appendix I for the different values of $\Psi_m(i,k)$ and $\Psi_m(i,l)$, it is possible to evaluate the parameter $\Psi_{mn}(i,j,k,l)$ in (45) before going on to find the stationary values of the mode amplitudes.

To make the calculation of $\Psi_{mn}(i,j,k,l)$ easier, we shall consider the two cases when the number of oscillators in each ring of the structure (m) is even or odd.

If the number of oscillators in each ring of the tube structure (m) is even, then the values of $\Psi_m(i,k)$ given by (I.6) and the values of $\Psi_n(j,l)$ given by (I.7) are substituted into (45), to obtain the general values of Ψ_{mn} as in Table 3 which are a function of the number of oscillators in each ring (m) and the number of rings in the structure (n). It is clear from Table 3 that the value $\Psi_{mn}(i,j,k,l)$ could then be easily calculated when we determine first which subspace the point (i,k) belongs to in the mxm elements of the space (i,k) as well as the point (j,l) in the nxn elements of space (j,l). As an example for the calculation of the parameter Ψ_{mn} for the case of the tube structure with m as an even number. Table 4 represents Ψ_{mn} when m=4, n=3.

If the number of oscillators in each ring of the tube system is odd the values of $\Psi_m(i,k)$ and $\Psi_n(j,l)$ are substituted into (45) to obtain the general values of Ψ_{mn} as in Table 5. Table 6 gives the values of Ψ_{mn} for the case of a tube structure when the number of oscillators in each ring m=3 and number of rings n=4.

Thus, using Tables 3 and 5 we are able, for any arbitrary m and n, to determine all the required values of Ψ_{mn} to be substituted into (42) using the condition (43) to obtain the amplitude stationary values.

5. Investigation of the Stability Problem

In order to determine which modes of the tubular structure are stable, it is necessary to determine first the stationary states of these oscillatory modes by reducing the first-order time derivatives in the averaged equations (42) to zero, as mentioned previously.

The stability of stationary states is then determined by linearising the average equations around the stationary values and investigating the characteristic equation of the linearised equation which is called the variational equation.

Introducing small disturbance ΔU_{kl} around the stationary state so that

$$U_{kl} = U_{kolo} + \Delta U_{kl} \tag{47}$$

leads to the variational equation

$$(\Delta \dot{U}_{ij}) = \sum_{k=1}^{m} \sum_{l=1}^{n} J_{ij}(k,l) \cdot \Delta U_{kl} \tag{48}$$

where $J_{ij}(k,l)$ is the Jacobian matrix of the structure, which is a square matrix of order equal to the multiplication of the number of oscillators in each ring by the number of rings in the whole structure, i.e. of order mn. The Jacobian matrix which is defined as

$$J_{ij}(k,l) = \frac{d}{dU_{kl}} \left(\frac{dU_{ij}}{d\tau}\right) = \frac{d(\dot{U}_{ij})}{dU_{kl}} \tag{49}$$

can be calculated from (42) for (i,j) = (k.l) as

$$J_{ij}(k,l) = \xi\left[1 - \tfrac{1}{2} \sum_{r=1}^{m} \sum_{s=1}^{n} \Psi_{mn}(i,j,r,s) U_{rs}\right]. \tag{50a}$$

But for $(i,j) \neq (k,l)$

$$J_{ij}(k,l) = - \tfrac{1}{2}\, \xi\, \Psi_{mn}(i,j,k,l) U_{ij} \tag{50b}$$

Consequently, one can distinguish the stability of a mode from the eigenvalues of the corresponding Jacobian matrix which are the roots of the characteristic equation

$$|\{J_{ij}(k,l)\} - sI| = 0 \tag{51}$$

where I is a unit matrix of order mnxmn.

If the real parts of the roots in (51) are all negative, the corresponding mode is regarded as stable. If at least one of the roots has positive real parts, the mode is unstable.

5.1 Stability of Single Modes

The mode stability for any mode (i_o, j_o) of the structure is now considered. Supposing that the mode (i_o, j_o) is the only mode which is excited, i.e.

$$U_{iojo} \neq 0 \quad ; \quad U_{ij} = 0$$

$$\text{for } i = 1,2, \ldots, m \quad ; \quad j = 1,2, \ldots, n \; (i,j) \neq (i_o, j_o) \tag{52}$$

From the transformation (14) and the condition (52) the mode x_{kl} can be written as

$$x_{kl} = p_{kio} q_{ljo} A_{iojo} \sin(\omega_{iojo}\tau + \phi_{iojo}) \tag{53}$$

where ω_{iojo} is calculated from (28) and the stationary amplitude A_{iojo} calculated from (44), (52) and (36) to be

$$A_{iojo} = \sqrt{U_{iojo}} = \frac{2}{\sqrt{\Psi_{mn}(i_o, j_o, i_o, j_o)}} \tag{54}$$

To have the condition of a stable mode, the single mode condition of (52) is substituted into (50), then $J_{ij}(k,l)$ takes the three different values

$$J_{ij}(k,l) = \xi\left[1 - \tfrac{1}{2}\Psi_{mn}(i,j,i_o,j_o)U_{iojo}\right], \text{ for } (i,j) = (k,l) \tag{55a}$$

$$= -\tfrac{1}{2}\xi\Psi_{mn}(i_o,j_o,k,l)\,U_{iojo}, \text{ for } (i,j) \neq (k,l); \; (i,j) = (i_o,j_o) \tag{55b}$$

or, = 0 $\qquad$ for $(i,j) \neq (k,l)$, (i_o,j_o) (55c)

where

$$i,k = 1,2, \ldots, m \quad ; \quad j,l = 1,2, \ldots, n$$

Substituting the value of stationary amplitude $\sqrt{U_{iojo}}$ (54) into the Jacobian matrix (55) gives the characteristic equation (51) in the following form

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (J_{ij} - s) = 0 \tag{56}$$

where

$$J_{ij} = \xi \left(1 - \frac{2\Psi_{mn}(i,j,i_o,j_o)}{\Psi_{mn}(i_o,j_o,i_o,j_o)}\right) \tag{57}$$

Therefore, the stability condition for the mode (i_o,j_o) is

$$\frac{\Psi_{mn}(i,j,i_o,j_o)}{\Psi_{mn}(i_o,j_o,i_o,j_o)} > \tfrac{1}{2} \tag{58}$$

for all $i = 1,2, \ldots,m$; $j = 1,2, \ldots,n$

Now, it is clear that the arbitrary mode (i_o,j_o) is stable only in the case of the values of Ψ_{mn} given by (45) and calculated using Table 3 or 5 which satisfy the condition of stability (58).

5.2 Stability of nonresonant double modes

To investigate the stability of nonresonant double modes, similar procedures to what have been used in investigating the stability of single modes are followed in Appendix II.

From the results in ~~this~~ Appendix 2, it is evident that for any nonresonant double mode (i_o,j_o) and (r_o,s_o), the nonzero elements of the characteristic equation (51) are restricted to the diagonal elements (i.e. for $(i,j)=(k,l)$) and to those for the two rows (i_o,j_o) and (r_o,s_o). Thus, the corresponding characteristic equation to the nonresonant double mode can be written as

$$\begin{bmatrix} J_{iojo}(i_o,j_o)-s & J_{iojo}(r_o,s_o) \\ J_{roso}(i_o,j_o) & J_{roso}(r_o,s_o)-s \end{bmatrix} \cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \left[\{J_{ij}(i,j)\}-s\right] = 0$$

where $i = 1,2, \ldots, m;\ j = 1,2, \ldots, n;\ (i,j) \neq (i_o,j_o),(r_o,s_o)$ (59)

Substituting the conditions (II.5) into (II.6) then

$$\left[J_{iojo}(i_o,j_o)-s+J_{iojo}(r_o,s_o)\right] \left[J_{iojo}(i_o,j_o)-s-J_{iojo}(r_o,s_o)\right] \prod_{i=1}^{m} \prod_{j=1}^{n} \left[\{J_{ij}(i,j)\} -s\right] = 0$$

$$(i,j) \neq (i_o,j_o),(r_o,s_o) \qquad (60)$$

Using the value of the stationary amplitude $\sqrt{U_{iojo}}$ given by (II.4) the roots of the characteristic equation can be calculated as

a. $\left[J_{iojo}(i_o,j_o)+J_{iojo}(r_o,s_o)\right] = -\ \xi$ (61a)

b. $\left[J_{iojo}(i_o,j_o)-J_{iojo}(r_o,s_o)\right]$

$$= \xi\left[1 - \frac{2\Psi_{mn}(i_o,j_o,i_o,j_o)}{\Psi_{mn}(i_o,j_o,i_o,j_o)+2\Psi_{mn}(i_o,j_o,r_o,s_o)}\right] \qquad (61b)$$

c. $J_{ij}(i,j) = \xi \left| 1-\tfrac{1}{2} \; \Psi_{mn}(i,j,i_o,j_o)+\Psi_{mn}(i,j,r_o,s_o)\} \; U_{iojo} \right|$

$$= \xi \left(1 - \frac{2\{\Psi_{mn}(i,j,i_o,j_o) + \Psi_{mn}(i,j,r_o,s_o)\}}{\Psi_{mn}(i_o,j_o,i_o,j_o)+2\Psi_{mn}(i_o,j_o,r_o,s_o)} \right)$$

for all $i = 1,2, \ldots, m$; $j = 1,2, \ldots, n$

except $(i,j) = (i_o,j_o), (r_o,s_o)$ (61c)

From the above equations (61) it is evident that the first root of the characteristic equation is always negative, the second root is negative only for

$$\frac{\Psi_{mn}(i_o,j_o,i_o,j_o)}{\Psi_{mn}(i_o,j_o,i_o,j_o) + 2\Psi_{mn}(i_o,j_o,r_o,s_o)} > \tfrac{1}{2} \tag{62a}$$

The rest of the roots which are (mn-2) roots, are only negative for

$$\frac{\Psi_{mn}(i,j,i_o,j_o) + \Psi_{mn}(i,j,r_o,s_o)}{\Psi_{mn}(i_o,j_o,i_o,j_o) + 2\Psi_{mn}(i_o,j_o,r_o,s_o)} > \tfrac{1}{2} \tag{62b}$$

for $i = 1,2, \ldots, m$: $j = 1,2, \ldots, n$

$(i,j) \neq (i_o,j_o), (r_o,s_o)$

and if that condition is applied for the (mn-2) combination of (i,j), we can conclude that all the (mn-2) roots are negative.

Thus, the stability of a nonresonant double mode (i_o,j_o) and mode (r_o,s_o) can be investigated through the values of Ψ_{mn} in the columns (i_o,j_o) and (r_o,s_o) of the Table 7.

Now, the solution of the nonresonant double mode can be given by (II.2) in which the elements of the matrices P and Q are given by (19) and (23) respectively, the angular frequencies ω_{iojo} and ω_{roso} given by (28), and the amplitudes U_{iojo} and U_{roso} given by II.4

From the mode stability criteria (58) and (62) it is advantageous to put the value $\Psi_{mn}(i,j,k,l)$ in the form which can be seen in Table 7.

Thus, we can investigate the stability of a mode by the values of Ψ_{mn} in the column corresponding to that mode only. Also, for nonresonant double modes we investigate their stability only by the values of Ψ_{mn} in the corresponding columns.

5.3 Stability of Degenerate Models

For investifating the stability of degenerate modes in this tube structure, the phase of the mode should be taken into consideration. Supposing two modes (i_1, j_1) and (i_2, j_2) are degenerate and there are no other degenerate modes except these, then in equation (32) for x_{ab}^3, the terms concerning the products between each of the degenerate mode, i.e. $y_{i1j1}^2\ y_{i2j2}$ and y_{i1j1}^2 from the second, and $y_{kl}\ y_{i1j1}\ y_{i2j2}$ from the third, should be calculated apart from the others as clarified in Appendix III.

Then, substituting the solution of the unperturbed equation (27) into the linearised equation which is obtained in Appendix III equations for the degenerate modes are

$$U^{\cdot}_{i1j1} = \xi U_{i1j1}\Big[1-\tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Psi_{mn}(i_1,j_1,k,l)U_{kl}+\tfrac{1}{4}\Psi_{mn}(i_1,j_1,i_2,j_2)U_{i1j1}$$

$$+\tfrac{1}{4}\Psi_{mn}(i_1,j_1,i_2,j_2)U_{i2j2}-\tfrac{1}{2}\Psi_{mn}(i_1,j_1,i_2,j_2)U_{i2j2}\cos^2\phi\Big] \tag{63a}$$

$$U^{\cdot}_{i2j2} = \xi U_{i2j2}\Big[1-\tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Psi_{mn}(i_2,j_2,k,l)U_{kl}+\tfrac{1}{4}\Psi_{mn}(i_2,j_2,i_2,j_2)U_{i2j2}$$

$$+\tfrac{1}{4}\Psi_{mn}(i_2,j_2,i_1,j_1)Ui_1j_1-\tfrac{1}{2}\Psi_{mn}(i_2,j_2,i_1,j_1)U_{i1j1}\cos^2\phi\Big] \tag{63b}$$

$$\phi^{\cdot} = \tfrac{1}{4}\,\xi U_{i2j2}\,\Psi_{mn}(i_1,j_1,i_2,j_2)\cos\phi\sin\phi \tag{64}$$

Investigating the stability of degenerate modes, by employing the same procedures for the nondegenerate modes,

$$U_{i1j1} = U_{i2j2} \neq 0 \quad , U_{ij} = 0 \; ; \text{ for all } i,j \tag{65}$$

except $(i,j) = (i_1,j_1), (i_2,j_2)$. The stationary values for the amplitudes and the phase are then determined by putting all the first-order time derivatives of averaged equations to zero, so

$$\Psi_{mn}(i_1,j_1,i_1,j_1)U_{i1j1} + \Psi_{mn}(i_1,j_1,i_2,j_2)\;(1+2\cos^2\phi)U_{i2j2} = 4$$

$$\Psi_{mn}(i_2,j_2,i_1,j_1)(1+2\cos^2\phi)U_{i1j1} + \Psi_{mn}(i_2,j_2,i_2,j_2)U_{i2j2} = 4$$

$$\Psi_{mn}(i_1,j_1,i_2,j_2)\sin\phi\cos\phi\;U_{i2j2} = 0 \tag{66}$$

From (37) it is clear that $\Psi_{mn}(i_1,j_1,i_2,j_2) = \Psi_{mn}(i_2,j_2,i_1,i_2)$. So, the relation $\Psi_{mn}(i_1,j_1,i_1,j_1) = \Psi_{mn}(i_2,j_2,i_2,j_2)$ should be satisfied to fulfill the requirement of the equal mode amplitude assumption. Now, two stationary states for the amplitudes of the degenerate mode can be obtained, one corresponds to a progressing wave, which is

$$U_{i1j1} = U_{i2j2} = \frac{4}{\Psi_{mn}(i_1,j_1,i_1,j_1)+\Psi_{mn}(i_1,j_1,i_2,j_2)}$$

$$\Phi = \pm\frac{\pi}{2}$$

$$x_{kl} = p_{ki2}\, q_{lj2}\, A_{i2j2} \sin \omega\tau \pm p_{ki1} q_{lj1} A_{i1j1} \cos \omega\tau \qquad (67)$$

The other solution corresponds to a standing wave and given by

$$U_{i1j1} = U_{i2j2} = \frac{4}{\Psi_{mn}(i_1,j_1,i_1,j_1)+3\Psi_{mn}(i_1,j_1,i_2,j_2)}$$

$$\phi = 0,\ \pi$$

$$x_{kl} = p_{ki2} q_{lj2} A_{i2j2} \sin \omega\tau \pm p_{ki1} q_{lj1} A_{i1j1} \sin \omega\tau \qquad (68)$$

where ω is the frequency of the degenerate mode, i.e. $= \omega_{i1j1} = \omega_{i2j2}$.

To investigate the stability of these two solutions, the characteristic equation of the corresponding Jacobian should be examined. This Jacobian matrix can be defined as

$$JB \equiv \frac{\partial(\dot U_{i1}, \dot U_{12}, \ldots, \dot U_{mn}; \dot U_{i1j1}, \dot U_{i2j2}, \dot\phi)}{\partial(U_{i1}, U_{12}, \ldots, U_{mn}, U_{i1j1}, U_{i2j2}, \phi)} \qquad (69)$$

Using (65), (67) and (68) the averaged equations give the Jacobian matrix as in

$$
\left(\begin{array}{ccccc|ccc}
J_{11} & & & & & & & \\
 & J_{12} & & 0 & & & & \\
 & 0 & & & & & 0 & \\
 & & J_{ij} & & & & & \\
 & & & J_{mn} & & & & \\
\hline
 & & & & & \frac{\partial \dot{U}_{i1j1}}{\partial U_{i1j1}} & \frac{\partial \dot{U}_{i1j1}}{\partial U_{i2j2}} & \frac{\partial \dot{U}_{i1j1}}{\partial \phi} \\
 & & & & & \frac{\partial \dot{U}_{i2j2}}{\partial U_{i1j1}} & \frac{\partial \dot{U}_{i2j2}}{\partial U_{i2j2}} & \frac{\partial \dot{U}_{i2j2}}{\partial \phi} \\
 & & & & & \frac{\partial \dot{\phi}}{\partial U_{i1j1}} & \frac{\partial \dot{\phi}}{\partial U_{i2j2}} & \frac{\partial \dot{\phi}}{\partial \phi}
\end{array}\right)
$$

where

$$J_{ij} = \frac{\partial \dot{U}_{ij}}{\partial U_{ij}} \;, \quad (i,j) \neq (i_1,j_1),\ (i_2,j_2) \tag{70}$$

So, the Jacobian matrix JB is a square matrix of order (mn+1). Therefore, the characteristic equation can be written by expanding the determinant $|JB - sI|$ with regard to all possible rows or columns, as

$$
\begin{pmatrix}
\frac{\partial \dot{U}_{i1j1}}{\partial U_{i1j1}} - s & \frac{\partial \dot{U}_{i1j1}}{\partial U_{i2j2}} & \frac{\partial \dot{U}_{i1j1}}{\partial \phi} \\
\frac{\partial \dot{U}_{i2j2}}{\partial U_{i1j1}} & \frac{\partial \dot{U}_{i2j2}}{\partial U_{i2j2}} - s & \frac{\partial \dot{U}_{i2j2}}{\partial \phi} \\
\frac{\partial \dot{\phi}}{\partial U_{i1j1}} & \frac{\partial \dot{\phi}}{\partial U_{i2j2}} & \frac{\partial \dot{\phi}}{\partial \phi} - s
\end{pmatrix}
\sum_{i=1}^{m} \sum_{j=1}^{n} (U_{ij} - s) = 0
$$

where $(i,j) \neq (i_1,j_1),\ (i_2,j_2)$

Following the same procedures as for the nonresonant double modes, the elements of the Jacobian matrix can be determined and the stability of the progressing wave can be examined by two inequalities; one is derived from the determinant in the characteristic equation (71) and the other from the J_{ij} part of the same equation. These two inequalities

$$\text{a.}\quad \Psi_{mn}(i_1,j_1,i_1,j_1) > \Psi_{mn}(i_1,j_1,i_2,j_2) \tag{72}$$

where

$$\Psi_{mn}(i_1,j_1,i_1,j_1) = \Psi_{mn}(i_2,j_2,i_2,j_2)$$

$$\text{b.}\quad \frac{\Psi_{mn}(i,j,i_1,j_1) + \Psi_{mn}(i,j,i_2,j_2)}{\Psi_{mn}(i_1,j_1,i_1,j_1) + \Psi_{mn}(i_1,j_1,i_2 \cdot j_2)} > \tfrac{1}{2}$$

$$\text{for all } i \text{ and } j \text{ except } (i,j) = (i_1,j_1),(i_2,j_2) \tag{73}$$

If the above two inequalities have been satisfied, then the corresponding degenerate mode (i_1,j_1) and (i_2,j_2) is regarded as a stable progressing wave, and its structural solution is given by (67) where the phase difference between the two components of the mode ϕ is equal to $\pm\pi/2$.

On the other hand, the element $\frac{\partial\dot{\phi}}{\partial\phi}$ for the standing wave is positive and hence, it can be easily concluded that this solution is always unstable.

The stationary amplitude of the stable degenerate mode can then be given by equation (67) as

$$A_{i1j1} = A_{i2j2} = \sqrt{U_{i1j1}} = \sqrt{\frac{2}{\Psi_{mn}(i_1,j_1,i_1,j_1)+\Psi_{mn}(i_1,j_1,i_2,j_2)}} \tag{74}$$

Also, from the two conditions of stabi ity (72) and (73) it is now clear that any degenerate mode in the tube structure can be examined to be stable or unstable only by the columns (i_1,j_1) and (i_2,j_2) in Table 7 which gives the values of the parameter Ψ_{mn}. As an example for degenerate modes, the modes (2,1) and (3,1); (2,2) and 3,2); (2,3) and (3,3); and (2,4) and (3,4) are four degenerate modes in the tube structure with m=3 and n=4, as already seen in Table 1. The stability of such degenerate modes can be investigated using the two conditions of stability (72) and (73), as will be shown in the following section.

6. Solved Examples with Different Numbers of Oscillators in Each Ring and Different Numbers of Rings in the Tubular Structure.

Now, we can analyse any tubular structure consisting of m oscillator in each ring and n rings. As an example, a tube structure

6.1 3 x 4 Tube Structure

This example consists of four rings and each ring contains three oscillators. Each oscillator has the values

$C = 0.1\ \mu F \pm 20\%$

$L = 68$ mH (it is a variable inductance varying from 54 to 82 mH).

The oscillators are coupled to each other by an inductor $L_c = 100$ mH. The active element of each oscillator is given by a cubic nonlinearity described by (1). It is required to analyse its various modes and investigate the stability of these modes.

The angular frequencies ω_{ij} are dependent on the coupling factor α which is calculated using (7a) to be

$\alpha = 0.51923 = \alpha'$ for L = 54 mH

$\quad = 0.57627 = \alpha$ for L = 68 mH

or $\quad = 0.60976 = \alpha'$ for L = 82 mH

The mode frequency f_{ij} can be determined using (56) as

$$L_{ij} = k_f\ \omega_{ij}$$

where

$$k_f = \frac{1}{2\pi}\ \sqrt{\frac{1}{2CL} + \frac{1}{CL_c}}$$

Substituting L = 54 mH, k_f becomes

$$k_f = (2.2087162)\ 10^3$$

and for L = 82 mH, k_f becomes

$$k_f = (2.0192969)\ 10^3$$

Therefore, the mode frequencies f_{ij} which correspond to the case when the inductance of each unit oscillator of the system is adjusted to its minimum value (L = 54 mH) are given by Table 8.

Using the above table, it is clear that the system has the following modes;

I The nondegenerate modes which are the modes (1,1), (1,2), (1,3) and (1,4)

II the degenerate modes which are the modes (2,1) and (3,1); (2,2) and (3,2); (2,3) and (3,3); and (2,4) and (3,4)

Before we are able to determine the structural solution of each mode it is necessary to first distinguish the nondegenerate modes to see if they are single ordinary or nonresonant double modes.

Looking at Table 6 which represents the values of the parameter $\Psi_{34}(i,j,k,l)$, where

$$i,k = 1,2,3 \quad ; \quad j,l = 1,2,3,4$$

and investigating the case for only a single mode by employing the stability conditions (58) it is seen that the two single ordinary modes (1,1) and (1,3) are stable. Their stationary amplitudes can be determined using (54) to be

$$A_{11} = \frac{2}{\sqrt{\Psi_{34}(1,1,1,1)}} = 2\sqrt{12} = 6.9282$$

and $$A_{13} = \frac{2}{\Psi_{34}(1,3,1,3)} = 2\sqrt{12} = 6.9282$$

Investigating now whether or not two modes of nondegenerate oscillations can be simultaneously excited, we employ the stability conditions in (62a). The resulting stable nonresonant double modes are mode (1,2) and mode(1,4).

Now the amplitude of the stable nonresonant double mode can be calculated using (II.4) to be

$$A_{12} = A_{14} = 2. \sqrt{\frac{24}{5}} = 4\sqrt{\frac{6}{5}} = 4.38178$$

Secondly, we are going to investigate the stability of the degenerate modes, of which each mode frequency is equal; these modes are the modes of class II. Employing the stability conditions in (72) and (73), it is found that the degenerate mode (2,1) and (3,1) are stable.

Similarly, it can be proven that the degenerate mode (2,3) and (3,3) are also stable; the other two degenerate modes are unstable since the stability conditions are not satisfied.

From (74) the mode amplitude of (2,1) and (3,1) is given as

$$A_{21} = A_{31} = \frac{2}{\sqrt{C_1 + C_2}} = 2\sqrt{6} = 4.89898$$

For the mode (2,3) and (3,3) its amplitude is exactly the same,

The theoretical spatial variation of the normalised voltage x_{ij} of the stable single mode (1,1) can be written using (53) as

$$[x_{ij}] = p_{i1} q_{j1} A_{11} \sin(\omega_{11}\tau + \phi_{11})$$

$$|x_{ij}| = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} 4\sqrt{3} \sin(0{,}98058\,\tau + \phi_{11})$$

$$= [2] \sin(0.98058\,\tau + \phi_{11})$$

for L = 54 mH, and the same for L = 82 mH but in this case $\omega_{11} = 0.870388$, and the phase ϕ_{11} is arbitrary.

For mode (1,3), x_{ij} similarly can be written as

$$= \begin{bmatrix} 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 \\ 2 & -2 & -2 & 2 \end{bmatrix} \sin (1.4142\ \tau + \phi_{13})$$

for both the two values of the inductance. The phase ϕ_{13} is arbitrary.

Similarly the structural solution of the nonresonant double mode (1,2) and (1,4), is given by (II.2) as

$$[x_{ij}] = \begin{bmatrix} 1.6527 & 0.6846 & -0.6846 & -1.6527 \\ 1.6527 & 0.6846 & -0.6846 & -1.6527 \\ 1.6527 & 0.6846 & -0.6846 & -1.6527 \end{bmatrix} \sin (1.1250\ \tau + \phi_{12})$$

$$+ \begin{bmatrix} 0.6846 & -1.6527 & 1.6527 & -0.6846 \\ 0.6846 & -1.6527 & 1.6527 & -0.6846 \\ 0.6846 & -1.6527 & 1.6527 & -0.6846 \end{bmatrix} \sin (1.6536\ \tau + \phi_{14})$$

for L = 54 mH, and for L = 82 mH it is the same but with $\omega_{12} = 1.0590$, $\omega_{14} = 1.6966$. The two phases ϕ_{12} and ϕ_{14} are both arbitrary.

Now, we find the structural solution of the degenerate modes. The first stable degenerate mode is (2,1) and (3,1). Substituting the mode amplitude $A_{21} = A_{31}$ and the angular frequency $\omega_{21} = \omega_{31}$ for L = 54 mH, into (67), the structural solution can be written as

$$[x_{ij}] = \begin{bmatrix} 2.0(240^{\circ}) & 2.0(240^{\circ}) & 2.0(240^{\circ}) & 2.0(240^{\circ}) \\ 2.0(240^{\circ}) & 2.0(240^{\circ}) & 2.0(240^{\circ}) & 2.0(240^{\circ}) \\ 2.0\ (0^{\circ}) & 2.0\ (0^{\circ}) & 2.0\ (0^{\circ}) & 2.0\ (0^{\circ}) \end{bmatrix} \sin 1.587\tau)$$

Similarly, for the degenerate mode (2,3) and (3,3) it can be written as

$$[x_{ij}] = \begin{bmatrix} 2.0(120^{\circ}) & 2.0(-60^{\circ}) & 2.0(-60^{\circ}) & 2.0(120^{\circ}) \\ 2.0(240^{\circ}) & 2.0(60^{\circ}) & 2.0(60^{\circ}) & 2.0(240^{\circ}) \\ 2.0(0^{\circ}) & 2.0(0^{\circ}) & 2.0(0^{\circ}) & 2.0(0^{\circ}) \end{bmatrix} \sin(1.8861\tau)$$

if the inductance L = 82 mH, the angluar frequency $\omega_{23} = \omega_{33} = 1.9656$.

To summarise the stability analysis of the tube structure under investigation, we can say that the stable modes are

1.a Two single ordinary modes which are mode (1,1) and (1,3)

1.b One nonresonant double mode which is (1,2) and (1,4)

2. Two degenerate double modes which are (2,1) and (3,1); and the second is (2,3) and (3,3).

But the other two degenerate modes (2,2) and (3,2); and (2,4) and (3,4) are both unstable.

6.2 4 x 3 Tube Structure

For a tube structure similar to that of the previous example, but with four oscillators in each ring and three rings, the mode frequencies are given by Table 9. Applying the stability conditions to the system modes, using Table 5.9, we conclude that the following modes are stable:

1.a Two single ordinary modes, which are mode (1,1) mode (3,1).

1.b One nonresonant double mode, which is mode (1,2) and (3,3).

2. One degenerate double mode, which is mode (2,1) and (4,1)

The structural solution of each mode of the above modes can be calculated using the same procedures which have been used for the previous examples.

7. Experimental Investigation of a Tube Oscillator System

In this section we briefly describe some results which have been obtained using an electronic model for a tube oscillator system. The mode has been initially developed by Davies (1977). It consists of sixteen oscillators arranged on four boards as seen in Fig. 2, such that the coupling and the layout can easily be altered. The outputs of any oscillator with its four adjacent oscillators in the structure can be displayed by using the selector and the five sockets on the right which are located at the upper part of the model. Also, the output of each oscillator can be displayed directly from the corresponding socket which lies on the right of the model and is marked by the oscillator's number. The circuit diagram of each oscillator is shown in Fig. 3. It produces the cubic nonlinear characteristic similar to the characteristic given by Fig. 4.

The schematic diagram for any oscillator on the board (e.g. the oscillator in position (i.j)), can be seen in Fig.5. From each oscillator output branches two coupling circuits, one in the horizontal direction, and the other in the vertical direction.

The above twelve oscillators are arranged in tube structure using connecting wires as shown in Fig. 2. The tube system consists of four rings, each comprising three oscillators, i.e. m = 3 and n = 4.

The circuit elements of each oscillator are exactly the same as in the solved examples. Therefore, the coupling coefficient and the frequency scalar $k_f (f_{ij} = k_f \, \omega_{ij})$ have the same values as solved example.

The cubic nonlinear conductance is constructed with an operational amplifier. It includes a certain amount of deviation from the theoretical cubic one. We should notice that there are more than three order terms included in the practical nonlinear characteristic. The average amplitude of all uncoupled oscillators is employed as the theoretical mode amplitude. This is reasonable because the coupled oscillator amplitude is theoretically equal to the uncoupled amplitude of every oscillator. Thus, the scalar of amplitudes is given by $k_v = 4.5$, i.e.

$$V_{ij} = 4.5 \, x_{ij}$$

The inductors of all oscillators in the tube system were adjusted to their minimum values 54 mH, and the output waveforms displayed on an oscilloscope. The results are as shown below:

a. Experimental value for the single mode (1,1) is

$V_{11} = 9.6 \sin (2\pi.2300\ t + \phi_{11})$

where the phase ϕ_{11} is arbitrary. The corresponding theoretical amplitude is 9.0V and the frequency is 2166 Hz as seen in Table 8.

b. Experimental value for the single mode (1,3) is

$V_{13} = 9.9 \sin (2\pi.3250\ t + \phi_{13})$

where the phase ϕ_{13} is arbitrary. The corresponding theoretical amplitude is 9.0V and the frequency is 3124 Hz as indicated in Table 8.

c. The nonresonant double mode (1,2) and (1,4), is described by its amplitude and frequency components. The absolute theoretical components of amplitudes in the standard space are 7.42 and 3.06 and the theoretical frequency components are $f_{12} = 2485$ Hz and $f_{14} = 3652$. Practically, there was difficulty in separating the two modes due to mode coupling. The waveform of the two modes is shown in Fig. 6.

d. Experimental values for the degenerate mode (2,1) and (3,1) are:

$V_{21} = 8.9 \sin 2\pi.3650\ t$; and

$V_{31} = 9.2 \sin 2\pi.3650\ t$

The corresponding theoretical amplitudes are 9.0V and the frequencies are $f = f_{21} = f_{31} = 3506$ Hz

e. The other stable degenerate mode (2,3) and (3,3) is also measured experimentally and found to be

$V_{23} = 8.7 \sin 2\pi.4230\ t$; and

$V_{33} = 8.9 \sin 2\pi.4230\ t$

The corresponding theoretical amplitudes are 9.OV and the frequencies are $f = f_{23} = f_{33} = 4166$ Hz.

All the above waveforms have been displayed on the screen of an electronic oscilloscope with different amplitude and time scales. Using the divisions on the screen and the time scaling, the mode frequencies have been measured.

The output waveforms which are reproduced from oscillators 1,2,3 and 4 in one column of the structure are shown by Fig. 7 and Fig. 8 respectively. Fig. 9 shows two waveforms of oscillators 1 and 5 which are in a boundary ring, while Fig. 10 shows the waveforms of oscillators 3 and 7 which are in an inner ring of the structure.

The tolerances of all inductors are within 5% of the nominal values and the tolerances of all capacitors are within 20% of the nominal values. The experimental results of the mode frequencies which are dependent on such passive elements are satisfactory. But, the results of the mode amplitudes do not agree so well with the theoretical results. This would be due to the deviation of the volt-ampere characteristic of the nonlinearity used in the experiments from the theoretical cubic nonlinearity. We ignored the higher terms under the condition that they were small compared with g_1 and g_3 in (1). If this fact has been taken into consideration, the amplitude errors could have been reduced.

It should be mentioned that there is difficulty in observing some of the multimode oscillations especially the nonresonant modes. This is perhaps due to the fact that the mode frequencies are not separated wide enough to be detected and hence mode coupling occurs. The mode coupling can be reduced considerably by choosing the inductive elemtns of the electronic model with higher Q-value.

8. Conclusions

An analytical mode analysis for a tubular structure of coupled non-linear oscillators has been presented in the paper. This model is of particular relevance to small-intestinal electrical slow-wave activity where entrainment or synchronisation is known to occur. In past studies, a one-dimensional ladder structure has been considered, whereas it is known that the pacemaker activity of smooth muscle cells exists in both transverse and longitudinal directions in gastro-intestinal tissue. For both small and large-intestines a tubular model is clearly preferable.

The analysis presented gives a general solution method for a tube comprising 'n' rings with 'm' oscillators per ring. The various possible mode frequencies are first determined from the unperturbed linear systems using two diagonalising matrices P and Q. The modes are then divided into categories comprising single modes, non-resonant double modes and degenerate modes. Stability criteria for each of these types of mode have been determined and shown to consist of terms which can be calculated using two general tables (Table 3 and 5). The method has been illustrated using 3 x 4 and 4 x 3 oscillator structures. In the 3 x 4 case two single modes are shown to be stable, the lower frequency case giving in-phase conditions throughout the tube. The higher frequency single mode has in-phase conditions around the tube, and some anti-phase relationships along the tube. Similarly, both components in the double non-resonant mode have in-phase relationships around the periphary and some anti-phase conditions along the axis of the tube. For the degenerate modes, one has in-phase conditions along the axis and the other has anti-phase conditions. In each case there is a phase shift of 120° per oscillator around the tube, but the latter mode produces a zero amplitude 'line' along the model.

For the case of 4 oscillators per ring, one single mode gives all in-phase relationships, while the other single mode produces complete anti-phase patterns around the ring. The double nonresonant mode likewise produces a mixture of anti-phase relationships, while there is only one degenerate mode. This mode gives 90^{o} phase shift per oscillator around the tube and in-phase conditions along the axis. Similar results have been found for the 4 x 4 case.

It is evident that, in contrast to the linear case, only a small number of the possible modes turn out to be stable in the non-linear case. This is consistent with the observation that large models tend to exhibit only a small number of stable limit cycle conditions. Although the mode analysis presented here determines mode stability, it should be noted that this does not indicate the relative ease with which different modes can be excited. For example, regions of atraction studies performed by simulation have indicated that some modes can be significantly harder to excite than others (Linkens, 1979). The number of modes observable in a model of this type is also affected by the type of coupling between oscillators. In the analysis presented here inductive coupling has been considered, while capacitive coupling gives dual effects as shown by Endo and Mori (1976a) for a ladder structure. The addition of resistive coupling tends to reduce the number of modes present unless there are delays present in the coupling pathways also (Linkens and Kitney, 1981). A further effect on the number of modes present is the degree of non-linearity in unit oscillators. Regions of atraction studies (Linkens, 1979) and simulation studies have, however, tended to show that these effects are not major. On the other hand, large assymmetries in the basic waveform may contribute a large effect on the number of possible stable modes.

The experimental results in this paper demonstrate that good prediction of mode frequencies, amplitudes and stability can be obtained from the theory using this matrix Kryloff and Bogoliouboff method. Some if the modes shown by calculation to be stable would be very difficult to observe via straight simulation studies without prior knowledge of the mode conditions of particular interest are the double degenerate modes which give a phase shift not equal to 0 or π and which gives the appearance of a progressive wave-pattern. This is relevant in intestinal modelling where progressing phase shifts are normal and have been interpreted as entrainment at non-zero phase shift of a chain of oscillators which an intrinsic frequency gradient. The degenerate modes demonstrate an apparently programming phase shift without an intrinsic frequency gradient.

The analysis has considered only single and double modes which cover all conditions in the illustrative examples. For larger models, higher order modes could be feasible, but the extension to the analysis would be involved. It has, in fact, been demonstrated for the ladder structure that triple and higher order modes are not stable (Endo and Mori, 1976a). Another extension to this work is the consideration of a tubular structure of 5th power van der Pol oscillators which has been hypothesised as a sutiable model for large-intestinal activity where periods of electrical silence appear to exist. The stability analysis in this case is considerably more involved, and will be presented in a later paper.

9. References

Davies, M.L. (1799) 'Electronic modelling of the human colon', M.Sc. Thesis, Department of Control Engineering, University of Sheffield, England.

Endo, T. and Mori, S. (1976a) 'Mode analysis of a multimode ladder oscillators', IEEE Trans. Cct. and Sys., 23, p. 100-113.

Endo, T. and Mori, S., (1976b) 'Mode analysis of a two-dimensional low-pass multimode oscillator', IEEE Trans. Cct. and Sys., 23, p. 517-530.

Endo, T. and Mori, S., (1978) 'Mode analysis of a ring of a large number of mutually coupled van der Pol oscillators', IEEE Trans. Cct. and Sys., 25, p. 7-18.

Gantmacher, F.R., (1960) 'The theory of matrices', Chelsea, New York.

Linkens, D.A., (1974), 'Analytical solution of large numbers of mutually coupled near-sinusoidal oscillators', IEEE Trans. Cct. and Sys., 21, p. 294-300.

Linkens, D.A., (1977), 'The stability of entrainment conditions for RLC coupled van der Pol oscillators', Bull. Math. Biol., 39, p. 359-372.

Linkens, D.A. (1979), 'Regions of attraction in coupled non-linear oscillators intercoupled with pure time delays', Bull. Math. Biol., in press.

Linkens, D.A., and Datardina, S.P. (1977), 'Frequency entrainment of coupled Hodgkin-Huxley type oscillators for modelling gastro-intestinal electrical activity', IEEE Trans. BME, 24, P. 362-365.

Linkens, D.A., and Kitney, R.I. (1981) Mode analysis of non-linear oscillators intercoupled with pure time delays', Bull. Math. biol. in press.

Linkens, D.A., Taylor, I and Duthie, H.L. (1976) 'Mathematical modelling of the colorectal myoelectrical activity in humans', IEEE Trans. BME, 23, p. 101-110.

Nelsen, T.S., and Becker, J.C. (1968), 'Simulation of the electrical and mechanical gradient of the small intestine', Am. J. Physiol., p. 749-757.

Patton, R.J. and Linkens, D.A., (1977), 'Phenonenological investigation of a distributed parameter model for co-ordinating the mechanical activity of the mammaliam gut'. IFAC 2nd Symposium on Control of Distributed Parameter Systems, Warwick, only.

Sarna, S.K., Daniel, E.E. and Kingma, Y.J. (1971), 'Simulation of slow-wave electrical activity of small intestine', Am. J. Physiol., 221, p. 166-175.

Sarna, S.K., Daniel, E.E., and Kingma, Y.J. (1972), 'Simulation of the electrical control activity of the stomach by an array of relaxation oscillators', Amer. J. Dig. Dis., 17, p. 299-310.

APPENDIX I

Calculation of the parameter $\Psi_{mn}(i,j,k,l)$

The parameter $\Psi_{mn}(i,j,k,l)$ can be calculated using equation (45) by evaluation of $\psi_m(i,k)$ and $\psi_n(j,l)$, to calculate $\Psi_m(i,k)$ as in (46a), the element p_{ai} of the matrix P is given by (19) hence p_{ai}^2 can be written as follows:

$$p_{ai}^2 = \frac{1}{m} \left| 1 + E_i \cos \frac{4\pi\ a(i-1)}{m} \right| \qquad (I.1)$$

if m is even, E_i is defined as

$$E_i = \begin{cases} 0, & \text{for } i=1, \frac{m}{2}+1 \\ 1, & \text{for } i=2,3, \ldots, \frac{m}{2} \\ -1, & \text{for } i=\frac{m}{2}+2, \ldots, m-1, m \end{cases} \qquad (I.2)$$

Similarly, p_{ak}^2 is obtained by permuting the variable i with k in (I.1) and (I.2)

But if m is odd, E_i is defined as

$$E_i = \begin{cases} 0, & \text{for } i=1 \\ 1, & \text{for } i=2,3, \ldots, \frac{m+1}{2} \\ -1, & \text{for } i=\frac{m+3}{2}, \ldots, m-1, m \end{cases} \qquad (I.3)$$

Defining $\beta_i = E_i \cos \dfrac{4\pi a(i-1)}{m}$ (I.4)

Similarly, p_{ak}^2 and β_k are obtained by permuting the variable i with k. Then Ψ_m can be written as

$$\Psi_m(i,k) = \frac{1}{m^2} \sum_{a=1}^{m} (1+\beta_i)(1+\beta_k)$$

$$= \frac{1}{m^2} \sum_{a=1}^{m} (1+\beta_i+\beta_k+\beta_i\ \beta_k) \qquad (I.5)$$

Calculating $\sum_{a=1}^{m} \beta_i$, $\sum_{a=1}^{m} \beta_k$ and $\sum_{a=1}^{m} \beta_i \beta_k$ in equation (I.5) using different mathematical deductions, the value of Ψ_m is determined. Thus, for an even m, we have five values for $\Psi_m(i,k)$ which correspond to the following five groups of points in the space (i,k) using equations (I.2), (I.4) and (I.5):

a. for the points (i,k) which satisfy

$$(i,k) = (s,s),\ (s+\frac{m}{2}, s+\frac{m}{2}),\ (s, \frac{m}{2}+2-s) \text{ or } (s+\frac{m}{2}, m+2-s)$$

where $s = 2,3, \ldots, \frac{m}{2}$; $s \neq \frac{m}{4}+1$

$$\Psi_m(i,k) = \frac{3}{2m} \tag{I.6a}$$

These points represent subspace A, so the value of Ψ_m corresponding to such points can be denoted as $(\Psi_m)_A$.

b. for the points (i,k) which satisfy

$$(i,k) = (s,m+2-s),\ (s,s+\frac{m}{2}),\ (s+\frac{m}{2}, \frac{m}{2}+2-s)$$

$$\text{or } (s+\frac{m}{2}, s)$$

where $s = 2,3, \ldots, \frac{m}{2}$; $s \neq \frac{m}{4}+1$

$$\Psi_m(i,k) = \frac{1}{2m} = (\Psi_m)_B \tag{I.6b}$$

These points represent subspace B.

c. for the points (i,k) which satisfy

$$(i,k) = (\frac{m}{4}+1, \frac{m}{4}+1) \text{ or } (\frac{3m}{4}+1, \frac{3m}{4}+1)$$

$$\Psi_m(i,k) = \frac{2}{m} = (\Psi_m)_C \tag{I.6c}$$

These two points represent subspace C.

d. for the points (i,k) which satisfy

$$(i,k) = (\frac{m}{4}+1, \frac{3m}{4}+1) \text{ or } (\frac{3m}{4}+1, \frac{m}{4}+1)$$

only for the case when m is a multiple of four

$$\Psi_m(i,k) = 0 = (\Psi_m)_D \tag{I.6d}$$

These two points represent subspace D.

e. for any point (i,k) not belonging to subspaces A,B,C or D

$$\psi_m(i,k) = \frac{1}{m} = (\Psi_m)_E \tag{I.6e}$$

From the above five cases, it is clear that any point in the space (i,k) which contains mxm points should belong either to A,B,C,D or E and the corresponding value of $\Psi_m(i,k)$ is either $(\Psi_m)_A$, $(\Psi_m)_B$, $(\Psi_m)_C$, $(\Psi_m)_D$ or $(\Psi_m)_E$ respectively.

For the case that m is an odd number, we have the following three values for $\Psi_m(i,k)$ which correspond to the different groups of points in the space (i,k):

a'. For the points, which satisfy

$(i,k) = (s,s)$

where $s = 2,3, \ldots, m$

$$\Psi_m\ (i,k) = \frac{3}{2m} \tag{I.7a}$$

These points represent subspace A', so the value of Ψ_m corresponding to such points can be denoted as $(\Psi_m)_{A'}$.

b'. For the points, which satisfy

$(i,k) = (s,m-s+2)$

where $s = 2,3, \ldots, m$

$$\Psi_m\ (i,k) = \frac{1}{2m} = (\Psi_m)_{B'} \tag{I.7b}$$

These points represent subspace B'.

c'. For any point (i,k), not belonging to subspaces A' nor B'

$$\Psi_m(i,k) = \frac{1}{m} = (\Psi_m)_{C'} \tag{I.7c}$$

Such points represent subspace C'.

Thus, we have only three different values of $\Psi_m(i,k)$ if the number of oscillators in each ring is an odd number.

Now, to calculate the different values of $\Psi_n(j,k)$ which are given by the elements of the matrix Q (I.2), we use the same deduction as for Ψ_m and put

$$q^2_{bj} = \frac{1}{n}\left[1 + F_j \cos \frac{(2b-1)(j-1)\pi}{n}\right] \qquad \text{(I.8)}$$

where F_i is defined as

$$F_j = \begin{cases} 0, & \text{for } j = 1 \\ 1, & \text{for } j \neq 1 \end{cases} \qquad \text{(I.9)}$$

Similarly, q_{bl} is obtained by permuting the variable j with l

Defining

$$\beta^*_j = F_j \cos \frac{(2b-1)(j-1)\pi}{n}$$

$$\beta^*_l = F_l \cos \frac{(2b-1)(l-1)\pi}{n} \qquad \text{(I.10)}$$

Then, Ψ_n can be written as

$$\Psi_n(j,l) = \frac{1}{n^2} \sum_{b=1}^{n} (1 + \beta^*_j)(1 + \beta^*_l)$$

$$= \frac{1}{n^2} \sum_{b=1}^{n} (1 + \beta^*_j + \beta^*_l + \beta^*_j\, \beta^*_l) \qquad \text{(I.11)}$$

Since

$$\sum_{b=1}^{n} \beta^*_j = \sum_{b=1}^{n} F_j \cos \frac{(2b-1)(j-1)}{n}\pi \;.\; \cos \frac{(2b-1)(l-1)\pi}{n}$$

$= \frac{n}{2}$ for j,l) = (s,s)

where $s = 2,3, \ldots, n,\ s \neq \frac{n}{2} + 1$

or,

$= -\frac{n}{2}$ for(j,l) = (s, n-s + 2)

where $s = 2,3, \ldots, n \quad s \neq \frac{n}{2} + 1$

or, = 0 for all points (j,l) not belonging to both the above two groups of points. (I.12)

If the first group of points (j,l) is called subspace A*, the second as subspace B* and the last as C*, and substitute the values given by (I.12) into (I.11), the following three values of $\Psi_n(j,l)$ (which correspond to the different groups of points in the space (j,l)), are given as

a. For the points (j,k) which are represented by A*

$$\Psi_n(j,l) = \frac{1}{n^2}\left(n + \frac{n}{2}\right) = \frac{1}{n^2} \cdot \frac{3n}{2} = \frac{3}{2n} \qquad \text{(I.13a)}$$

Thus, such value can be denoted as $(\Psi_n)_{A^*}$.

b. For the points (j,k) which are represented by B*

$$\Psi_n(j,l) = \frac{1}{n^2}\left(n - \frac{n}{2}\right) = \frac{1}{2n} = (\Psi_n)_{B^*} \qquad \text{(I.13b)}$$

c. For the points (j,k) which are represented by C*

$$\Psi_n(j,l) = \frac{1}{n} = (\Psi_n)_{C^*} \qquad \text{(I.13c)}$$

APPENDIX II

The Jacobian elements for the case of the nonresonant double mode.

It is assumed that any two arbitrary modes (i_o, j_o) and (r_o, s_o) are nonresonant double modes, i.e. the two oscillations are excited simultaneously and the ratio between their frequencies is an irrational number, so

$$U_{iojo} \neq \; ; \; U_{roso} \neq 0 \; ; \; U_{ij} = 0$$

for $i = 1,2, \ldots, m$; $j = 1,2, \ldots, n$

$$(i,j) \neq (1_o, j_o), (r_o, s_o) \quad ; \quad (i_o, j_o) \neq (r_o, s_o) \qquad \text{(II.1)}$$

Since the tubular structure is symmetrical the amplitudes U_{iojo} and U_{roso} must be equal (Utkin, 1959).

Using the above conditions of amplitudes the mode x_{kl} can be written as

$$x_{kl} = p_{kio} q_{ljo} A_{iojo} \sin(\omega_{iojo}\tau + \phi_{iojo}) + p_{kro} q_{lso} A_{roso} \sin(\omega_{roso}\tau + \phi_{roso}) \qquad \text{(II.2)}$$

Substituting the amplitude condition (II.1) into (44), to obtain the stationary values, then η_{ij} from (36) for $(i.k) = (i_o, j_o)$ gives

$$\eta_{iojo}(i_o, j_o) = \frac{3}{2}\Psi_{mn}(i_o, j_o, r_o, s_o)U_{roso} + \frac{3}{4}\Psi_{mn}(i_o, j_o, i_o, j_o)U_{iojo} = 3 \qquad \text{(II.3a)}$$

and for $(i.j) = (r_o, s_o)$

$$\eta_{iojo}(r_o, s_o) = \frac{3}{4}\Psi_{mn}(r_o, s_o, r_o, s_o)U_{roso} + \frac{3}{2}\Psi_{mn}(r_o, s_o, i_o, j_o)U_{iojo} = 3 \qquad \text{(II.3b)}$$

when the two mode amplitudes U_{iojo} and U_{roso} are equal, then

$$U_{iojo} = U_{roso} = \frac{4}{\Psi_{mn}(i_o, j_o, i_o, j_o) + 2\Psi_{mn}(i_o, j_o, r_o, s_o)}$$

$$= \frac{4}{\Psi_{mn}(r_o, s_o, r_o, s_o) + 2\Psi_{mn}(r_o, s_o, i_o, j_o)} \qquad \text{(II.4)}$$

$$\Psi_{mn}(i_o,j_o,i_o,j_o) = \Psi_{mn}(r_o,s_o,r_o,s_o) \quad \text{(II.5a)}$$

$$\Psi_{mn}(i_o,j_o,r_o,s_o) = \Psi_{mn}(r_o,s_o,i_o,j_o) \quad \text{(II.5b)}$$

Applying the amplitude condition (II.1), the Jacobi's matrix of (50a) becomes

$$J_{ij}(k,l) = \xi\left[1-\tfrac{1}{2}\Psi_{mn}(i,j,i_o,j_o)U_{iojo} - \tfrac{1}{2}\Psi_{mn}(i,j,r_o,s_o)U_{roso}\right]$$

$$\text{for } (i,j) = (k,l) \quad \text{(II.6a)}$$

But for $(i,j) \neq (k,l)$ from (50b)

$$J_{ij}(k,l) = -\tfrac{1}{2}\,\xi\,\Psi_{mn}(i,j,k,l)U_{iojo} \quad \text{for } (i,j) = (i_o,j_o)$$

or $$= -\tfrac{1}{2}\,\xi\,\Psi_{mn}(r_o,s_o,k,l)U_{roso} \quad \text{for } (i,j) = (r_o s_o)$$

or $$= 0 \quad \text{for all other } i,j$$

where $i = 1,2,\ldots, m$; $j = 1,2, \ldots, n$ (II.6b)

APPENDIX III

The averaged equations of the system when considering the degenerate modes

Ignoring the harmonics in the quasi-harmonic approximation when expanding $\sin^3 \theta_{i1j1}$ and $\sin^2\theta_{i1j1} \cos \theta_{i1j1}$ in the higher power terms, and putting ϕ_{i2j2} (i.e. $\phi = - \phi_{i1j1}$) for simplicity, then the equivalent linearised x^3_{ab} can be written as

$$x^3_{ab} = \frac{3}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} p^3_{ak}\, q^3_{bl}\, A^2_{kl}\, y_{kl}$$

$$+ \frac{3}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} p^2_{ak}\, q^2_{bl}\, p_{ar}\, q_{bs} \cdot A^2_{kl}\, y_{rs}$$

$$+ K_1 y_{i1j1} + K_2 y_{i2j2} + K_3{}' \sum_{k=1}^{m} \sum_{l=1}^{n} p_{ak}\, q_{bl}\, y_{kl}$$

$$(k,l) \neq (r,s) \;;\; (r,s) \neq (i_1,j_1),(i_2,j_2)$$

with

$$K_1 = \frac{3}{2} K_3\, p_{ai1}\, q_{bj1} + \frac{3}{4} K_3 \frac{A_{i2j2}}{A_{i1j1} \cos\phi}$$

$$K_2 = \frac{3}{2} K_3\, p_{ai2}\, q_{bj2} + \frac{3}{4} K_3 \frac{A_{i1j1}}{A_{i2j2} \cos\phi}$$

$$K_3 = p_{ai1}\, q_{bj1} p_{ai2} q_{bj2} A_{i1j1} A_{i2j2} \cos \phi \qquad \text{(III.1)}$$

In this case h_{ij} of (30) is counted and can be indicated in a linear combination as in (35). The necessary term in h_{ij} is only y_{ij} for the ordinary modes, while those for degenerate modes are y_{ij} and its degenerate pair term because the two components of a degenerate mode have the same frequency ω.

Hence, the equivalent linearised equations, which are given by (38) should be distinquished as two cases:

a. for the case of nondegenerate modes in which no two mode frequencies are equal

$$h_{ij} = \eta_{ij}(i,j)\ y_{ij}$$

$$(i,j) \neq (i_1,j_1),\ (i_2,j_2) \tag{III.2}$$

where

$$\eta_{ij}(i,j) = \eta_{ij} = \frac{3}{2}\sum_{k=1}^{m}\sum_{l=1}^{n} \Psi_{mn}(i,j,k,l)A_{kl}^2$$

$$-\frac{3}{4}\Psi_{mn}(i,j,i,j)A_{ij}^2 + 3\sigma_{(i,j)}^{i_1,j_1,i_2j_2} A_{i1j1}A_{i2j2}\cos\ \phi$$

$$\sigma_{(i,j)}^{i_1j_1-i_2j_2} = \sum_{a=1}^{m}\sum_{b=1}^{n} p_{ai}^2\ q_{bj}^2\ p_{ai_1}q_{bj_1}p_{ai_2}q_{bj_2} \tag{III.3}$$

From III.2, the linearised equation (38) becomes

$$\ddot{y}_{ij} + \omega_{ij}^2\ y_{ij} = \xi\dot{y}_{ij} - \frac{1}{3}\xi\eta_{ij}\ y_{ij}$$

$$(i,j) \neq (i_1,j_1),\ (i_2,j_2) \tag{III.4}$$

The averaged equations for nondegenerate modes are

$$\dot{U}_{ij} = \xi\ U_{ij}\left|1-\tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Psi_{mn}(i,j,k,l)U_{kl}+\tfrac{1}{4}\Psi_{mn}(i,j,i,j)U_{ij}\right| \tag{III.5a}$$

$$\dot{\phi}_{ij} = 0,\ (i,j) \neq (i_1,j_1),\ (i_2,j_2) \tag{III.5b}$$

b. for the case of degenerate modes where the two modes (i_1,j_1), (i_2j_2) are of equal frequencies

$$h_{i1j1} = \eta_{i1j1}(i1,j1)y_{i1j1} + \eta_{i1j1}(i_2,j_2)\ y_{i2j2} \tag{III.6a}$$

and

$$h_{i2j2} = \eta_{i2j2}(i_1,j_1)y_{i1j1} + \eta_{i2j2}(i_2,j_2)y_{i2j2} \qquad \text{(III.6b)}$$

with

$$\eta_{i1j1}(i_1,j_1) = \eta_{i1j1} = \frac{3}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Psi_{mn}(i_1,j_1,k,l)A_{kl}^2$$

$$-\frac{3}{4}\Psi_{mn}(i_1,j_1,i_1,j_1)A_{i1j1}^2 - \frac{3}{4}\Psi_{mn}(i_1,j_1,i_2,j_2)A_{i2j2}^2 \qquad \text{(III.7a)}$$

$$\eta_{i1j1}(i_2,j_2) = \frac{3}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\sigma_{(k,l)}^{i_1j_1-i_2j_2}A_{kl}^2$$

$$+\frac{3}{2}\Psi_{mn}(i_1,j_1,i_2,j_2)\cos\phi\, A_{i1j1}A_{i2j2} \qquad \text{(III.7b)}$$

$$\eta_{i2j2}(i_1,j_1) = \eta_{i1j1}(i_2,j_2) \qquad \text{(III.7c)}$$

$$\eta_{i2j2}(i_2,j_2) = \eta_{i2j2} = \frac{3}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Psi_{mn}(i_2,j_2,k,l)A_{kl}^2$$

$$-\frac{3}{4}\Psi_{mn}(i_2,j_2,i_2,j_2)A_{i2j2}^2 - \frac{3}{4}\Psi_{mn}(i_2,j_2,i_1,j_1)A_{i1j1}^2 \qquad \text{(III.7d)}$$

Substituting (III.6) into the equivalent linearised equation (38), it becomes

$$\ddot{y}_{ij}+\omega_{ij}^2 y_{ij} = \xi\dot{y}_{ij} - \frac{1}{3}\xi\left[\{\eta_{i1j1}+\eta_{i2j2}(i_1,j_1)\}y_{i1j1}+\{\eta_{i2j2}+\eta_{i1j1}(i_2,j_2)\}y_{i2j2}\right]$$

(III.8)

i \ j	1	2	3	4
1	$\sqrt{2-2\alpha}$	$\sqrt{2-\sqrt{2}\alpha}$	$\sqrt{2}$	$\sqrt{2+\sqrt{2}\alpha}$
2	$\sqrt{2+\alpha}$	$\sqrt{2+(3-\sqrt{2})\alpha}$	$\sqrt{2+3\alpha}$	$\sqrt{2+(3+\sqrt{2})\alpha}$
3	$\sqrt{2+\alpha}$	$\sqrt{2+(3-\sqrt{2\alpha})}$	$\sqrt{2+3\alpha}$	$\sqrt{2+(3+\sqrt{2})\alpha}$

Table 1 Angular frequencies ω_{ij} of the tube oscillator system with m=3 and n=4.

i \ j	1	2	3
1	$\sqrt{2-2\alpha}$	$\sqrt{2-\alpha}$	$\sqrt{2+\alpha}$
2	$\sqrt{2}$	$\sqrt{2+\alpha}$	$\sqrt{2+3\alpha}$
3	$\sqrt{2+2\alpha}$	$\sqrt{2+3\alpha}$	$\sqrt{2+5\alpha}$
4	$\sqrt{2}$	$\sqrt{2+\alpha}$	$\sqrt{2+3\alpha}$

Table 2 Angular frequencies ω_{ij} of the tube oscillator system with m=4 and n=3

(i,k)	$\psi_m(i,k)$	(j,l)	$\psi_n(j,l)$	$\psi_{mn}(i,j,k,l)$
A	$\frac{3}{2m}$	A*	$\frac{3}{2n}$	$\frac{9}{4mn}$
A	$\frac{3}{2m}$	B*	$\frac{1}{2n}$	$\frac{3}{4mn}$
A	$\frac{3}{2m}$	C*	$\frac{1}{n}$	$\frac{3}{2mn}$
B	$\frac{1}{2m}$	A*	$\frac{3}{2n}$	$\frac{3}{4mn}$
B	$\frac{1}{2m}$	B*	$\frac{1}{2n}$	$\frac{1}{4mn}$
B	$\frac{1}{2m}$	C*	$\frac{1}{n}$	$\frac{1}{2mn}$
C	$\frac{2}{m}$	A*	$\frac{3}{2n}$	$\frac{3}{mn}$
C	$\frac{2}{m}$	B*	$\frac{1}{2n}$	$\frac{1}{mn}$
C	$\frac{2}{m}$	C*	$\frac{1}{n}$	$\frac{2}{mn}$
D	0	A*,B*,C*	$\frac{3}{2n}$, $\frac{1}{2n}$, $\frac{1}{n}$	0
E	$\frac{1}{m}$	A*	$\frac{3}{2n}$	$\frac{3}{2mn}$
E	$\frac{1}{m}$	B*	$\frac{1}{2n}$	$\frac{1}{2mn}$
E	$\frac{1}{m}$	C*	$\frac{1}{n}$	$\frac{1}{mn}$

Table 3 Determination of the value $\psi_{mn}(i,j,k,l)$ when the number of oscillators m in each ring of the structure is an even number

(i,j) \ (k,l)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)	(4,1)	(4,2)	(4,3)
(1,1)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,2)	1/12	1/8	1/24	1/12	1/8	1/24	1/12	1/8	1/24	1/12	1/8	1/24
(1,3)	1/12	1/24	1/8	1/12	1/24	1/18	1/12	1/24	1/8	1/12	1/24	1/8
(2,1)	1/12	1/12	1/12	1/6	1/6	1/6	1/12	1/12	1/12	0	0	0
(2,2)	1/12	1/8	1/24	1/6	1/4	1/12	1/12	1/8	1/24	0	0	0
(2,3)	1/12	1/24	1/8	1/6	1/12	1/4	1/12	1/24	1/8	0	0	0
(3,1)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(3,2)	1/12	1/8	1/24	1/12	1/8	1/24	1/12	1/8	1/24	1/12	1/8	1/24
(3,3)	1/12	1/24	1/8	1/12	1/24	1/8	1/12	1/24	1/8	1/12	1/24	1/8
(4,1)	1/12	1/12	1/12	0	0	0	1/12	1/12	1/12	1/6	1/6	1/6
(4,2)	1/12	1/8	1/24	0	0	0	1/12	1/8	1/24	1/6	1/4	1/12
(4,3)	1/12	1/24	1/8	0	0	0	1/12	1/24	1/8	1/6	1/12	1/4

Table 4 The values $\psi_{mn}(i,j,k,l)$ corresponding to the structure with m=4, n=3

(i,k)	$\psi_m(i,k)$	(j,l)	$\psi_n(j,l)$	$\psi_{mn}(i,j,k,l)$
A'	$\frac{3}{2m}$	A*	$\frac{3}{2n}$	$\frac{9}{4mn}$
A'	$\frac{3}{2m}$	B*	$\frac{1}{2n}$	$\frac{3}{4mn}$
A'	$\frac{3}{2m}$	C*	$\frac{1}{n}$	$\frac{3}{2mn}$
B'	$\frac{1}{2m}$	A*	$\frac{3}{2n}$	$\frac{3}{4mn}$
B'	$\frac{1}{2m}$	B*	$\frac{1}{2n}$	$\frac{1}{4mn}$
B'	$\frac{1}{2m}$	C*	$\frac{1}{n}$	$\frac{1}{2mn}$
C'	$\frac{1}{m}$	A*	$\frac{3}{2n}$	$\frac{3}{2mn}$
C'	$\frac{1}{m}$	B*	$\frac{1}{2n}$	$\frac{1}{2mn}$
C'	$\frac{1}{m}$	C*	$\frac{1}{n}$	$\frac{1}{mn}$

Table 5 Determination of the value ψ_{mn} when the number of oscillators in each ring of the structure m is an odd number

(k,l) / (i,j)	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)
(1,1)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,2)	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24
(1,3)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,4)	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8
(2,1)	1/12	1/12	1/12	1/12	1/8	1/8	1/8	1/8	1/24	1/24	1/24	1/24
(2,2)	1/12	1/8	1/12	1/24	1/8	3/16	1/8	1/16	1/24	1/16	1/24	1/48
(2,3)	1/12	1/12	1/12	1/12	1/8	1/8	1/8	1/8	1/24	1/24	1/24	1/24
(2,4)	1/12	1/24	1/12	1/8	1/8	1/16	1/8	3/16	1/24	1/48	1/24	1/16
(3,1)	1/12	1/12	1/12	1/24	1/24	1/24	1/24	1/24	1/8	1/8	1/8	1/8
(3,2)	1/12	1/8	1/12	1/24	1/24	1/16	1/24	1/48	1/8	1/16	1/8	1/16
(3,3)	1/12	1/12	1/12	1/12	1/24	1/24	1/24	1/24	1/8	1/8	1/8	1/8
(3,4)	1/12	1/24	1/12	1/8	1/24	1/48	1/24	1/16	1/8	1/16	1/8	3/16

Table 6 The values $\psi_{mn}(i,j,k,l)$ when m=3, n=4

(k,l) / (i,j)	(1,1)	(1,2)	(1,n)	(2,n)	(m,1)	(m,n)
(1,1)	$\psi_{mn}(1,1,1,1)$	$\psi_{mn}(1,1,1,2)$	$\psi_{mn}(1,1,1,n)$	$\psi_{mn}(1,1,2,n)$	$\psi_{mn}(1,1,n,1)$	$\psi_{mn}(1,1,m,n)$
(1,2)	$\psi_{mn}(1,2,1,1)$	$\psi_{mn}(1,2,1,2)$	$\psi_{mn}(1,2,1,n)$			
(1,n)	$\psi_{mn}(1,n,1,1)$	$\psi_{mn}(1,n,1,2)$				
(2,1)	$\psi_{mn}(2,1,1,1)$	$\psi_{mn}(2,1,1,2)$				
(2,2)	$\psi_{mn}(2,2,1,1)$	$\psi_{mn}(2,2,1,2)$				
(2,n)	$\psi_{mn}(2,n,1,1)$	$\psi_{mn}(2,n,1,2)$				
(3,1)	$\psi_{mn}(3,1,1,1)$	$\psi_{mn}(3,1,1,2)$				
(3,2)	$\psi_{mn}(3,2,1,1)$	$\psi_{mn}(3,2,1,2)$				
(3,n)	$\psi_{mn}(3,n,1,1)$	$\psi_{mn}(3,n,1,2)$				
(m,1)	$\psi_{mn}(m,1,1,1)$	$\psi_{mn}(m,1,1,2)$				
(m,2)	$\psi_{mn}(m,2,1,1)$	$\psi_{mn}(m,2,1,2)$				
(m,n)	$\psi_{mn}(m,n,1,1)$	$\psi_{mn}(m,n,1,2)$	$\psi_{mn}(m,n,1,n)$	$\psi_{mn}(m,n,2,n)$	$\psi_{mn}(m,n,m,1)$	$\psi_{mn}(m,n,m,n)$

Table 7 Distribution of the values ψ_{mn} (i,j,k,l) for simple investigation of mode stability

j / i	1	2	3	4
1	2.16582	2.48487	3.12359	3.65226
2	3.50567	3.71129	4.16605	4.57582
3	3.50567	3.71129	4.16605	4.57582

Table 8 Mode frequencies f_{ij} of the oscillator system with $L = 54$ mH (f_{ij} is calculated on KHz)

j / i	1	2	3
1	2.46193	2.58792	3.57595
2	3.12360	3.57595	4.34148
3	3.97718	4.34148	4.99095
4	3.12360	3.57595	4.34148

Table 9 Mode frequencies f_{ij} of the oscillator system with $L = 54$ mH (in KHz)

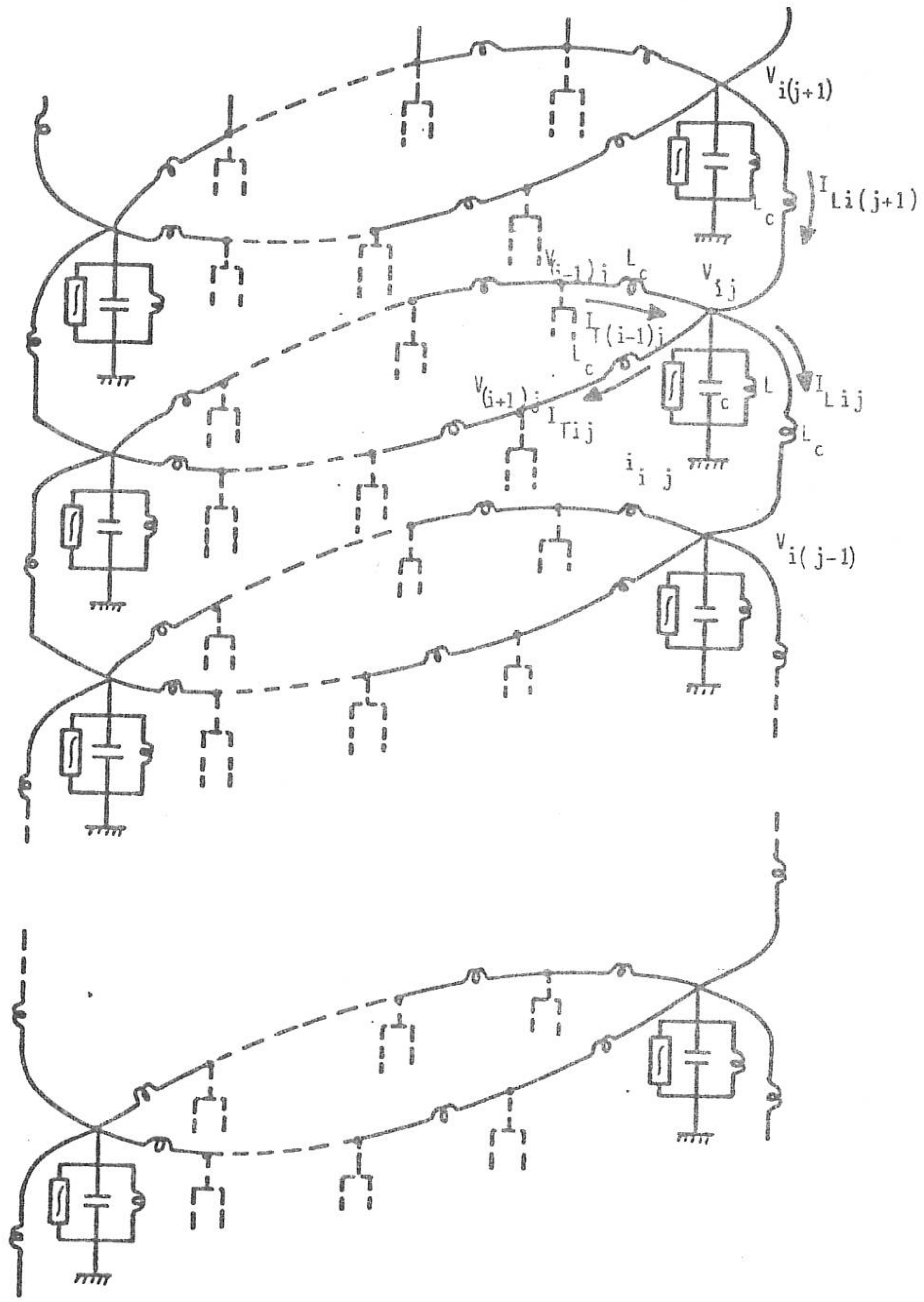

Fig. 1 Tube structure with an arbitrary number of oscillators in each ring and an arbitrary number of rings

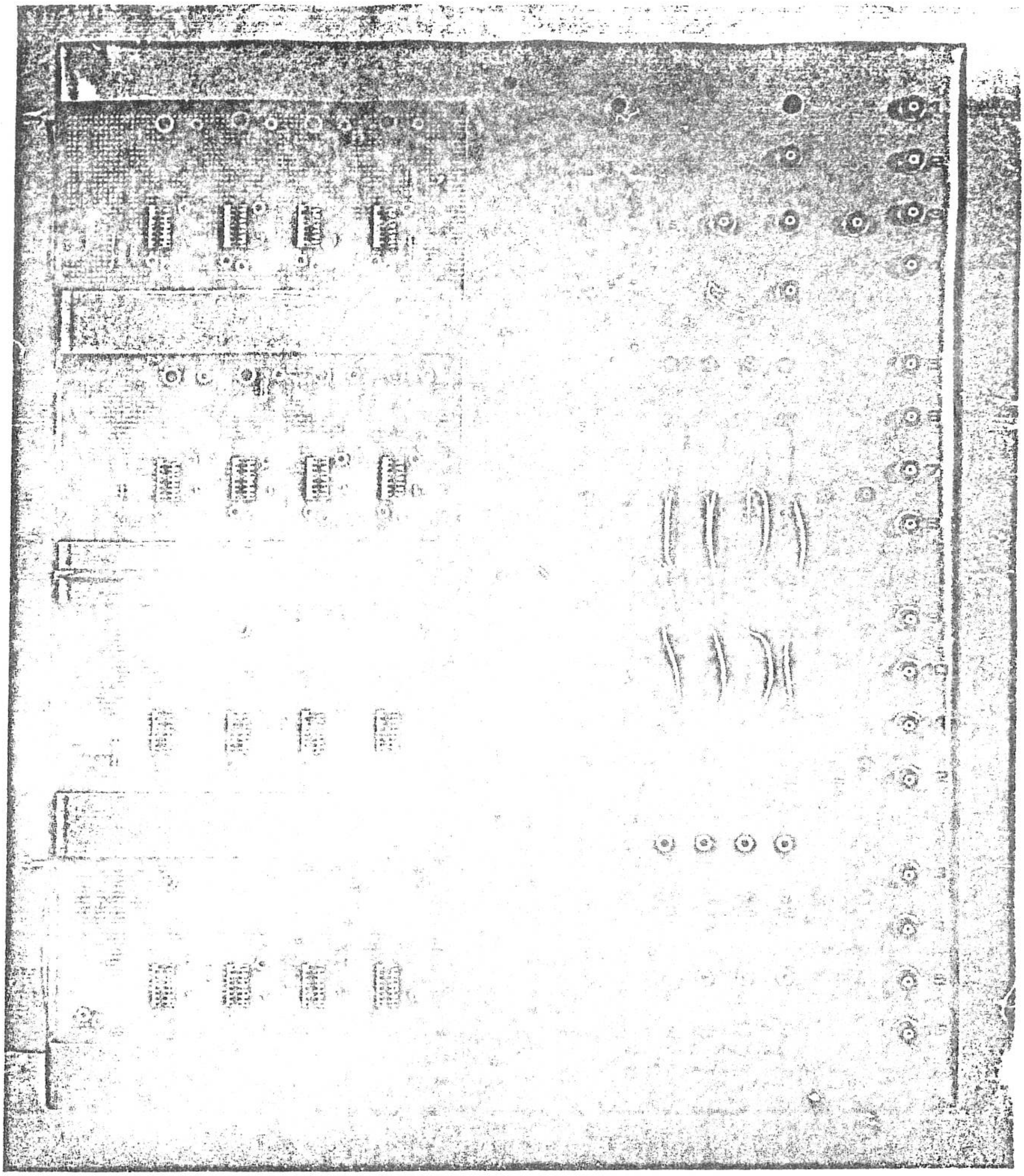

Fig. 2 The electronic model with sixteen oscillators arranged on four boards

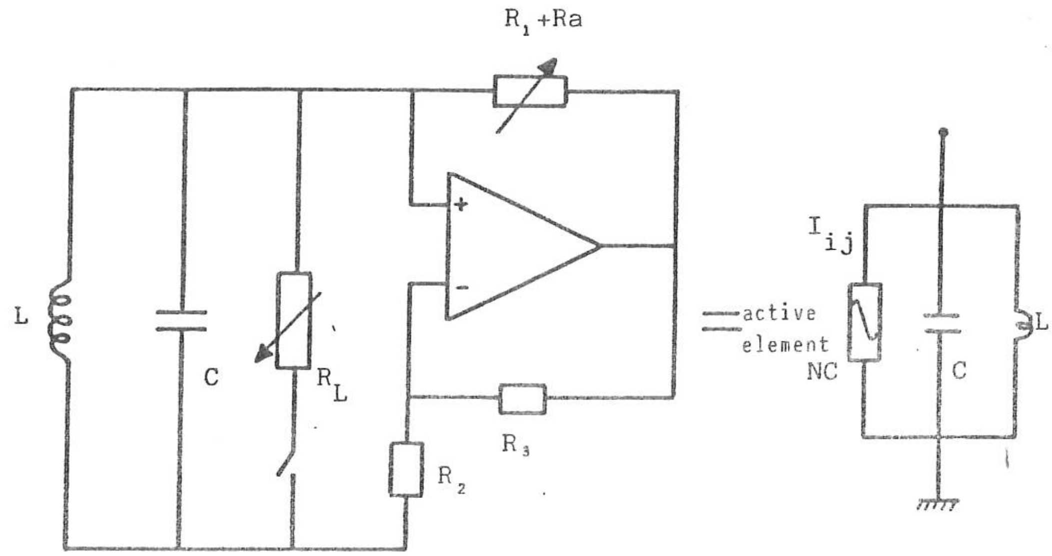

Fig. 3 Oscillator circuit with cubic nonlinear characteristic

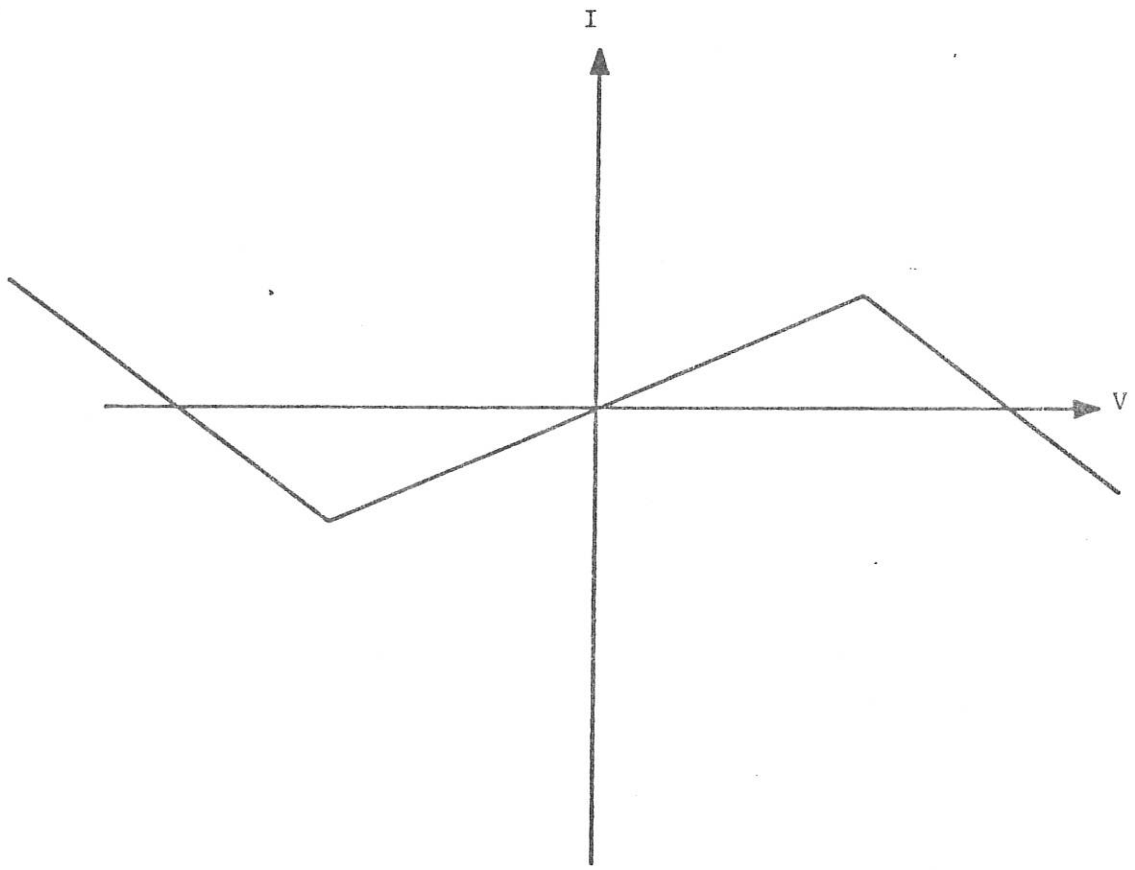

Fig. 4 Cubic characteristic of the nonlinear conductance of the oscillator

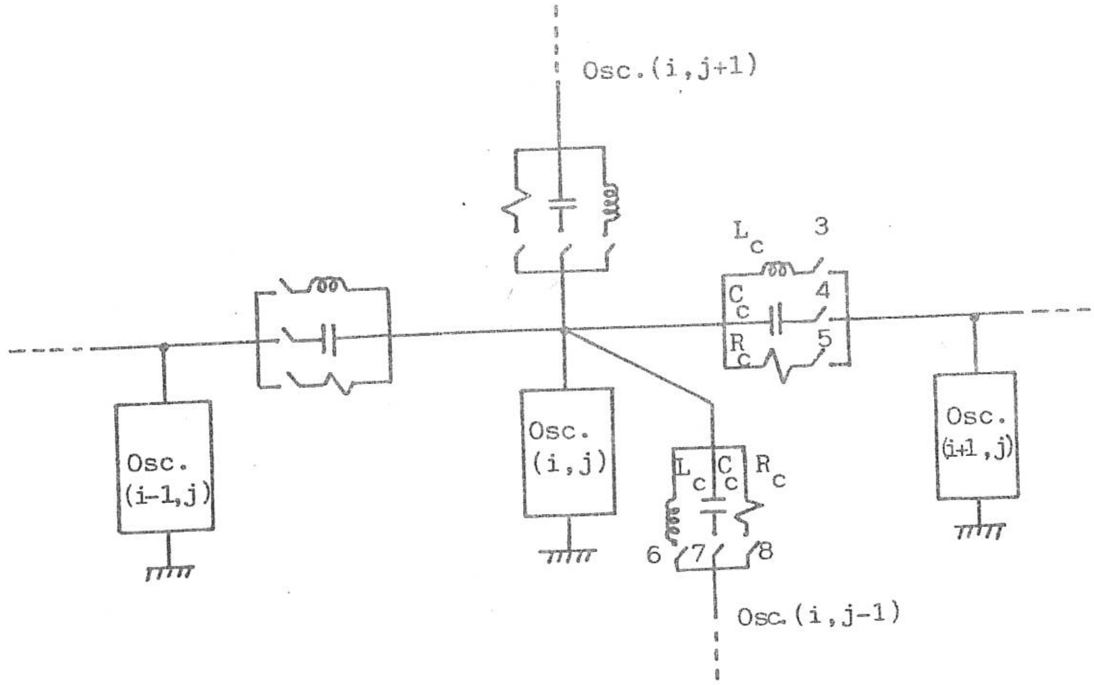

Fig. 5 Schematic diagram for oscillator located in position (i,j) on the electronic board of oscillators

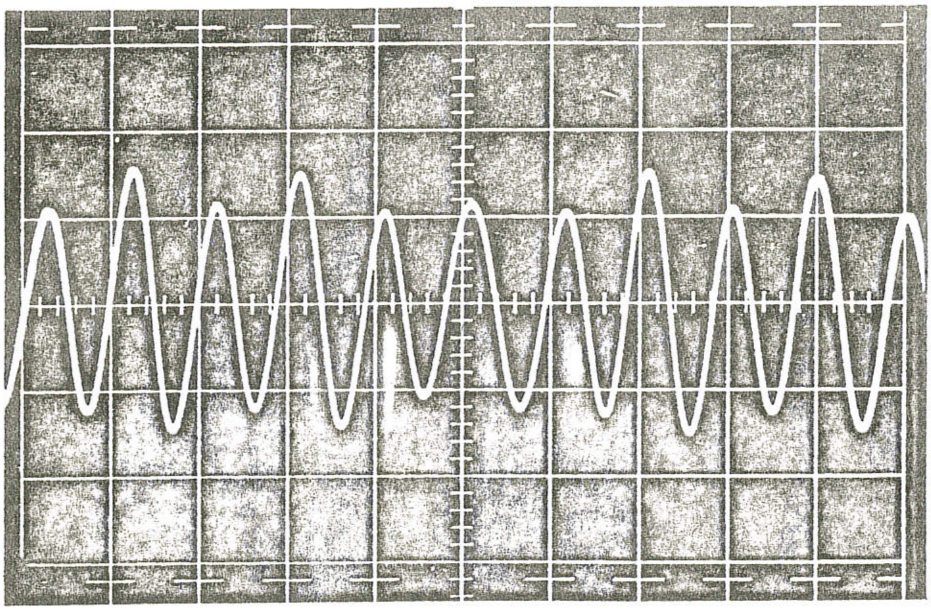

Fig. 6 Nonresonant double mode in mutual inductive tube oscillator system

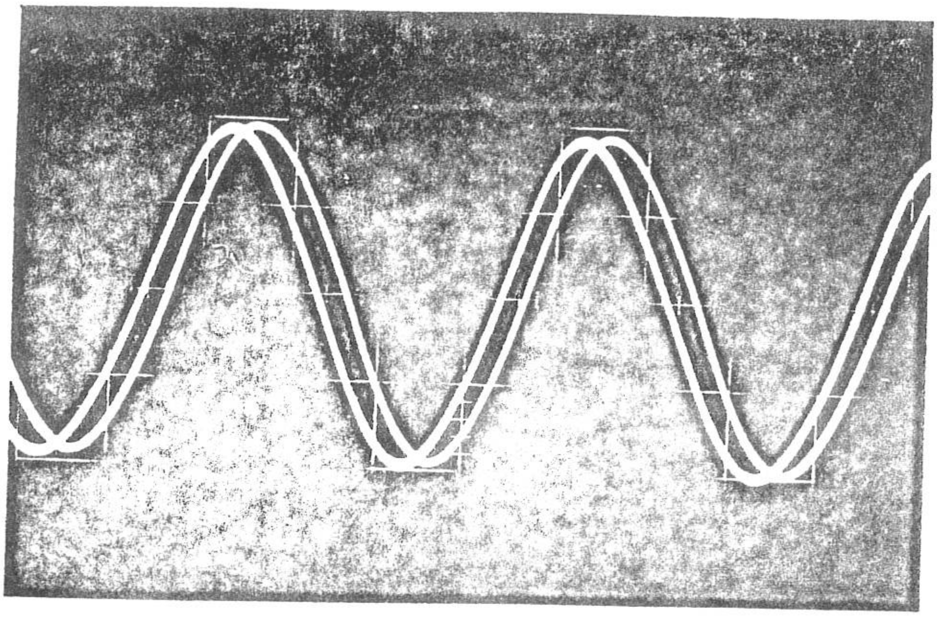

Fig. 7 Output waveforms of oscillators 1 and 2 in the coupled tube oscillator model

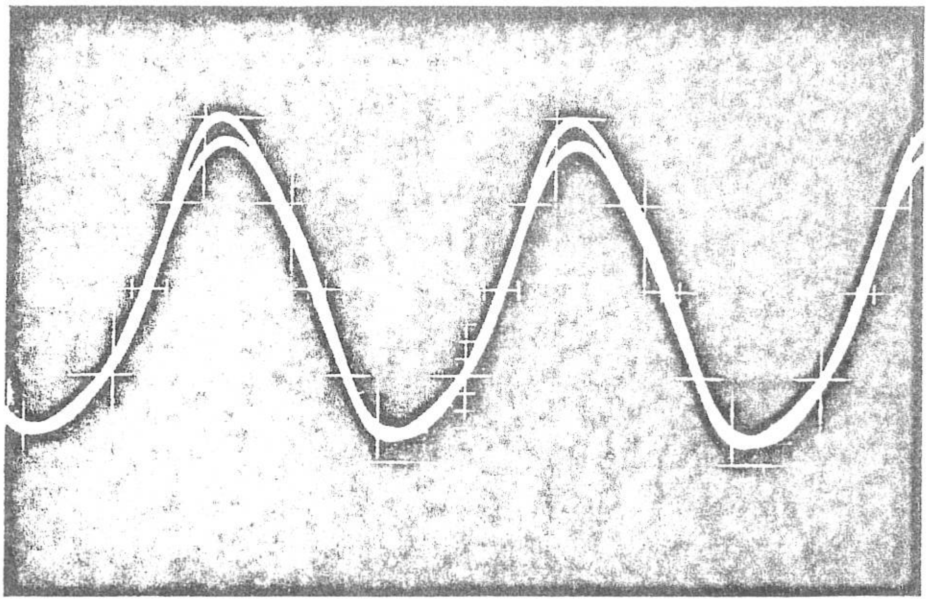

Fig. 8 Output waveforms of oscillators 3 and 4 in the coupled tube oscillator system

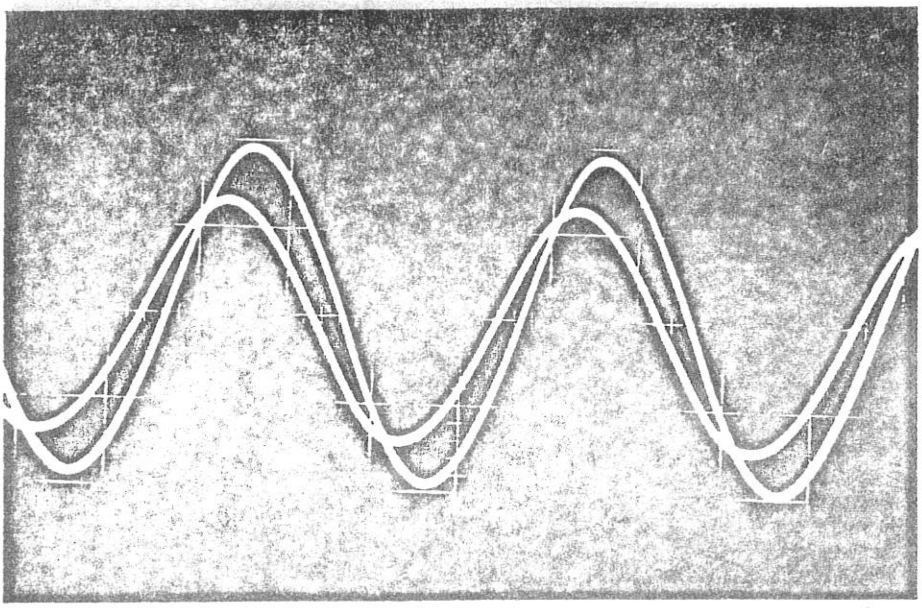

Fig. 9 Output waveforms of oscillators 1 and 5 which lie in a boundary ring of the structure

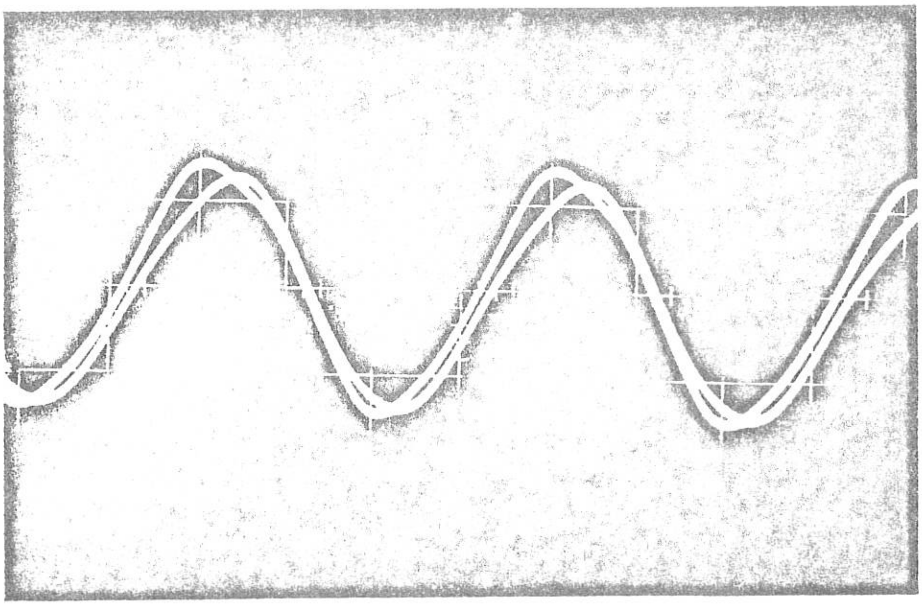

Fig. 10 Waveforms of oscillators 3 and 7 which lie in an inner ring of the tubular structure