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EXPANSION OF LINEAR AND SEMI-LINEAR SYSTEMS AND THE POPOV CRITERION

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1. Introduction

In this paper we shall consider the operation of system expansion in the case of linear and semilinear systems. The emphasis here is on stability theory with particular attention being given to an extension of the Popov criterion in Anderson's multivariable form (see Anderson (1965) and Tokumaru and Saito (1965)). The latter condition is very strong in requiring the positive reality of a certain matrix function. In the present work we shall seek to generalize this result by using system expansions so as to require only certain submatrices, which are to some extent open to choice, to be positive real.

There are, of course, many papers on large scale systems, for example Banks (1976), Callier (1978) where graph theoretic decompositions are used Banks (1979) and Araki (1975) where nonlinearly connected subsystems are considered. The idea of overlapping decompositions and system extension was introduced in many distinct areas of application: for example, in traffic control (Athans, 1967), economics (Aoki 1976) and power systems Siljak (1978).

In Section 2 the basic terminology is introduced and in Section 3 the extension of linear systems is considered leading to a definition of generalized extensions via 'expansive resolutions'. The notion of system extension is generalised in Section 4 to the case of semilinear systems where consideration is given to the multivariable Popov criterion. Finally an example is presented, which illustrates the application of system expansion to absolute stability.

2. Terminology

Throughout the paper we shall use the standard state-space representations of systems. However, in discussing expansions we shall introduce partitioned matrices and vectors. In the case of the vector $x = (x^1, x^2, \ldots x^n) \in R^n$, we shall use superscripts to denote individual components of x, to distinguish them from the subvectors of a partition of x, which will be denoted by subscripts. For example, if x is partitioned into α subvectors each of dimension less than (or equal to) n, then we write

$$x = (x_1, \ldots, x_\alpha).$$

Only when each x_i is of dimension 1, i.e. $\alpha = n$ will we write

$$x = (x_1, \ldots, x_n) = (x^1, \ldots, x^n),$$

these being no risk of confusion in this case. Similar remarks apply to partitioned matrices.

In the following discussion, T will denote a matrix of full column rank and T^I will denote its pseudoinverse (Nashed 1976). In this case,

$$T^I = (T^T T)^{-1} T^T$$

where $(.)^T$ denotes matrix transposition.

Recall (Anderson and Moore, 1968) that a matrix function Z(s) of a complex variable s is positive real if $Z(\infty) < \infty$, Z has a decomposition in the form

$$Z(s) = J + H^T(sI-F)^{-1}G$$

where F,G,H,J are real matrices and

$$Z(j\omega) + Z^T(-j\omega) \geqslant o \text{ for almost all real } \omega.$$

(i.e. the matrix on the left is non-negative definite for all real ω, apart, possibly, from poles of elements of Z(.)).

We shall also mention, briefly, the notion of Category, which is a collection (class) of objects $\mathcal{C}$ together with sets of morphisms $[A,B]$ for each pair of objects $A,B \in \mathcal{C}$, such that composition of morphisms $\beta\alpha$ is defined (if $\alpha \in [A,B]$, $\beta \in [B,C]$) and

(i) $\exists\ 1_A \in [A,A]$ such that $1_A\beta = \beta$, $\gamma 1_A = \gamma$

for $\beta \in [B,A]$, $\gamma \in [A,B]$ (existence of identities)

(ii) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ whenever both sides are defined. (MacLane, 1971).

An isomorphism in $\mathcal{C}$ is a morphism $\alpha \in [A,B]$ such that there exists an inverse morphism $\beta \in [B,A]$ such that $\alpha\beta = 1_B$, $\beta\alpha = 1_A$.

3. Expansion of Linear Systems

Consider the linear system

$$S: \dot{x} = Ax \ , \ x(o) = x_o \in R^n \tag{3.1}$$

and introduce the expansion $\tilde{x}$ of x by

$$\tilde{x} = Tx$$

where T is of full column rank. Then we consider systems

$$\tilde{S}: \dot{\tilde{x}} = \tilde{A}\tilde{x}, \ \tilde{x}(o) = \tilde{x}_o \in R^{\tilde{n}} \tag{3.2}$$

where $\tilde{x}_o = Tx_o$, and

$$\tilde{A} = T\,A\,T^I + M \tag{3.3}$$

for some matrix $M \in R^{\tilde{n}^2}$, as in Ikeda and Siljak (1979). For a given extension matrix T, it is desirable to have a characterization of the matrices M such that

$$x(t;x_o) = T^I\,\tilde{x}(t;\tilde{x}_o) \tag{3.4}$$

whenever $\tilde{x}_o = Tx_o$. (In other words, if the initial condition of system $\tilde{S}$ is obtained from that of system S by expanding with T, then the solution of S may be obtained fromthat of $\tilde{S}$ by applying T^I). Such a criterion is given by Ikeda and Siljak (1979) in the form

$$T^I\,M^i T = O \qquad \text{for} \quad 1 \leq i \leq n \ . \tag{3.5}$$

However, this condition is not sufficient for (3.4), as can be seen by considering the scalar equation

$$\dot{x} = x \ , \ x(o) = x_o \in R$$

and the trivial expansion

$$\dot{\tilde{x}} = (T\,T^I + M)\tilde{x} \ , \ \tilde{x}(o) = \tilde{x}_o \in R^2 \tag{3.6a}$$

defined by $T = (1,1)^T$. Then the conditions (3.5) reduce to the single condition (since n = 1):

$$m_1 + m_2 + m_3 + m_4 = O, \text{ for } M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in R^4. \tag{3.6b}$$

In particular, $M = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ satisfies the condition (3.6b), but $\{\exp(T\,T^I + M)t\}\tilde{x}_o = e^t\, x_o\left[\cos \tfrac{1}{2}t - \sin \tfrac{1}{2}t,\ \sin \tfrac{1}{2}t + \cos \tfrac{1}{2}t\right]$, if $\tilde{x}_o = T\,x_o$ and so the solution of (3.6a), when 'contracted' by T^I is

$$T^I\,\tilde{x}(t;\tilde{x}_o) = e^t\,x_o\,\cos \tfrac{1}{2}t \neq e^t\,x_o = x(t\,;x_o).$$

A complete characterization of the matrices M is given in the next result.

<u>Proposition 3.1</u> The system $\tilde{S}$ is an expansion of the system S if and only if

$$\det M = 0 \text{ and } T^I\,M^i\,T = 0\,,\ 1 \leq i \leq \tilde{n}-1\,. \tag{3.7}$$

<u>Proof:</u> $\tilde{S}$ is an expansion of S if $\tilde{x}_o = T\,x_o$ implies that

$$x(t;x_o) = T^I\,\tilde{x}(t;\tilde{x}_o)\ ,\ \text{for all } t \geq 0,$$

which holds if and only if

$$e^{At} = T^I\,e^{\tilde{A}t}\,T\ ,\ \text{for all } t \geq 0\,.$$

Now,

$$T^I\,e^{\tilde{A}t}\,T = T^I\,e^{(TAT^I + M)t}\,T$$

$$= T^I\,\{I_{\tilde{n}} + (TAT^I + M)\,t + \frac{1}{2!}\,(TAT^I + M)^2 t^2 + \ldots\}\,T$$

$$= I_n + At + \frac{1}{2!}\,A^2 t^2 + \ldots$$

$$+ (T^I MT)\,t + \frac{1}{2!}\,(T^I TAT^I MT + T^I MTAT^I\,T + T^I M^2 T)\,t^2$$

$$+ \ldots.$$

If the right hand side equals e^{At}, then, by equating coefficients of t, it follows that

$$T^I MT = 0\,.$$

However, the coefficient of t^2 is now just $T^I M^2 T$ which must also be zero. Continuing in this way up to the ciefficient of $t^{\tilde{n}-1}$, it follows that

$$T^I M^i T = 0\ ,\ 1 \leq i \leq \tilde{n}-1\,. \tag{3.8}$$

Now the coefficient of $t^{\tilde{n}}$ must be $T^I M^{\tilde{n}} T = (-1)^{\tilde{n}-1} T^I (\det M) T$, by (3.8) and the Cayley-Hamilton theorem, and so detM = 0. The conditions (3.7) are therefore necessary. They are clearly sufficient and so the result is proved. □

The application of system expansion to the study of stability has been demonstrated by Ikeda and Siljak (1979), where it is shown that if the system

$$S: \frac{d}{dt}\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_m \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{m1} & \cdots & A_{mm} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_m \end{pmatrix}$$

where $x = (x_1 \dots x_m)^T$ is a partition of $x \in R^n$ and $A = (A_{ij})$ is the corresponding partition of A, is expanded to the system

$$\tilde{S}: \frac{d}{dt}\begin{pmatrix} \tilde{x}_1 \\ \cdot \\ \cdot \\ \tilde{x}_{\tilde{m}} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{1\tilde{m}} \\ \cdot & \cdots & \cdot \\ \tilde{A}_{\tilde{m}1} & \cdots & \tilde{A}_{\tilde{m}\tilde{m}} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \cdot \\ \cdot \\ \tilde{x}_{\tilde{m}} \end{pmatrix},$$

then the stability of $\tilde{S}$ (and therefore that of S) may be demonstrated by a Lyapunov function

$$V = \sum_{i=1}^{\tilde{m}} \tilde{x}_i^T H_i \tilde{x}_i$$

where H_i is chosen to be the identity matrix in the transformed state space where A_i is diagonal. (c.f. Šiljak (1978)). It is also shown that this method will work when that of disjoint decomposition due to Araki (1975) does not.

The method therefore has genuine applications and it is natural to ask if one may expand the system $\tilde{S}$ further. Suppose then that

$$\tilde{S} : T_1 A T_1^I + M_1 = \tilde{A}$$

is an expansion of the system S defined by A, and

$$\tilde{\tilde{S}} : T_2 \tilde{A} T_2^I + M_2 = \tilde{\tilde{A}}$$

is an expansion of $\tilde{S}$. Then

$$\tilde{\tilde{A}} = T_2 T_1 A T_1^I T_2^I + T_2 M_1 T_2^I + M_2 . \qquad (3.9)$$

Of course, if $\tilde{x}_o = T_1 x_o$ and $\tilde{\tilde{x}}_o = T_2 \tilde{x}_o = T_2 T_1 x_o$, then

$$x(t ; x_o) = T_1^I \tilde{x} (t; \tilde{x}_o) = T_1^I T_2^I \tilde{\tilde{x}}(t ; \tilde{\tilde{x}}_o) ,$$

and so $\tilde{\tilde{S}}$ is a generalized extension of S.

<u>Proposition 3.2</u> If

$$(T_2 T_1)^I = T_1^I T_2^I \qquad (3.10)$$

holds for the expansion matrices T_1, T_2, then $\tilde{\tilde{S}}$ is an expansion of S; i.e. $\tilde{\tilde{A}}$ may be written in the form

$$\tilde{\tilde{A}} = T A T^I + M .$$

<u>Proof</u> From (3.9)

$$\tilde{\tilde{A}} = T A T^I + M$$

where

$$T = T_2 T_1 , M = T_2 M_1 T_2^I + M_2 .$$

If the orders of S , $\tilde{S}$ and $\tilde{\tilde{S}}$ are n , $\tilde{n}$ and $\tilde{\tilde{n}}$ respectively, then we must show that

$$T^I M^i T = 0 \text{ for } 1 \leq i \leq \tilde{n}-1 \qquad (3.11)$$

and

$$\det M = 0, \qquad (3.12)$$

by proposition 3.1. Consider (3.11); for i=1,

$$T^I M T = T_1^I T_2^I T_2 M_1 T_2^I T_2 T_1 + T_1^I T_2^I M_2 T_2 T_1$$

$$= T_1^I M_1 T_1 + T_1^I 0 T_1$$

$$= 0 .$$

The remaining conditions for i > 1 follow in a similar way by induction.

To prove (3.12) note that since, in fact, $T_1^I M_1^i T = 0$ and $T_2^I M_2^i T = 0$ for all i, it follows as above that $T^I M^i T = 0$ for $i \geq 1$ and hence by the Cayley Hamilton thm. $T^I \det M\, T = 0$ and so $\det M = 0$. □

<u>Remark 3.3</u> A simple case where (3.10) holds is when $T_2^T T_2 = I$; for example, with

$$T_2 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_2 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

, for appropriate identity matrices I_m. For,

$$\begin{aligned} (T_2T_1)^I &= ((T_2T_1)^T T_2T_1)^{-1}(T_2T_1)^T \\ &= (T_1^T T_2^T T_2 T_1)^{-1}(T_1^T T_2^T) \\ &= (T_1^T T_1)^{-1} T_1^T T_2^T \\ &= T_1^I T_2^I . \end{aligned}$$

It should be noted, however, that (3.10) does not hold for all matrices T_1, T_2 of full rank. For example, if

$$T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

then $(T_2T_1)^I = \frac{1}{14}(1\ 2\ 3)$, whereas $T_1^I T_2^I = (0, \frac{1}{5}, \frac{1}{5})$.

It has been shown in proposition 3.2 that a generalized extension of S by T_1 followed by T_2 is equivalent to an (ordinary) extension of S by T_2T_1, provided $(T_2T_1)^I = T_1^I T_2^I$. If (3.10) does not hold, then it may be advantageous to consider further extensions of S. We therefore introduce the following

<u>Definition 3.4</u> Let S be a linear system defined by the matrix A of order nxn.

Then, by an <u>expansive resolution of S</u> (of length ℓ), we shall mean a sequence of linear maps (i.e. matrices) T_i

$$R^n \xrightarrow{T_1} R^{n_1} \xrightarrow{T_2} R^{n_2} \xrightarrow{T_3} \;\cdots\; \xrightarrow{T_\ell} R^{n_\ell}$$

where $n < n_1 < n_2 < \ldots < n_\ell$ and each T_i is of full rank, together with matrices M_i of order $n_i \times n_i$ such that

$$\det M_i = 0 \;, \quad T_i^I M_i^j T_i = 0 \quad \text{for} \quad 1 \leq j \leq n_i - 1 \;, \quad 1 \leq i \leq \ell .$$

In view of proposition 3.2, it is also natural to require in this definition that

$$(T_i T_{i-1})^I \neq T_{i-1}^I \; T_i^I$$

for all $i \in \{2,\ldots,\ell\}$. The expanded systems with respect to this resolution are given inductively by

$$A_i = T_i \; A_{i-1} \; T_i^I + M_i \;, \; 1 \leq i \leq \ell,$$

where $A_0 = A$.

There is a category $\mathcal{C}$ (cf. MacLane 1971) of expansive resolutions whose morphisms are commutative diagrams of the form

$$\begin{array}{ccccccccc} R^n & \xrightarrow{T_1} & R^{n_1} & \xrightarrow{T_2} & R^{n_2} & \xrightarrow{T_3} & \cdots & \xrightarrow{T_\ell} & R^{n_\ell} \\ \downarrow Q_0 & & \downarrow Q_1 & & \downarrow Q_2 & & & & \downarrow Q_\ell \\ R^n & \xrightarrow{S_1} & R^{n_2} & \xrightarrow{S_2} & R^{n_2} & \xrightarrow{S_3} & \cdots & \xrightarrow{S_\ell} & R^{n_\ell} \end{array} \qquad \begin{array}{ccc} R^{n_i} & \xrightarrow{M_i} & R^{n_i} \\ \downarrow Q_i & & \downarrow Q_i \\ R^{n_i} & \xrightarrow{L_i} & R^{n_i} \\ & (1 \leq i \leq \ell) & \end{array} \qquad (3.13)$$

Equivalences in this category are morphisms where $Q, Q_i, 1 \leq i \leq \ell$ are invertible. To determine the structure of an equivalent expansion for a given expansion let

$$\dot{x} = A\,x$$

be a given system with expansive resolution

$$R^n \xrightarrow{T_1} R^{n_1} \xrightarrow{T_2} R^{n_2} \;\ldots\; \xrightarrow{T_\ell} R^{n_\ell}$$

and let $(Q_o, Q_1, \ldots, Q_\ell)$ be an isomorphism in $\mathcal{C}$. Then

$$\dot{x}_i = T_i A_{i-1} T_i^I x_i + M_i x_i \quad , \quad 1 \leq i \leq \ell$$

(where $x_1 = x$) is the i^{th} expansion and if we put $y_i = Q_i^{-1} x_i$ then we obtain the system

$$\dot{y}_i = Q_i^{-1} T_i A_{i-1} T_i^I Q_i y_i + Q_i^{-1} M_i Q_i$$

$$= Q_i^{-1} T_i Q_{i-1} Q_{i-1}^{-1} A_{i-1} Q_{i-1} Q_{i-1}^{-1} T_i^I Q_i y_i + Q_i^{-1} M_i Q_i$$

for $1 \leq i \leq \ell$. Then, by commutativity of (3.13) (with the rôles of S and T reversed), we have

$$\dot{y}_i = S_i Q_{i-1}^{-1} A_{i-1} Q_{i-1} S_i^I y_i + Q_i^{-1} M_i Q_i y_i$$

provided

$$(Q_i^{-1} T_i Q_{i-1})^I = Q_{i-1}^{-1} T_i^I Q_i \, . \tag{3.14}$$

This condition will be satisfied, for example, if

$$Q_i^T Q_i = I \quad , \quad o \leq i \leq \ell \tag{3.15}$$

i.e. all the matrices Q in the isomorphism are orthogonal, since then, for any matrix H of full rank and appropriate size,

$$(H\,Q_i)^I = ((HQ_i)^T HQ_i)^{-1} (HQ_i)^T$$

$$= (Q_i^T H^T H\,Q_i)^{-1} Q_i^T H^T$$

$$= Q_i^{-1} (H^T H)^{-1} (Q_i^{-1})^T Q_i^T H^T$$

$$= Q_i^{-1} H^I \, ,$$

and

$$(Q_iH)^I = ((Q_iH)^T Q_iH)^{-1}(Q_iH)^T$$

$$= (H^TQ_i^TQ_iH)^{-1}\ H^TQ_i^T$$

$$= (H^TH)^{-1}H^TQ_i^{-1}$$

$$= H^IQ_i^{-1}\ .$$

Hence,

$$\dot{y}_i = S_iA'_{i-1}S_i^Ty_i + L_iy_i$$

where $A'_{i-1} = Q_{i-1}^{-1}A_{i-1}Q_{i-1}$. Thus, an isomorphism in the category $\mathcal{C}$ is a set of commutative diagrams of the form (3.13) for which (3.14) holds and the expanded systems have matrices which are similar and states which are related by the coordinate transformations $y_i = Q_i^{-1}x_i$. This therefore gives us a generalisation of the classical notion of equivalence of systems, namely those whose defining matrices are similar.

Clearly the main application to systems theory of expansive resolutions is to 'unlock' certain properties of a lower order representation and present them in a tractible form in higher order expansions. In the next section we shall go on to discuss the expansion of nonlinear systems, but before doing this we mention an elementary application of expansions to instability conditions for certain large scale systems. We have mentioned above that the stability of an expansion $\tilde{S}$ of a system S implies the stability of S; it follows that any unstable system S can only have unstable extensions. This allows us to generate large scale systems which can be recognised immediately as being unstable. As a trivial example consider the scalar system.

$$\dot{x} = x.$$

Consider the extension defined by $T = (\alpha\ \beta\ \gamma)^T$ and $M = 0$.

Then

$$T^I = (T^T T)^{-1} T^T = (\alpha \ \beta \ \gamma) \frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

and

$$\tilde{A} = T \ a \ T^I + M = \frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix} \tag{3.16}$$

and so any system of the form

$$\dot{\tilde{x}} = \tilde{A} \ \tilde{x}$$

where $\tilde{A}$ is of the form (3.16) is unstable. In this trivial case, of course, it is easy to see that the eigenvalues of (3.16) are $0, 0, (\alpha^2+\beta^2+\gamma^2)$. However by choosing, for example,

$$M = \begin{pmatrix} 0 & m_1 & 0 \\ 0 & m_2 & 0 \\ 0 & m_3 & 0 \end{pmatrix}$$

where $\alpha m_1 + \beta m_2 + \gamma m_3 = 0$, then we see that the system

$$\dot{\tilde{x}} = \begin{pmatrix} \alpha^2 & \alpha\beta + m_1 & \alpha\gamma \\ \alpha\beta & \beta^2 + m_2 & \beta\gamma \\ \alpha\gamma & \beta\gamma + m_3 & \gamma^2 \end{pmatrix} \tilde{x}$$

is also unstable and this is not so obvious without a tedious computation of the eigenvalues.

4. Expansion of Semi-Linear Systems

Having considered the expansion of linear systems, we now would like to see to what extent this technique can be applied to nonlinear systems. Suppose then that

$$S : \dot{x} = A x \tag{4.1}$$

is a linear system and let

$$S_p : \dot{x} = Ax + f(x) \ , \ x(o) = x_o \in R^n \tag{4.2}$$

be a nonlinear perturbation of S, where $f: R^n \to R^n$ is assumed to satisfy sufficient conditions for (4.2) to have unique and (for simplicity of exposition) global solutions for any initial value x_o. As in Section 3, let

$$\tilde{S} : \dot{\tilde{x}} = \tilde{A}\,\tilde{x} \tag{4.3}$$

be an expansion of S together with a nonlinear perturbation

$$\tilde{S}_p : \dot{\tilde{x}} = \tilde{A}\,\tilde{x} + \tilde{f}(\tilde{x}) \;,\; \tilde{x}(o) = \tilde{x}_o = T\,x_o \tag{4.4}$$

where the notations are as in §3. S and $\tilde{S}$ will be called the linearizations of S_p and $\tilde{S}_p$, respectively. The next result gives a sufficient condition for $\tilde{S}_p$ to be an expansion of S_p.

<u>Proposition 4.1</u> Suppose that $\tilde{S}$ is an expansion of S. Then $\tilde{S}_p$ is an expansion of S_p (i.e. $\tilde{x}_o = T\,x_o \Rightarrow x(t) = T^I\,\tilde{x}(t)\,,\; t \geq o$) if $\tilde{f}(\tilde{x}) = T\,f(T^I\tilde{x})$, which means that the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\;f\;} & R^n \\ {\scriptstyle T^I}\uparrow & & \downarrow{\scriptstyle T} \\ R^{\tilde{n}} & \xrightarrow{\;\tilde{f}\;} & R^{\tilde{n}} \end{array} \tag{4.5}$$

is commutative.

<u>Remark 4.2</u> It should be recalled that $T^I T = I_n$, but $T\,T^I \neq I_{\tilde{n}}$ and so the condition of proposition 4.1 implies, but is not implied by, the condition $T^I\tilde{f}\,(\tilde{x}) = f(T^I\tilde{x})$; hence the right-hand vertical arrow in (4.5) cannot be reversed.

<u>Proof of Proposition 4.1</u> As we have seen, the linearized system $\tilde{S}$ is an expansion of S only if $e^{At} = T^I\,e^{\tilde{A}t}\,T$ for all $t \geq o$. Now, by the variation of constants formula, the solution of (4.4) is given by

the integral equation

$$\tilde{x}(t) = e^{\tilde{A}t}\,\tilde{x}_o + \int_o^t e^{\tilde{A}(t-s)}\,\tilde{f}(\tilde{x})\,ds \ .$$

Hence,

$$T^I\,\tilde{x}(t) = T^I\,e^{\tilde{A}t}\,T\,x_o + \int_o^t T^I\,e^{\tilde{A}(t-s)}\,T\,f(T^I\tilde{x})\,ds$$

$$= e^{At}\,x_o + \int_o^t e^{A(t-s)}\,f(T^I\tilde{x})\,ds$$

if $\tilde{x}_o = T\,x_o$. However, by the uniqueness of solutions, it follows that $x(t) = T^I\,\tilde{x}(t)$ for all $t \geqslant o$, which proves the result. □

<u>Remark 4.3</u> We have seen that the solutions $\tilde{x}(t)$, $x(t)$ of systems $\tilde{S}_p$ and S_p are related by $x(t) = T^I\tilde{x}(t)$ provided $\tilde{x}(t_o) = T\,x(t_o)$. One could also consider the connection between $\tilde{x}(t)$ and $T\,x(t)$. In general, of course, $\tilde{x}(t) \neq T\,x(t)$; however, they are equal if $MT = 0$, since

$$\dot{x}(t) = Ax + f(x)$$

and so

$$T\dot{x}(t) = T\,AT^ITx + T\,f(x)$$

since $T^IT = I$. Now, $x(t) = T^I\tilde{x}(t)$ for any expansion, so

$$\begin{aligned} T\dot{x}(t) &= (T\,AT^I)Tx + T\,f(T^I\,\tilde{x}) \\ &= (T\,AT^I)Tx + \tilde{f}(\tilde{x}) \\ &= (T\,AT^I)Tx + MTx + \tilde{f}(\tilde{x}) \quad (MT = 0) \\ &= \tilde{A}\,Tx + \tilde{f}(\tilde{x}) \end{aligned}$$

which implies, by uniqueness of solutions, that

$$T\,x(t) = \tilde{x}(t) . \ □$$

As an example of the condition (4.5) consider the expansion defined by

$$T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} , \quad T^I = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_2 & \frac{1}{2}I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$$

where the I's are identity matrices of various sizes, and partition the states and nonlinearities correspondingly, as follows:

$$x = (x_1, x_2, x_3)^T \ , \ \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)^T$$
$$f(x) = (f_1(x), f_2(x), f_3(x))^T, \ \tilde{f}(\tilde{x}) = (\tilde{f}_1(\tilde{x}), \tilde{f}_2(\tilde{x}), \tilde{f}_3(\tilde{x}), \tilde{f}_4(\tilde{x}))^T.$$

Then the diagram (4.5) commutes if

$$\begin{aligned}
\tilde{f}_1(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) &= f_1(\tilde{x}_1, \tfrac{1}{2}\tilde{x}_2 + \tfrac{1}{2}\tilde{x}_3, \tilde{x}_4) \\
\tilde{f}_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) &= f_2(\tilde{x}_1, \tfrac{1}{2}\tilde{x}_2 + \tfrac{1}{2}\tilde{x}_3, \tilde{x}_4) \\
\tilde{f}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) &= f_2(\tilde{x}_1, \tfrac{1}{2}\tilde{x}_2 + \tfrac{1}{2}\tilde{x}_3, \tilde{x}_4) \\
\tilde{f}_4(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) &= f_3(\tilde{x}_1, \tfrac{1}{2}\tilde{x}_2 + \tfrac{1}{2}\tilde{x}_3, \tilde{x}_4) \ ,
\end{aligned}$$

i.e. if each function $\tilde{f}_i$ is in fact a function of the three variables $\tilde{x}_1, \tilde{x}_2 + \tilde{x}_3, \tilde{x}_4$ and $\tilde{f}_2 \equiv \tilde{f}_3$.

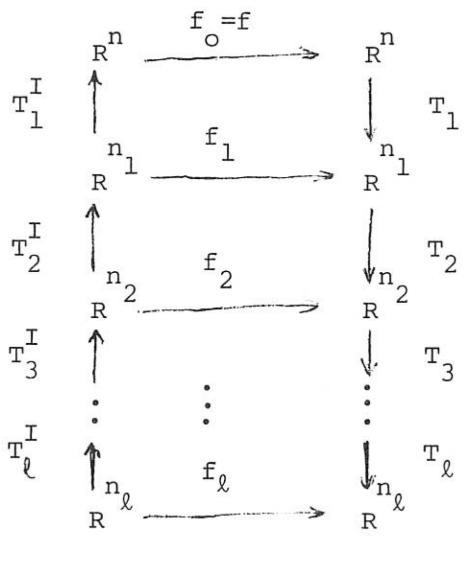
As in Section 3 we can consider generalized extensions of the nonlinear systems S_p, which will consist, as before, of an expansive resolution

$$R^n \xrightarrow{T_1} R^{n_1} \xrightarrow{T_2} R^{n_2} \quad \cdots \xrightarrow{T_\ell} R^n$$

together with matrices $M_i \in R^{n_i \times n_i}$ for which

$$\det M_i = 0 \ , \ T_i^I M_i^j T_i = 0 \ , \ 1 \leq j \leq n_i - 1 \ , \ 1 \leq i \leq \ell$$

and also a sequence of nonlinearities f_i, $0 \leq i \leq \ell$, where $f_0 = f$, such that the diagram



(4.6)

commutes.

In studying the stability of a system of the form (4.2) it is common to use a Lyapunov function of the general type

$$V = x^T Hx + \int_0^x f(x')dx' ,$$

for some positive definite matrix H, where

$$\int_0^x f(x')dx' = \sum_{i=1}^n \int_0^{x_i} f_i(x_i') \, dx_i' \; , \; x' = (x_1',\ldots,x_n')^T$$

provided $f_i(x') \equiv f_i(x_i')$, i.e. the i^{th} component function of f is a function of x_i' only. This construction is well-known in the scalar case, but its application in the n-dimensional case requires the following result of Anderson (1967).

Lemma 4.4 Let Z(.) be a matrix of rational transfer functions such that $Z(\infty)$ is finite and Z has poles which lie in Re $s < 0$ or are simple on Re $s = 0$. Let $\{F,G,H,Z(\infty)\}$ be a minimal realization of Z. Then Z(.) is positive real if and only if there exists a symmetric positive definite P and matrices W_o and L such that

$$PF + F'P = - L L'$$

$$PG = H - L W_o$$

$$W_o' W_o = Z(\infty) + Z'(\infty) . \;\square$$

Note that a matrix Z(.) is positive real if

(i) Z(s) has elements which are analytic for Re $s > 0$

(ii) $Z^*(s) = Z(s^*)$ for Re $s > 0$ (4.7)

(iii) $Z'(s^*) + Z(s)$ is nonnegative definite for Re $s > 0$.

Then we have the following theorem (c.f. Csáki (1972)):

Theorem 4.5 Consider the system defined by the equations

$$\dot{x} = A x + B g(e)$$

$$y = -e = Cx \qquad (4.8)$$

where $x(t) \in R^n$, A,B,C are nxn, nxm and mxn matrices respectively which form a minimal realization of G(s) and suppose that, for some $\delta_o > 0$ there exist diagonal matrices $R = \text{diag}\,[r_1,\ldots,r_m]$ and $Q = \text{diag}\,[q_1,\ldots,q_m]$ such that $r_i \geq 0$, $q_i \geq 0$, $r_i + q_i > 0$ where $-r_i/q_i$ is no pole of any one of the elements in the i^{th} row of the matrix $G(s-\delta_o)$ and the poles of G(s) are within the region $\text{Re}\, s < -\delta_o$. If

$$\Pi_o(s) = [R + Qs]\, G(-\delta_o + s) + RK^{-1}$$

is positive real, where $K = \text{diag}\,[K_1,\ldots,K_m]$ is such that

$$0 < \frac{g_i(e_i)}{e_i} < K_i \quad (1 \leq i \leq m),$$

then the system is absolutely stable and its transients are damped no slower than $\exp(-\delta_o t)$.

Proof. An outline proof of this theorem will be given for convenience. We have

$$G(s) = C(sI-A)^{-1}B$$

and

$$\begin{aligned}\Pi_o(s) &= (R + Qs)G(s-\delta_o) + RK^{-1}\\ &= RC(sI-A-\delta_o I)^{-1}B + QC[(sI-A-\delta_o I)+A+\delta_o I]\,[sI-A-\delta_o I]^{-1}B+RK^{-1}\\ &=[RC + QC(A+\delta_o I)]\,[sI-A-\delta_o I]^{-1}B+[RK^{-1}+QCB].\end{aligned}$$

Hence, the matrices

$$A + \delta_o I,\ B,\ RC + QC[A + \delta_o I]$$

form a minimal realization of $\Pi_o(s) - \Pi_\infty(s)$, where

$$\Pi_\infty(s) = RK^{-1} + QCB\ .$$

It follows from lemma 4.4 that there exist matrices P,L and G_o such that

$$P[A + \delta_o I] + [A + \delta_o I]^T P = - L L^T$$

$$RC + QC[A + \delta_o I] - G_o^T L^T = B^T P$$

$$2RK^{-1} + QCB + B^T C^T Q = G_o^T G_o .$$

Defining the Lyapunov function

$$V(x) = x^T Px + 2 \int_o^e g^T(e)\ Qde \tag{4.9}$$

it follows that

$$\dot{V}(x) = -[x^T L + g^T(e) G_o^T][L^T x + G_o g(e)]$$

$$- 2g^T(e)\ R[e - K^{-1} g(e)] - 2\delta_o [x^T Px + g^T(e) Qe],$$

and the result is proved. □

The condition of positive reality on the matrix function $\Pi_o(s)$ above is a strong restriction and is unlikely to hold for an arbitrary system under investigation. However, using a judicious expansion of the linear part of the system, the condition may be made to hold for the diagonal matrices in the block structure of the expanded linearized system A_ℓ. We shall now examine the possibility of such an expansion. Consideration will be restricted to (simple) expansions $\tilde{S}_p$ of S_p as in (4.1 - 4.4), the results being easily extended to expansive resolutions as in (4.6). Also, for convenience of exposition we shall assume complete observation so that the system (4.8) takes the form of (4.2) with B = I, C = I. Again, the more general version can easily be obtained from the following considerations.

Suppose, then, that as above x and $\tilde{x}$ are partitioned in the form

$$x = (x_1, \ldots, x_\alpha)\ ,\ \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_\beta)\ ,\quad 1 \leq \alpha \leq n,\ 1 \leq \beta \leq \tilde{n}.$$

We shall also suppose that the corresponding matrices and nonlinearities are similarly partitioned, e.g.

$$\tilde{A} = (\tilde{A}_{ij}) \ , \ \tilde{f} = (\tilde{f}_i) \ , \ 1 \leq i, j \leq \beta .$$

In choosing a Lyapunov function for the expanded system consider first the term $x^T P x$ in (4.9), corresponding to the linear part of the system. For each matrix $\tilde{A}_{ii}$ on the diagonal of $\tilde{A}$ we consider the transfer function

$$G_{ii}(s) = (sI-\tilde{A}_{ii})^{-1} \ , \ 1 \leq i \leq \beta$$

where I is an identity matrix of appropriate size. Suppose that there exist diagonal matrices R_i and K_i, matrices $\bar{Q}_{ii}$ (not necessarily diagonal) and numbers δ_o^i such that the matrix function

$$\Pi_o^i(s) = [R_i + \bar{Q}_{ii}s]G(-\delta_o^i + s) + R_i K_i^{-1} \ , \ 1 \leq i \leq \beta$$

is positive real, and $(\tilde{A}_{ii} + \delta_o^i I \ , \ \bar{Q}_{ii}[\tilde{A}_{ii} + \delta_o^i I])$ is an observable pair.

Then, again by lemma 4.4, there exists symmetric positive definite matrices P_i and matrices G_o^i and L_i such that

$$P_i[\tilde{A}_{ii} + \delta_o^i I] + [\tilde{A}_{ii} + \delta_o^i I]^T P_i = - L_i L_i^T$$

$$R_i + \bar{Q}_{ii} [\tilde{A}_{ii} + \delta_o^i I] - (G_o^i)^T L_i^T = P_i \qquad (4.10)$$

$$2R_i K_i^{-1} + 2\bar{Q}_{ii} = (G_o^i)^T G_o^i \ ,$$

for $i \in \{1,\ldots,\beta\}$. We shall then take

$$V = \sum_{i=1}^{\beta} \tilde{x}_i^T P_i \tilde{x}_i + 2 \int_o^{-x} f^T(x) \ Q \ dx$$

as a tentative Lyapunov function for the system $\tilde{S}_p$. Note that, in order to bring $\tilde{S}$ and $\tilde{S}_p$ in line with equations (4.8), we shall replace

$f(x)$ by $f(-x)$ and write

$$S_p : \dot{x} = Ax + f(-x)$$

$$\tilde{S}_p : \dot{\tilde{x}} = \tilde{A}x + \tilde{f}(-\tilde{x})$$

This is, of course, merely a notational change. In the expression for V, Q is a diagonal matrix which will be chosen later, except the diagonal elements are assumed positive. If each f_i is in the first and third x^i quadrants, then V is clearly positive. Now,

$$\dot{V} = \sum_{i=1}^{\beta} \dot{\tilde{x}}_i^T P_i \tilde{x}_i + \sum_{i=1}^{\beta} x_i^T P_i \dot{\tilde{x}}_i - 2 f^T(-x) Q \dot{x}.$$

However, if we assume that M is chosen to satisfy $MT = 0$, then by Remark 4.3, we have

$$2 f^T(x) Q\dot{x} = 2 \tilde{f}^T(\tilde{x}) (T^I)^T Q T^I T \dot{x}$$

$$= 2 \tilde{f}^T(\tilde{x}) \overline{Q} \dot{\tilde{x}}$$

since, as we have seen, $\tilde{x} = Tx$, $\tilde{x} = T^I x$ for any solutions $x, \tilde{x}$ of $S_p, \tilde{S}_p$, respectively, when $MT = 0$ holds. Here,

$$\overline{Q} = (T^I)^T Q T^I.$$

Put

$$\overline{Q} = (\overline{Q}_{ij}) \quad (1 \leq i,j \leq \beta).$$

Then,

$$\dot{V} = \sum_{i=1}^{\beta} \dot{\tilde{x}}_i^T P_i \tilde{x}_i + \sum_{i=1}^{\beta} \tilde{x}_i^T P_i \dot{\tilde{x}}_i - 2\tilde{f}^T(-\tilde{x}) \overline{Q} \dot{\tilde{x}}$$

$$= \sum_{i=1}^{\beta} \sum_{j=1}^{\beta} \tilde{x}_j^T \tilde{A}_{ij}^T P_i \tilde{x}_i + \sum_{i=1}^{\beta} \sum_{j=1}^{\beta} \tilde{x}_i^T P_i \tilde{A}_{ij} \tilde{x}_j$$

$$+ \sum_{i=1}^{\beta} \tilde{f}_i^T(-\tilde{x}) P_i \tilde{x}_i + \sum_{i=1}^{\beta} \tilde{x}_i^T P_i \tilde{f}_i(-\tilde{x})$$

$$- 2 \sum_{i=1}^{\beta} \sum_{j=1}^{\beta} \tilde{f}_i^T(-\tilde{x}) \overline{Q}_{ij} \left(\sum_{k=1}^{\beta} \tilde{A}_{jk} \tilde{x}_k + \tilde{f}_j(-\tilde{x}) \right)$$

$$= \sum_{i=1}^{\beta} \{-[\tilde{x}_i^T L_i + \tilde{f}_i^T(-\tilde{x})(G_o^i)^T][L_i^T\tilde{x}_i + G_o^i\tilde{f}_i(-\tilde{x})]$$

$$- 2\,\tilde{f}_i^T(-\tilde{x})R_i[-\tilde{x}_i - K_i^{-1}\tilde{f}_i(-\tilde{x})] - 2\,\delta_o^i\tilde{x}_i^T P_i\tilde{x}_i\}$$

$$- 2\sum_{i\neq j=1}^{\beta} \tilde{f}_i^T(-\tilde{x})\overline{Q}_{ij}\,\tilde{f}_j(-\tilde{x})$$

$$+ \sum_{i\neq j=1}^{\beta} \tilde{x}_j^T\tilde{A}_{ij}^T\,P_i\tilde{x}_i + \sum_{i\neq j=1}^{\beta} \tilde{x}_i^T P_i\tilde{A}_{ij}\tilde{x}_j$$

$$- 2\sum_{\Omega} \tilde{f}_i^T(-\tilde{x})\overline{Q}_{ij}\tilde{A}_{jk}\tilde{x}_k + 2\sum_{i=1}^{\beta}\delta_o^i\,\tilde{f}_i^T(-\tilde{x})\overline{Q}_{ii}\tilde{x}_i \qquad (4.11)$$

where $\Omega = \{i,j,k: 1 \leq i,j,k \leq \beta,\ i,j,k \text{ not all equal}\}$.

However,

$$\sum_{i=1}^{\beta} \tilde{f}_i^T R_i[-\tilde{x}_i - K_i^{-1}\tilde{f}_i(-\tilde{x})] = \tilde{f}^T R\,[-\tilde{x}-K^{-1}\tilde{f}(-\tilde{x})]$$

$$= f^T(-x)T^T R\,[-Tx - K^{-1}T\,f(-x)].$$

$$= f^T(-x)T^T R\,T\,[-x - K'^{-1}f(-x)]$$

$$= f^T(-x)R'[-x - K'^{-1}f(-x)],$$

where $R = \text{diag}\,\{R_1,\ldots,R_\beta\}$, $K^{-1} = \text{diag}\,\{K_1^{-1},\ldots,K_\beta^{-1}\}$ and we assume that there exists a diagonal matrix $K' = \text{diag}\{k_1',\ldots,k_n'\}$ such that

$$K^{-1}T = T\,K'^{-1}$$

and $R' = T^T RT$. Suppose that all the diagonal elements of R' are positive and let R_o' be the matrix obtained from R' by setting the diagonal elements to zero. Then, if $o < f^i(x^i)/x^i < k_i'$,

$$\dot{V} < |f^T(-x)R_o'\,[-x - K'^{-1}\,f(-x)]| - \sum_{i=1}^{\beta} 2\,\delta_o^i\,\tilde{x}_i^T\,P_i\tilde{x}_i$$

$$+ \; 2 \sum_{i \neq j=1}^{\beta} ||\tilde{f}_i^T(-\tilde{x})|| \, ||\overline{Q}_{ij}|| \;\; ||\tilde{f}_j(-\tilde{x})||$$

$$+ \; 2 \sum_{i \neq j=1}^{\beta} ||\tilde{x}_j|| \;\; ||\tilde{A}_{ij}|| \;\; ||P_i|| \;\; ||\tilde{x}_i||$$

$$+ \; 2 \sum_{\Omega} ||\tilde{f}_i^T(-\tilde{x})|| \, ||\overline{Q}_{ij}|| \, ||\tilde{A}_{jk}|| \, ||\tilde{x}_k|| \;\; + 2 \sum_{i=1}^{\beta} \delta_o^i \; \tilde{f}_i^T(-\tilde{x})\overline{Q}_{ii}\tilde{x}_i \; .$$

Consider the terms $||\tilde{f}_i^T(-\tilde{x})||$; we have

$$\tilde{f}^T(-\tilde{x}) = f^T(-x)T^T$$

and so

$$\tilde{f}_i^T(-\tilde{x}) = \sum_{\ell=1}^{\alpha} f_\ell^T(-x)(T^T)_{\ell i} .$$

However, $f_\ell^T(-x)$ depends only on x_ℓ and so

$$||\tilde{f}_i^T(-\tilde{x})|| \leq \sum_{\ell=1}^{\alpha} ||f_\ell^T(-x)|| \, ||(T^T)_{\ell i}|| \leq \sum_{\ell=1}^{\alpha} ||K_\ell'|| \, ||x_\ell|| \, ||(T^T)_{\ell i}||$$

$$= \sum_{\ell=1}^{\alpha} ||K_\ell'|| \, || \sum_{m=1}^{\beta} (T^I)_{\ell m}\tilde{x}_m || \, ||(T^T)_{\ell i}||$$

$$\leq \sum_{\ell=1}^{\alpha} \sum_{m=1}^{\beta} ||K_\ell'|| \, ||(T^I)_{\ell m}|| \, ||(T^T)_{\ell i}|| \, ||\tilde{x}_m|| .$$

Since each matrix P_i is positive definite, there exist numbers ρ_i such that

$$\tilde{x}_i^T P_i \tilde{x}_i \geq \rho_i ||\tilde{x}_i||^2$$

and so

$$\dot{V} \leq - 2 \sum_{i=1}^{\beta} \delta_o^i \rho_i ||\tilde{x}_i||^2 + |f^T(-x)R_o'[-x-K'^{-1}f(-x)]| + 2 \sum_{i=1}^{\beta} \delta_o^i \tilde{f}_i^T(-\tilde{x})\overline{Q}_{ii}\tilde{x}_i$$

$$+ 2\kappa^2 \sum_{i\neq j=1}^{\beta} \sum_{m=1}^{\beta} \sum_{\ell=1}^{\beta} \tau_{mi} ||\tilde{x}_m|| \tau_{\ell j} ||\tilde{x}_\ell|| \, ||\bar{Q}_{ij}||$$

$$+ 2 \sum_{i\neq j=1}^{\beta} ||\tilde{x}_j|| \, ||\tilde{A}_{ij}|| \, ||P_i|| \, ||\tilde{x}_i||$$

$$+ 2\kappa \sum_{\Omega} \sum_{m=1}^{\beta} \tau_{mi} ||\tilde{x}_m|| \, ||\bar{Q}_{ij}|| \, ||\tilde{A}_{jk}|| \, ||\tilde{x}_k||, \qquad (4.12)$$

where $\kappa = \max \{||K_\ell'||\}$ and $\tau_{mi} = \sum_{\ell=1}^{\alpha} ||(T^I)_{\ell m}|| \, ||(T^T)_{\ell i}||$.

If the δ_o^i can be chosen independently of i, then the third term on the right of (4.12) is

$$2\delta_o \tilde{f}^T(-\tilde{x})\bar{Q}\tilde{x} - 2\delta_o \sum_{i\neq j=1}^{\beta} \tilde{f}_i^T(-\tilde{x})\bar{Q}_{ij}\tilde{x}_j$$

$$= 2\delta_o f^T(-x)Q x - 2\delta_o \sum_{i\neq j=1}^{\beta} \tilde{f}_i^T(-\tilde{x})\bar{Q}_{ij}\tilde{x}_j$$

$$\lesssim |2\delta_o \sum_{i\neq j=1}^{\beta} \tilde{f}_i^T(-\tilde{x})\bar{Q}_{ij}\tilde{x}_j| \lesssim 2\delta_o \sum_{i\neq j=1}^{\beta} \bar{\kappa} \sum_{m=1}^{\beta} \tau_{mi} ||\bar{Q}_{ij}|| \, ||\tilde{x}_m|| \, ||\tilde{x}_j||$$

and this term may be estimated in terms of κ and $||Q_{ij}||$ as above. The second term on the right hand side of (4.12) containing R_o' can also be written as a function of $\bar{\kappa}$ and $||\tilde{x}_i||$ and so can be put in the form of the remaining terms in (4.12). However, in the example we shall give shortly, $R_o' = 0$ and so, for simplicity, we shall omit this term. Of course, in case $R_o' \neq 0$ the term can be accounted for as above. It follows, therefore, from (4.12) that $\dot{V}$ may be bounded as follows:

$$\dot{V} \lesssim -2 \sum_{i=1}^{\beta} (\delta_o^i \rho_i - \varepsilon_i(\bar{\kappa})) ||\tilde{x}_i||^2 + \sum_{i\neq j=1}^{\beta} ||\tilde{x}_i|| \xi_{ij} ||\tilde{x}_j||,$$

$$= F(\kappa, ||\tilde{x}_i||), \text{ say,}$$

for some functions $\varepsilon_i(\kappa) \to 0$ as $\kappa \to 0$ and some numbers ξ_{ij}.

Let

$$E(\kappa) = \{\tilde{x} : F(\kappa, ||\tilde{x}_i||) < 0\} \cup \{0\}$$

If we can show that $E(\kappa)$ contains some open set D containing the origin, then the absolute stability of the system on D will follow and D will be an estimate of the domain of attraction. Of course, whether or not $E(\kappa)$ is empty will depend on the particular system and a judicious choice of expansion. We shall present next a simple example to illustrate the use of this theory. However, we shall first collect together the assumptions made above and state the result formally (notation as before):

<u>Theorem 4.6</u>. Consider a nonlinear system S_p and an expansion $\tilde{S}_p$ where $MT = 0$. Suppose that there exist diagonal positive matrices R_i, K_i, matrices $\overline{Q}_{ij}$ (not necessarily diagonal, where $\overline{Q} = (\overline{Q}_{ij})$ is of the form $(T^I)^T Q T^I$ for some diagonal positive matrix Q) such that

$$\left[R_i + \overline{Q}_{ii} s\right] G(-\delta_o^i + s) + R_i K_i^{-1}$$

(for some positive numbers δ_o^i) is positive real, and satisfies the above observability and controllability conditions. Suppose also that there exists a diagonal matrix K' such that $K^{-1}T = TK'^{-1}$ and

$$0 < f^i(x^i)/x^i < k_i' \quad , \quad 1 \leq i \leq n.$$

Then, if there exists a number κ such that $E(\kappa)$ contains a connected open set containing the origin, then the system $\tilde{S}_p$ (and therefore S_p) is absolutely stable and $E(\kappa)$ is contained in the domain of attraction. □

<u>Example 4.7</u> Consider the three-dimensional system

$$\dot{x} = \begin{pmatrix} -1 & a & b \\ o & -1 & o \\ c & d & -1 \end{pmatrix} x + f(x) = Ax + f(x)$$

where $x \in R^3$ and $f : R^3 \longrightarrow R^3$. Also, $f^i(x^1,x^2,x^3) = f^i(x^i)$ and

$$o < \frac{f^i(x^i)}{x^i} < k.$$

Let

$$T = \begin{pmatrix} 1 & o & o \\ o & 1 & o \\ o & 1 & o \\ o & o & 1 \end{pmatrix} \quad \text{and } M = \begin{pmatrix} o & -a/2 & a/2 & o \\ o & -1/2 & 1/2 & o \\ o & 1/2 & -1/2 & o \\ o & -d/2 & d/2 & o \end{pmatrix}$$

Then $MT = O$ and

$$\tilde{A} = T^I AT + M = \begin{pmatrix} -1 & o & a & b \\ o & -1 & o & o \\ o & o & -1 & o \\ c & d & o & -1 \end{pmatrix}$$

We shall partition the vectors x and $\tilde{x}$ (=Tx) as follows:

$$x = ((x^1),(x^2),(x^3))^T = (x_1,x_2,x_3)^T$$

$$\tilde{x} = ((\tilde{x}^1,\tilde{x}^2)^T,(\tilde{x}^3,\tilde{x}^4)^T)^T = (\tilde{x}_1,\tilde{x}_2)^T,$$

and consider the corresponding partitions of $\tilde{A}$, T etc. We have first to derive equations similar to (4.10) for $\tilde{A}_{11}$ and $\tilde{A}_{22}$ (which are identical in this case). Note that if $0<\delta < 1$, then

$$G_{11}(s-\delta) = ((s-\delta)I - \tilde{A}_{11})^{-1} = \frac{1}{1-\delta+s} \quad I,$$

so for matrices $R_1 = \text{diag}\ (r,r)$, $K_1 = \text{diag}\ (k,k)$, $\bar{Q}_{11} = \text{diag}(q,q)$ we have

$$\Pi_o^1(s) = \frac{r + qs}{s+1-\delta} \quad I.$$

Hence $\Pi_o^1(j\omega) + (\Pi_o^1(-j\omega))^T = 2\frac{(r(1-\delta) + \omega^2 q)}{\omega^2+(1-\delta)^2} I$

and so Π_o^1 is positive real for any r and q provided $\delta < 1$. Therefore, the equations (4.10) certainly have solutions P_1, P_2 which we may again take to be equal and moreover we shall find a diagonal solution $P = \text{diag}(p,p)$. Setting i=1 therefore in (4.10) (the solution for i=2 being the same) and dropping the indices i for convenience, we have

$$p(-1+\delta) + (-1+\delta)\, p = -\ell^2$$

$$r + q(-1+\delta) - g\ell = p$$

$$2r/k + 2q = g^2$$

where we have set $L_i = \text{diag}(\ell,\ell)$, $G_o^i = \text{diag}(g,g)$.
In order to simplify matters (although these may not be the best values, but will demonstrate the method) we shall take

$$r = k\ ,\ q = 1\ .$$

Then,

$$g = 2$$

and

$$2p(-1+\delta) = -\ell^2$$

$$k + (-1+\delta) - 2\ell = p\ .$$

Hence, $\ell^2 + 2(k + (-1+\delta) - 2\ell)(-1+\delta) = 0$,

and so

$$2\ell = 4(-1+\delta) \pm \sqrt{8(-1+\delta)((-1+\delta)-k)}$$

Then,

$$p = k - 3(-1+\delta) \pm \sqrt{8(-1+\delta)((-1+\delta)-k)}$$

Again, for simplicity choose $\delta = 1/2$ and let p be the larger value, i.e. $p = k + 3/2 + 2\sqrt{k + 1/2}$.

If $Q = \text{diag}\ (1,4,1)$, then $\overline{Q} = (T^I)^T Q T^I$

and so

$$\overline{Q} = \begin{pmatrix} 1 & o & o & o \\ o & 1 & 1 & o \\ o & 1 & 1 & o \\ o & o & o & 1 \end{pmatrix}.$$

In order to apply (4.12), suppose that $a \gtrsim b \gtrsim o$ and $c \gtrsim d \gtrsim o$. Then $||\tilde{A}_{12}|| < a$, $||\tilde{A}_{21}|| < c$. Also $||\overline{Q}_{ij}|| = 1$, $\forall_{i,j}$ and

$$(\tau_{mi}) = \left(\sum_{\ell=1}^{3} ||(T^I)_{\ell m}||\ ||(T^T)_{\ell i}|| \right)$$

$$= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}.$$

Note finally that $K = \text{diag}\ (k,k,k,k)$, $K' = \text{diag}(k,k,k)$, $R = \text{diag}(r,r,r,r)$ and $R' = \text{diag}(r,2r,r)$. We can now apply (4.12) to obtain

$$\dot{V} \lesssim - \sum_{i=1}^{2} p\ ||\tilde{x}_i||^2 + k \sum_{i\neq j=1}^{2} \sum_{m=1}^{2} \tau_{mi}\ ||\tilde{x}_m||\ ||\tilde{x}_j||$$

$$+ 2k^2 \sum_{i\neq j=1}^{2} \sum_{m=1}^{2} \sum_{\ell=1}^{2} \tau_{mi}\ ||\tilde{x}_m|| \tau_{\ell i} ||\tilde{x}_\ell||$$

$$+ ||\tilde{x}_1||\ ||\tilde{x}_2|| p(a+c) + 2k \sum_{\Omega} \sum_{m=1}^{2} \tau_{mi} ||\tilde{x}_m||\ ||\tilde{A}_{jk}||\ ||\tilde{x}_k||$$

$$= -p(||\tilde{x}_1||^2 + ||\tilde{x}_2||) + 2k(3||\tilde{x}_1||\ ||\tilde{x}_2|| + 1/2(||\tilde{x}_1||^2 + ||\tilde{x}_2||^2))$$

$$+ k^2(3||\tilde{x}_1||^2 + 10\ ||\tilde{x}_1||\ ||\tilde{x}_2|| + 3||\tilde{x}_2||^2)$$

$$+ p(a+c)||\tilde{x}_1||\ ||\tilde{x}_2|| + 2k(1/2+2c)||\tilde{x}_1||^2 + 2k(2a+2c+3)||\tilde{x}_1||\ ||\tilde{x}_2||$$

$$+ 2k(2a + 1/2)||\tilde{x}_2||^2.$$

$$=(-p+k+3k^2+2k(1/2+2c))||\tilde{x}_1||^2 + (12k+10k^2 + (4k+p)(a+c))(||\tilde{x}_1||\cdot||\tilde{x}_2||)$$

$$+ (-p+k+3k^2+2k(1/2+2a))||\tilde{x}_2||^2.$$

Hence, $\dot{V}$ is negative definite if

$$3/2 > 3k^2 +2k(1/2+2c) -2\sqrt{k+1/2}$$

$$3/2 > 3k^2 +2k(1/2+2a) -2\sqrt{k+1/2}$$

and

$$\frac{17}{4} - 16k^4+(8a+8c-55)k^3 + (-\frac{91}{2} + 16ac - 21(a+b) - \frac{45}{4}(a+c)^2)k^2$$

$$+(1 - \frac{27}{2}(a+c) - \frac{17}{4}(a+c)^2)k - \frac{13}{16}(a+c)^2$$

$$+(6 - \frac{3}{2}(a+c)^2)\sqrt{k+1/2} - k\sqrt{k+1}\,(4+18(a+c)+\frac{3}{2}(a+c)^2)$$

$$- 14k^2\sqrt{k+1/2} > 0.$$

Thus, if for example $a+c < 2$, then for sufficiently small k the original system is absolutely stable.

This example indicates that although in many cases theorem 4.5 will not be applicable, since the requirement that $\Pi_o(s)$ is positive real is a strong condition, the more general theorem 4.6 may apply if one can find an extension T such that the diagonal submatrices of $\tilde{A}$ have corresponding positive real functions $\Pi_o^i(s)$. The example has a particularly simple form and, as we have seen, the relations (4.10) reduce to scalar equations. In general, of course, one would not be able to achieve such a simplification and estimates of $||P_i||$, $1 \leq i \leq \beta$ would have to be determined from the nonlinear matrix equations (4.10).

5. Conclusions

In this paper we have discussed the effects of expansion of the state vector for linear and semilinear systems. The basic idea of expansion is to separate out certain properties in the system which are not readily apparent from the contracted form. The method is particularly useful in nonlinear systems where we have shown that Anderson's generalization of the Popov criterion can itself be extended by removing the condition that the complete linear part generates the positive real matrix $\Pi_o(s)$ and replacing it by the weaker condition that diagonal submatrices of the extended linear part have corresponding positive real matrices $\Pi_o^i(s)$.

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