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ROBUST CONTROLLER DESIGN FOR UNCERTAIN DYNAMIC SYSTEMS USING APPROXIMATE MODELS

PART II: THE MULTIVARIABLE CASE

by

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Abstract

Controller design for continuous and discrete multivariable systems whose models are unknown or highly complex are frequently based upon the use of a simple, approximate and, very often, rough-and-ready model. This paper continues the theme of its first part(1) by quantifying the degree of uncertainty to be expected in multivariable feedback design studies due to observed differences in plant and model open-loop transient behaviour. Frequency domain design techniques are extended to quantify this uncertainty such that closed-loop stability and tracking of step demands is guaranteed. Functional analytic methods in partially ordered Banach space are then used to provide easily computed graphical estimates of the error involved in prediction of closed-loop transient performance. A detailed model of plant dynamics is not needed at any point in the design, all calculations being based upon graphical analyses of plant step response data deduced from plant trials or complex model simulations.

1. Introduction

This paper aims to continue the theme introduced in its companion[1] of providing a theoretical framework within which non-adaptive control systems design for uncertain dynamic systems can be achieved using an approximate plant model of desired simplicity. Approximate models of engineering plant are a fact of life as, although a detailed (and usually high order) linear model may be agood approximation to observed plant dynamics, it never matches the (usually high order) plant exactly. In fact, even if a high order model is a good fit, there are many conceptual, computational and design advantages[1] of using a low order, simple structure approximate model during the design stage even if the chosen model is rather crude or 'rough-and-ready' (see, for example, refs. (1)-(4)). If a simple model is used however, it is of vital practical importance to take account of the observed differences between plant and model open-loop responses during the design stage to ensure that the stability and acceptable performance of the model in the presence of the controller guarantees the stability and acceptable performance of the real plant when the control scheme is implemented. This problem is the subject of this paper which aims to generalize the results of its companion[1] to cope with multivariable systems. The development follows closely that of ref. (1) but requires a more sophisticated mathematical approach and introduces new problems and possibilities of an essentially multivariable flavour. The graphical interpretation of the results is emphasized, proofs being relegated in the main to the appendices.

The paper logically divides into several sections. The frequency domain design technique is described in section 2 together with its interpretation in terms of 'smudging' of inverse Nyquist array[5-7] plots. Stability assessment based on simulation data alone is described in Section 3 and graphical bounds on deterioration in predicted transient performance are derived. In Section 4, the extension of the ideas to multivariable sampled-data plant is indicated.

2. Frequency Domain Design based on Approximate Models

Consider an m-output/ℓ-input, strictly proper, linear system G with output measurements generated from an m-input/m-output proper linear system F and the problem of the design of an m-input/ℓ-output proper linear forward path controller K to ensure the stability and acceptable transient performance of the feedback system of Fig. 1(a). All elements are assumed continuous and linear, F is known, K is to be designed and G is either unknown or is regarded as unnecessarily or inconveniently complex for the design exercise under consideration. It is supposed however that, for each pair of indices (i,j), the response $Y_{ij}(t)$ from zero initial conditions of the i^{th} output to a unit step in the j^{th} input has been found from plant trials or model simulations. It is convenient to define the plant 'step-response matrix' Y(t) as

$$Y(t) = \begin{bmatrix} Y_{11}(t) & \cdots\cdots & Y_{1\ell}(t) \\ & & \\ Y_{m1}(t) & \cdots\cdots & Y_{m\ell}(t) \end{bmatrix} \tag{1}$$

Let G_A be an approximate model of G and suppose that the step response matrix $Y_A(t)$ of G_A has been obtained by simulation. The observed open-loop mismatch between plant and model is described by the $m\times\ell$ 'error matrix'

$$E(t) \triangleq Y(t) - Y_A(t) = [E^{(1)}(t),\ldots,E^{(\ell)}(t)] \tag{2}$$

with columns $E^{(j)}(t)$, $1 \leq j \leq \ell$. The error is not necessarily assumed to be small! If the controller K is designed on the basis of the approximate model G_A to ensure the stability of the approximating feedback scheme of Fig. 1(b) then we consider the problem of how to use the graphical properties of E(t) to ensure that the resultant design guarantees the stability of the real configuration of Fig. 1(a).

As in ref (1) G and G_A are assumed to be linear convolution mappings of the form

$$y(t) = \int_o^t H(t')u(t-t')dt', \quad y_A(t) = \int_o^t H_A(t')u_A(t-t')dt' \qquad (3)$$

where, for all (i,j), $H_{ij}(t)$ and $(H_A(t))_{ij}$ have well-defined Laplace transforms and the modelling error $G - G_A$ is stable in the sense that

$$\int_o^\infty ||H(t) - H_A(t)||_m dt < + \infty \qquad (4)$$

(Note: $||.||_m \triangleq \max_i \sum_j |(.)_{ij}|$ is the matrix norm induced by the vector norm $||.||_m \triangleq \max_i |(.)_i|$ in the linear vector space C^m of complex mx1 column vectors).

Finally, note that

$$Y(t) = \int_o^t H(t')dt', \quad Y_A(t) = \int_o^t H_A(t')dt' \qquad (5)$$

and that the signal $Y - Y_A$ is stable due to equation (4).

2.1 Frequency Domain Stability Theory

If the designed controller K ensures the input/output stability of the configuration of Fig. 1(b) then[8] we must have

$$\inf_{Res>o} |\det(I_m + G_A(s)K(s)F(s))| > o \qquad (6)$$

Moreover, K will also stabilize the configuration of Fig. 1(a) if

$$\inf_{Res>o} |\det(I_m + G(s)K(s)F(s))| > o \qquad (7)$$

or equivalently, using the identity[6]

$$\det(I_m + M_1M_2) = \det(I_\ell + M_2M_1) \qquad (8)$$

valid for any $m\times\ell$ matrix M_1 and $\ell\times m$ matrix M_2, if the relation

$$\inf_{Res>o} |\det(I_\ell + K(s)F(s)G(s))| > o \qquad (9)$$

holds.

Following the development in ref. (1) we can combine (6) and (9) using the identity

$$\det(I_\ell + KFG) \equiv \det(I + KFG_A + KF(G-G_A))$$
$$\equiv \det(I_\ell + KFG_A)\det(I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A))$$
$$\equiv \det(I_m + G_AKF)\det(I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A)) \qquad (10)$$

to replace (9) by the sufficient condition

$$\inf_{Re s \geq o} |\det(I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A))| > o \qquad (11)$$

or, noting that the stability assumptions guarantee that $(I+KFG_A)^{-1}KF(G-G_A)$ is analytic and bounded in Re s $\geq$ o, by the equivalent relation

$$\inf_{s \in D} |\det(I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A)| > o \qquad (12)$$

where D is the usual Nyquist 'infinite' semi-circle in the closed-right-half complex plane. (Note: Although these relations bear a superficial similarity to those seen in the single-input/single-output case, note that the ordering of the terms in (11) and (12) is important as matrices, in general, do not commute).

Using (12) the following stability result is easily proven:

Lemma 1: If the controller K stabilizes the approximate model G_A in the configuration of Fig. 1(b), then it will also stabilize the real uncertain plant G in the configuration of Fig. 1(a) if

(a) the composite system GKF is both controllable and observable, and

(b) $$\lambda_o \triangleq \sup_{s \in D} r((I_\ell + KFG_A)^{-1}KF(G-G_A)) < 1 \qquad (13)$$

(Note: the spectral radius r(M) of an $\ell x \ell$ matrix M with eigenvalues $m_1, m_2, \ldots, m_\ell$ is defined by[(9),(10)]

$$r(M) = \max_i |m_i| \qquad (14)$$
)

Proof: Condition (a) ensures that asymptotic stability is implied by input/output stability whilst (b) ensures that (12) holds. More precisely, if $(I + KFG_A)^{-1}KF(G-G_A)$ has eignevalues $\eta_1, \eta_2, \ldots, \eta_\ell$, then, for all $s \in D$,

$$|\det(I_\ell + (I_\ell + KFG_A)^{-1}KF(G-G_A)|$$
$$= |(1+\eta_1) \; .... \; (1+\eta_\ell)|$$
$$\geq (1-|\eta_1|)(1-|\eta_2|) \; ... \; (1-|\eta_\ell|)$$
$$\geq (1-\lambda_o)^\ell > 0 \qquad (15)$$

The computation of λ_o presents a problem as it depends upon the detailed frequency domain structure of the modelling error $G - G_A$ whereas, by assumption, we only have available (or only wish to use) the time-domain data $E(t)$. To circumvent this problem we introduce the partial ordering[9,10] on the space of $n_1 x n_2$ real matrices defined by the relation

$$A \leq B \quad \text{iff } A_{ij} \leq B_{ij} \; \forall \; i,j \qquad (16)$$

and define the 'absolute value' of a complex $n_1 x n_2$ matrix A to be the $n_1 x n_2$ real matrix

$$||A||_p \triangleq \begin{pmatrix} |A_{11}| & \cdots & |A_{1n_2}| \\ & & \\ |A_{n_1 1}| & \cdots & |A_{n_1 n_2}| \end{pmatrix} \qquad (17)$$

The following 'norm-like' properties of the absolute value are easily proven:

<u>Fact 1</u>: $||A||_P \geq 0$ (18)

<u>Fact 2</u>: If α is any complex number, then

$$||\alpha A||_P = |\alpha| . ||A||_P \qquad (19)$$

<u>Fact 3</u>: $||A+B||_P \leq ||A||_P + ||B||_P$ (20)

<u>Fact 4</u>: $||AB||_P \leq ||A||_P \cdot ||B||_P$ (21)

We will also need the following simple spectral radius results for square matrices. The proofs are elementary and can be based on theorem 2.4.9 in reference (9).

<u>Fact 5</u>: $0 \leq A \leq B \quad r(A) \leq r(B)$ (22)

<u>Fact 6</u>: $r(A) \leq r(||A||_P)$ (23)

Finally we will need the following matrix measure of the 'magnitude' of an $n_1 x n_2$ continuous matrix function of time defined by

$$\breve{F}(t) = F_o + \int_o^t F_1(t')dt' \qquad (24)$$

(regarded as the step response matrix of a proper n_2-input/n_1-output system) with F_o constant and the elements of $F_1(t)$ smooth enough to ensure that the elements of $\breve{F}(t)$ have local maxima and minima at a finite number of points only on any finite subinterval of $[0, +\infty[$. The measure used is the $n_1 x n_2$ matrix (c.f. (15) of ref.(1))

$$N_T^P(\breve{F}) \triangleq ||F_o||_P + \int_o^T ||F_1(t)||_P dt \quad , \quad T \geq 0 \qquad (25)$$

which can be written in the element form

$$N_T^P(\breve{F}) = \begin{pmatrix} N_T(\breve{F}_{11}) & \cdots & N_T(\breve{F}_{1n_2}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ N_T(\breve{F}_{n_1 1}) & \cdots & N_T(\breve{F}_{n_1 n_2}) \end{pmatrix} \qquad (26)$$

where $N_T(\breve{F}_{ij})$ is the scalar measure of $\breve{F}_{ij}$ introduced in Proposition 1 of reference 1. Note[1] that $N_T^P(\breve{F})$ can hence be deduced by graphical analysis of the time-variation of the elements of $\breve{F}(t)$ without the need to compute F_o or $F_1(t)$ explicitly.

The importance of N_T^P is expressed by the following lemma which follows trivially from Lemma 2 in reference (1) by considering elements.

Lemma 2: If the plant modelling error is stable in the sense of (4), then

$$||G(s) - G_A(s)||_P \leq N_\infty^P(E) \; \forall \; \text{Re } s \geq 0 \qquad (27)$$

2.2 A Graphical Stability Criterion for Uncertain Systems

The following result is a generalization of theorem 1 in reference (1).

Theorem 1: If the controller K stabilizes the approximate model G_A in the configuration of Fig. 1(b), then it will also stabilize the real uncertain plant in the configuration of Fig. 1(a) if,

(a) The plant modelling error is stable in the sense of equation (4),

(b) the composite system GKF is both controllable and observable and

(c) the inequality

$$\lambda_o = \sup_{s\in D} r(||(I_\ell + KFG_A)^{-1}KF||_P \; N_\infty^P(E)) < 1 \qquad (28)$$

The result is proved below but it is of interest to compare this result with the single-variable equivalent described by theorem 1 of reference (1). Conditions (a) and (b) are identical but condition (c) does not have, in general, a Nyquist-like interpretation unless[1] $m = \ell = 1$ or K,F and G_A are diagonal (see Corollary 1.2 below). It does have the common quality, however, that it enables the stability of the uncertain system G to be assessed in terms of the known dynamics of K, F and G_A and the computable measure $N_\infty^P(E)$ of plant/model mismatch. It is probably best checked in general in a point-wise frequency sense by evaluation of $r(||(I_\ell+KFG_A)^{-1}KF||_P \; N_\infty^P(E))$ at a selected number of frequency points covering the bandwidth of interest.

Bearing in mind that repetitive eigenvalue (and hence spectral radius) calculations can be time-consuming, the following relaxed versions of theorem 1 are stated. In general, they generate distinct stability conditions unless $m = \ell = 1$.

Corollary 1.1: The result of theorem 1 remains valid if (28) is replaced by any one of the following computable conditions,

$$\text{(i)} \quad r(\sup_{s\in D}\{||(I_\ell+KFG_A)^{-1}KF||_P N_\infty^P(E)\}) < 1 \qquad (29)$$

$$\text{(ii)}\quad r(\{\sup_{s\in D} \| (I_\ell + KFG_A)^{-1} KF \|_P \} N_\infty^P(E)) < 1 \tag{30}$$

$$\text{(iii)}\quad \sup_{s\in D} \sigma_\ell (\| (I_\ell + KFG_A)^{-1} KF \|_P N_\infty^P(E)) < 1 \tag{31}$$

$$\text{(iv)}\quad \sup_{s\in D} \| (\| (I_\ell + KFG_A)^{-1} KF \|_P N_\infty^P(E)) \|_m < 1 \tag{32}$$

$$\text{(v)}\quad \lambda_o'' \triangleq \sup_{s\in D} \| (\| (I_\ell + KFG_A)^{-1} KF \|_P) \|_m \cdot \| N_\infty^P(E) \|_m < 1 \tag{33}$$

(Notes: In (i) and (ii) the supremum is interpreted as a least upper bound in terms of the partial ordering. In (iii), the singular values[11] $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_\ell$ of a complex $\ell x \ell$ matrix M are the ordered positive square roots of the eigenvalues of $M^* M$ where M^* is the conjugate transpose of M).

Proof: (i) and (ii) follows from fact 5 and the ordering, for any $s \in D$,

$$0 \leq \| (I_\ell + K(s)F(s)G_A(s))^{-1} K(s)F(s) \|_P \; N_\infty^P(E)$$

$$\leq \sup_{s\in D} \{ \| (I_\ell + KFG_A)^{-1} KF \|_P \; N_\infty^P(E) \}$$

$$\leq \sup_{s\in D} \| (I_\ell + KFG_A)^{-1} KF \|_P \} \; N_\infty^P(E) \tag{34}$$

Condition (iii) follows from the observation[11] that all eigenvalues of a $\ell x \ell$ matrix M are bounded in modulus by its largest singular value. Conditions (iv) and (v) follow from the related fact that the eigenvalues of a matrix are also bounded in modulus by any induced norm.

The above alternative expositions of theorem 1 can lead to more convenient graphical representations of the stability criterion. For example, the following special case represents a generalization of both theorem 1 of reference (1) to the multivariable case and of Proposiition 6 in reference 4 and is proved by replacing (28) by (32).

Corollary 1.2: The conclusions of theorem 1 remain valid if $m = \ell$ and G_A, K and F are all diagonal (non-interacting systems) and (28) is replaced by the condition

$$\left| \frac{K_{kk}(s)F_{kk}(s)}{1 + K_{kk}(s)F_{kk}(s)(G_A(s))_{kk}} \right| < \frac{1}{\sum_{j=1}^{m} N_\infty(E_{kj})}$$

$$\triangleq R_k \qquad , \forall s \in D \quad , \quad 1 \leq k \leq m \qquad (35)$$

In particular, if $(G_A)_{kk}$, F_{kk} and K_{kk} have no zeros on D, (35) can be written as

$$|1 + ((G_A(s))_{kk}K_{kk}(s)F_{kk}(s))^{-1}| > |(G_A(s))_{kk}^{-1}| \sum_{j=1}^{m} N_\infty(E_{kj})$$

$$\triangleq d_k(s) \qquad \forall \; s \in D \qquad , 1 \leq k \leq m \qquad (36)$$

The condition (35) is probably best checked[4] by splitting the procedure. Firstly, the 'infinite' semi-circular component of D requires that

$$\lim_{\substack{|s| \to \infty \\ Res \geq o}} \sup |K_{kk}(s)F_{kk}(s)| < R_k \quad , \quad 1 \leq k \leq m \qquad (37)$$

whilst the imaginary axis component of D requires that the normal frequency response locus of $K_{kk}F_{kk}/(1+(G_A)_{kk}K_{kk}F_{kk})$ lies in the interior of the circle of centre the origin and radius R_k, $1 \leq k \leq m$. Verification of (36) reduces to verification of (37) and a graphical check that, for $1 \leq k \leq m$, the 'confidence band' generated by the inverse Nyquist locus of $(G_A)_{kk}K_{kk}F_{kk}$ for $s = i\omega(\omega \geq o)$ with superimosed 'confidence circles' of centre $((G_A(s))_{kk}K_{kk}(s)F_{kk}(s))^{-1}$ and radius $d_k(s)$ at each frequency point does not contain or touch the (-1,0) point of the complex plane. These ideas have been discussed in special cases previously[1,4] and will not be expanded on here except to note that they enable diagonal/non-interacting models of multivariable plant to be used as the basis of controller design provided that the interaction effects observed in the time-domain are small enough to produce confidence circles that are not so large as to make (35) or (36) invalid. This possibility has clear advantages in practice and has loose connections with the Gershgorin-circle based method of the inverse Nyquist array and dyadic expansion techniques[5-7].

Proof of Theorem 1: Condition (a) is required for the validity of lemma 1. Conditions (b) and (c) imply conditions (a) and (b) of lemma 1 as, for $\mathrm{Re}s \geq 0$,

$$\begin{aligned} 0 &\leq \|(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)(G(s)-G_A(s))\|_P \\ &\leq \|(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)\|_P \ \|G(s) - G_A(s)\|_P \\ &\leq \|(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)\|_P \ N_\infty^P(E) \end{aligned} \tag{38}$$

by Fact 4 and lemma 2. Facts 5 and 6 then indicate that, for $\mathrm{Re}s \geq 0$,

$$\begin{aligned} & r((I_\ell + KFG_A)^{-1}KF(G-G_A)) \\ &\leq r(\|(I_\ell + KFG_A)^{-1}KF(G-G_A)\|_P) \\ &\leq r(\|(I_\ell + KFG_A)^{-1}KF\|_P N_\infty^P(E)) \end{aligned} \tag{39}$$

The proof is completed by noting that (28) clearly implies (13).

2.3 Discussion and Robustnes Analysis

A comparison of the above results with the single-input/single-output theory[1] indicate a basic structural similarity with (a) the multivariable case introducing many more possibilities both in choice of approximate model and choice of stability criterion (see Corollary 1.1) and (b) a general increase in mathematical complexity. Both of these observations, in the authors opinion, merit regarding the multivariable case as a distinct field of study possessing characteristics that cannot be found in scalar systems. This will be further supported by the time-domain analysis in the next section where non-commutation of multivariable convolution systems requires a fundamental change in the analyses. It can be simply illustrated here by noting[12] that input/output transformations (with suitable safeguards) can be used to simplify a design problem yet both $||.||_P$ and $r(||.||_P)$ are not invariant under such transformations. It is possible therefore to generate an infinity of stability criteria simply by choosing different transformations. Unfortunately there are no explicit guidelines available to simplify the choice of 'best' transformation.

Finally, we note that the design based on theorem 1 is inherently a robust design[11,13] in the sense that, if the plant G changes over a period of time to the plant $\tilde{G}$ with step response matrix $\tilde{Y}$, stability will be retained provided that the change $\tilde{G} - G$ is 'small enough'. This is obvious from theorem 1 when it is noted that the spectral radius will only change by a 'small amount' if the transient error E changes by a 'small amount'. A computable measure of the size of the permissible change is not easily obtained from this expression however. Suppose, for simplicity, therefore that the more conservative conditions (33) holds. We can prove the following result paralleling Proposition 2 of reference (1).

Proposition 1: If the conditions of theorem 1 hold with (33) replacing (28), then the closed-loop system of Fig. 1(a) will retain its stability if GKF is both controllable and observable, $\tilde{G} - G$ is stable and

$$||N_\infty^P(\tilde{Y} - Y)||_m < \frac{1 - \lambda_o''}{\sup_{s \in D} ||(||(I + KFG_A)^{-1}KF||_P)||_m} \tag{40}$$

Proof: The conditions of theorem 1 are satisfied as GKF is controllable and observable by assumption, $\tilde{G} - G_A = (\tilde{G} - G) + (G - G_A)$ is stable and (33) holds with E replaced by $\tilde{Y} - Y_A$. More precisely, by considering elements, note that $N_\infty^P(\tilde{Y} - Y_A) \leq N_\infty^P(\tilde{Y} - Y) + N_\infty^P(E)$ and hence that

$$\sup_{s \in D} ||(||(I_\ell + KFG_A)^{-1}KF||_P)||_m \cdot ||N_\infty^P(\tilde{Y} - Y_A)||_m$$

$$\leq \sup_{s\in D} ||(||(I_\ell + KFG_A)^{-1}KF||_p)||_m \cdot \{||N_\infty^P(\tilde{Y} - Y)||_m$$

$$+ ||N_\infty^P(E)||_m\}$$

$$< 1 \qquad (41)$$

by (40) and the definition of λ_o''.

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3. Time Domain Design Based on Approximate Models

It is the purpose of this section to generalize the results of section 3 of reference (1) and hence provide techniques for stability assessment of uncertain multivariable dynamic systems based upon time-domain data alone. The use of time-domain data in stability assessment is unusual but it may have a number of advantages over frequency domain calculations, particularly in the multivariable case. For example, the checking of the frequency domain stability condition (13) requires the calculation of the inverse of the $\ell \times \ell$ complex matrix $I + KFG_A$ at a large number of frequency points. This is a feasible proposition even if ℓ is large but the corresponding time-domain result (see theorem 2 for example) is simpler requiring only system simulations and one eigenvalue calculation. A more important benefit of time-domain analysis is, however, the possibility of providing bounds on the deterioration in predicted transient performance to be expected due to the approximation used. The benefits of such bounds have been illustrated in the scalar case[1] but there are several technical problems here due to the non-commutation/convolution of systems that require substantial modifications in the form and proof of the design results.

3.1 Mathematical Background

The contraction mapping theorem was a basic tool in the scalar case[1] but for multivariable studies the extra degrees of freedom available merit the use of a generalized contraction theorem that reflects the multi-input/multi-output nature of the problem. The required mathematics is outlined below.

Let X be a Banach space (we will take $X = L_\infty(0,t)$ in the following sections) and X^d be the d^{th} Cartesian product of X regarded as the linear vector space of columns $x = (x_1, x_2, \ldots, x_d)^T$ of elements of X. The absolute value of $x \in X^d$ will be denoted (c.f. equation (17))

$$||x||_P \triangleq \begin{pmatrix} ||x_1|| \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ ||x_d|| \end{pmatrix} \in R^d \tag{42}$$

where $||.||$ denotes the norm in X. If L is a bounded linear operator mapping X^{d_2} into X^{d_1}, it can be represented as the operator $y = Lx$ with $y_i = \sum_j L_{ij} x_j$ and L_{ij} bounded, linear operators in X. The absolute value of L is defined to be

$$||L||_P \triangleq \begin{pmatrix} ||L_{11}|| & \cdots & ||L_{1d_2}|| \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ ||L_{d_1 1}|| & \cdots & ||L_{d_1 d_2}|| \end{pmatrix} \tag{43}$$

where $||.||$ is the operator norm induced by the vector norm in X. It is easily shown that $y = L\,u$ implies $||y||_P \leq ||L||_P\, ||u||_P$ and that, if $||y||_P \leq M||u||_P$ for all u, then $||L||_P \leq M$.

Let W be a mapping of X^d into itself, then[9] W is a global P-contraction if there exists a real dxd matrix $P \geq 0$ with the property that $r(P) < 1$ and, for all $x, y \in X^d$,

$$||W(x) - W(y)||_P \leq P||x - y||_P \tag{44}$$

The example of greatest relevance to this paper is an operator W of the form $x \to Lx + x_o$ with $x_o \in X^d$ and L bounded and linear. It is easily verified that W satisfies (44) with $P = ||L||_P$ and hence is a P-contraction if $r(||L||_P) < 1$. Moreover, if (44) holds for any other P, it is easily seen that $||L||_P \leqslant P$ and, using fact 5, that the condition $r(||L||_P) < 1$ is hence both necessary and sufficient for W to be a P-contraction.

We now state the natural generalization of the global version of the contraction mapping theorem under partial ordering given in ref. 9. (p. 433).

Lemma 3: Suppose that W is a global P-contraction in the (Banach) product space X^d. Then, for any $x^{(o)} \in X^d$, the sequence $x^{(k+1)} = W(x^{(k)})$, $k \geqslant o$, converges to the unique solution of the equation $x = Wx$ in X^d. Moreover, we have the error estimate

$$||x - x^{(1)}||_P \leqslant (I_d - P)^{-1} P \, ||x^{(1)} - x^{(o)}||_P \tag{45}$$

(Note: the proof of the result is given for X = R in ref.(9) but it carries through with no change to an arbitrary Banach space. It can also be deduced from section 12.1 of ref. (10)).

The following corollary follows by taking norms in R^d of (45).

Corollary: If $||P||_m < 1$, then

$$\max_k ||x_k - x_k^{(1)}|| \leqslant \frac{||P||_m}{1-||P||_m} \max_k ||x_k^{(1)} - x_k^{(o)}|| \tag{46}$$

3.2 Time Domain Stability Theory: Output-based Formulation

Due to the noncommutation of multivariable convolution operators the main result (Theorem 2) of reference (1) does not carry through to the multivariable case unless commutative controllers[6,7,14] are designed. As commutative controls[6,7] suffer from severe realizability difficulties unless, for example, the plant has a degree of symmetry, this possibility is not considered here. The nearest generalization of the scalar case appears to be the following result:

Theorem 2: Suppose that the controller K has been designed to stabilize the approximate model G_A in the feedback configuration of Fig. 1(b) and that the mxm matrix function $V_A(t)$ is defined by, $t \geq o$, the convolution

$$V_A(t) \triangleq \int_o^t H_E(t-t')H_{KF}(t')dt' \qquad (47)$$

where

(1) $H_{KF}(t)$ is the ℓxm impulse response matrix of the composite system KF and

(2) the mxℓ matrix $H_E(t)$ has the form

$$H_E(t) = \left[H_E^{(1)}(t),\ldots,H_E^{\ell}(t)\right] \qquad (48)$$

where the j^{th} columns $H_E^{(j)}(t)$ is the response from zero initial conditions of the proper system $(I + G_AKF)^{-1}$ to the j^{th} column $E^{(j)}(t)$ of the known error matrix $E(t)$.

Then the controller K will stabilize the real uncertain system in the configuration of Fig. 1(a) if

(a) the modelling error is stable in the sense of (4),

(b) the composite system GKF is controllable and observable,

(c) both of the systems $(I + G_AKF)^{-1}(G - G_A)KF$ and $(I+G_AKF)^{-1}(G-G_A)K$ are input/output stable and

(d) if the following spectral inequality holds

$$r(N_\infty^P(V_A)) < 1 \qquad (49)$$

Moreover, under these conditions, suppose also that

(i) the demand signal r is the response from zero initial conditions of a mxm stable system H_o to the step $\hat{r}(t) = \alpha$, $t > o$,

(ii) $y^{(o)}(t)$ is the response from zero initial conditions of an mxm stable, proper system H_1 to the step $\hat{y}(t) = \beta$, $t > o$, and

(iii) $\eta(t)$ is the mxl vector defined by the convolution

$$\eta(t) = \int_o^t H_E(t-t')\{H_{KH_o}(t')\alpha - H_{KFH_1}(t')\beta\}dt' \tag{50}$$

where H_{KH_o} and H_{KFH_1} are the impulse response matrices of KH_o and KFH_1 respectively. Then, for all $t \geq o$, the response $y(t)$ of the real feedback scheme Fig. 1(a) from zero initial conditions to the demand $r(t)$ satisfies the bound

$$|y_j(t) - y_j^{(1)}(t)| \leq \varepsilon_j(t) \quad , \quad 1 \leq j \leq m \tag{51}$$

where

$$\varepsilon(t) = \begin{pmatrix} \varepsilon_1(t) \\ \cdot \\ \circ \\ \varepsilon_m^{\cdot}(t) \end{pmatrix} \triangleq (I_m - N_t^P(V_A))^{-1} N_t^P(V_A) \sup_{o \leq t' \leq t} ||y^{(1)}(t') - y^{(o)}(t')||_P \tag{52}$$

and $y^{(1)}(t) = y_A(t) + \eta(t)$ where $y_A(t)$ is the known response of Fig. 1(b) from zero initial conditions to the demand r.

Finally, (51) holds with $\varepsilon(t)$ replaced by $\varepsilon_\infty(t)$ obtained from (52) by replacing $N_t^P(V_A)$ by $N_\infty^P(V_A)$ and with $\varepsilon(t)$ replaced by $\varepsilon^\mu(t)$ obtained from (52) by replacing $N_t^P(V_A)$ by the matrix

$$N_t^{P,\mu}(V_A) \triangleq \begin{pmatrix} N_{\mu_{11}(t)}((V_A)_{11}) & \cdots & N_{\mu_{1m}(t)}((V_A)_{1m}) \\ \\ N_{\mu_{m1}(t)}((V_A)_{m1}) & \cdots & N_{\mu_{mm}(t)}((V_A)_{mm}) \end{pmatrix} \tag{53}$$

where, for each pair of indices (i,j), μ_{ij} is some function satisfying $\mu_{ij}(t) \geq t$, for all $t \geq o$.

The theorem is proved in Appendix 8 but despite its complex structure, careful scrutiny indicates that, subject to the conditions (a) - (c), both stability and performance of the implemented feedback scheme can be assessed in terms of quantities defined in terms of known data

G_A, K, F, E, H_o and H_1. More precisely, stability calculations require the evaluation of the matrix function $V_A(t)$, and an eigenvalue calculation based on $N_\infty^P(V_A)$ to check (49). Knowledge of $E(t)$ and $N_t^P(V_A)$ (subject to conditions (i) and (ii)) then enables the calculation of $\eta(t)$ and $\varepsilon(t)$, $\varepsilon_\infty(t)$ or $\varepsilon^\mu(t)$ from which the error bound (51) is obtained. The graphical interpretation of bounds such as (51) have been discussed in reference (1) and will not be discussed here. Note, however, that evaluation of $\varepsilon(t)$ requires matrix inversion. This may not be a problem but, if it is, it can be avoided in the following special case.

Corollary 2.1: Under the conditions of theorem 2, suppose also that $||N_\infty^P(V_A)||_m < 1$, then, defining $\lambda_1(t) = ||N_t^{P,\mu}(V_A)||_m$ for any μ,

$$|y_j(t) - y_j^{(1)}(t)| \leq \frac{\lambda_1(t)}{1-\lambda_1(t)} \max_{o \leq t' \leq t} ||y^{(1)}(t') - y^{(O)}(t')||_m$$

$$, \ 1 \leq j \leq m \tag{54}$$

Proof: From the definition, $N_t^{P,\mu}(V_A) \leq N_\infty^P(V_A)$ and hence $\lambda_1(t) < 1$, $t \geq 0$. The result follows from (51) and (52) by bounding $||\varepsilon(t)||_m$ by the right-hand-side of (54).

The application of the result as part of a design procedure paralleling that outlined in reference (1) is a clear possibility provided that conditions (a) - (d) can be satisfied. Conditions (a) and (b) are required in the scalar case and will not be discussed further here. Condition (d) provides a computable measure of whether or not the modelling error E is small enough to allow stability prediction based on transient data only and presents no problem in principle. There can be a major problem in satisfying (c) in the multivariable case however as, even if K stabilizes the approximate model G_A and the error $G-G_A$ is stable, the systems

$(I + G_A KF)^{-1}(G-G_A)K$ and $(I + G_A KF)^{-1}(G - G_A)KF$ could be unstable if integral action is included in the controller. To illustrate this suppose that $m = \ell = 2$, $F = I_2$ and G_A, K and $G - G_A$ are defined by the transfer function matrices

$$G_A(s) = \frac{1}{s+2} I_2 \quad , \quad K(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{1}{s} \end{pmatrix}$$

$$G(s) - G_A(s) = \frac{1}{s+2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{55}$$

then it is easily verified that $(I + G_A KF)^{-1}(G - G_A)K$ has transfer function matrix

$$(I_2 + G_A(s)K(s)F(s))^{-1}(G(s) - G_A(s)K(s)$$

$$= \begin{pmatrix} 0 & \frac{(s+1)}{s(s+3)} \\ 0 & 0 \end{pmatrix} \tag{56}$$

which is clearly unstable due to the pole at $s = o$ introduced by the integrator in K. This problem is avoided in the analysis of the next section but at the expense of producing more conservative bounds on performance degradation.

To complete this section, we made the following observations concerning the application of the result:

Spectral Radius Evaluation: If m is large then (49) may be more conveniently checked by evaluation of any upper bound $\hat{r}$ of $r(N_\infty^P(V_A))$. If $\hat{r} < 1$, then (49) is clearly true. Obvious choices of $\hat{r}$ are $\hat{r} = ||N_\infty^P(V_A)||_m$ and $\hat{r} = \sigma_m(N_\infty^P(V_A))$, but there are many others that could be more convenient.

Choice of H_1: This operator plays the role of H in Theorem 2 of reference (1). It can be used to decrease the uncertainty in the prediction of y by reducing $\varepsilon(t)$. The obvious choices are

(i) $H_1 = 0$ yields $y^{(o)}(t) \equiv 0$, and, when $F = H_o$, yields $\eta = V_A\alpha$,

(ii) $H_1 = F^{-1}H_o$ (if F has a proper stable inverse) yields $\eta = 0$ and $y^{(1)} = y_A$ and

(iii) $H_1 = (I + G_A KF)^{-1} G_A K$ and $\beta = \alpha$ yields $y(o) = y_A$. This choice probably produces the smallest prediction error as $y^{(o)}$ is the first guess in a successive approximation scheme and y_A is our best available estimate of y.

Choice of ε and μ: This has been discussed elsewhere[1]. There is no change in the multivariable case.

Restrictions due to H_o: The result is only valid for demands r that are the step responses of the proper stable system H_o. If $H_o = I_m$ we obtain standard step responses but the result also holds for inputs of the form, say, $r(t) = \alpha$, $o < t \leq T$, and $r(t) = 0$, $t > T$ by identifying H_o as the map $f(t) \rightarrow f(t) - f(t-T)$.

3.3 Time-Domain Stability Theory: Input-based Formulation

The problems associated with condition (c) of theorem 2 can be eliminated by focussing attention on the behaviour of the input u to the plant G. The following result is proved in Appendix 9.

Theorem 3: Suppose that the controller K stabilizes the model G_A in the configuration of Fig. 1(b) and that

$$W_A(t) = \left[W_A^{(1)}(t), \ldots, W_A^{(\ell)}(t)\right] \tag{57}$$

where $W_A^{(j)}(t)$ is the response from zero initial conditions of the system $(I + KFG_A)^{-1}KF$ to the input $E^{(j)}(t)$. Then the controller K will stabilize the real uncertain system G in the configuration of Fig. 1(a) if

(a) the modelling error is stable in the sense of (4),

(b) the composite system GKF is controllable and observable and

(c) the following inequality holds

$$r(N_\infty^P(W_A)) < 1 \tag{58}$$

Moreover, under these conditions, suppose also that

(i) $u^{(o)}(t)$ is the response from zero initial conditions of an $\ell x\ell$ stable, proper system H_2 to the step $\hat{u}(t) = \beta, t > o$ and

(ii) $\xi(t)$ is the ℓx1 vector defined by the convolution

$$\xi(t) = -(\int_o^t W_A(t-t')\ H_{H_2}(t')dt')\beta \tag{59}$$

where $H_{H_2}(t)$ is the impulse response matrix of H_2. Then the input response $u(t)$ of the real feedback system of Fig. 1(a) from zero initial conditions to the demand $r(t)$ satisfies the bound

$$|u_j(t) - u_j^{(1)}(t)| \le \tilde{\varepsilon}_j(t) \quad , \quad 1 \le j \le \ell \tag{60}$$

where

$$\tilde{\varepsilon}(t) = \begin{pmatrix} \tilde{\varepsilon}_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\varepsilon}_\ell(t) \end{pmatrix} \triangleq (I_\ell - N_t^P(W_A))^{-1} N_t^P(W_A) \sup_{o \le t' \le t} ||u^{(1)}(t') - u^{(o)}(t')||_P \tag{61}$$

and $u^{(1)}(t) = u_A(t) + \xi(t)$

(Note: As in theorem 2, $\tilde{\varepsilon}(t)$ can be replaced by $\tilde{\varepsilon}_\infty(t)$ or $\tilde{\varepsilon}^\mu(t)$ in (60) by replacing $N_t^P(W_A)$ by $N^{P,\mu}(W_A)$ given by (53). The details are ommitted for brevity).

Comparing theorems 2 and 3 indicates that the difficult condition (c) of theorem 2 has vanished but that theorem 3 provides bounds on the input u rather than the output y. It is of course of value to have input estimates to avoid excessive input magnitudes but output estimates are probably more important in general. The input estimate can be converted into an output estimate under certain conditions stated in the following corollary (proved also in Appendix 9):

Corollary 3.1: With the conditions of theorem 3 suppose also that G is stable. Then (60) can be replaced by the bounds, $t \geq o$,

$$||y(t) - y^{(1)}(t)||_P \leq N_t^P(Y)\,\tilde{\varepsilon}(t) + N_t^{P(E)} \max_{o \leq t' \leq t} ||u^{(1)}(t')||_P \qquad (62)$$

where $y^{(1)}(t)$ is the open-loop response of G_A from zero initial conditions to the input $u^{(1)}(t)$.

This bound is not expected to be as tight as (51) however, as it is deduced from (61) via norm inequalities

The interpretation of theorem 3 is similar to that of theorem 2 with ε and μ as before and H_2 playing the role of H_1. The two choices that suggest themselves are:

(i) $H_2 = o$ when $u^{(o)}(t) \equiv o$ and $u^{(1)}(t) = u_A(t)$,

(ii) $H_2 = (I_\ell + KFG_A)^{-1} K\, H_o$ where H_o is a stable proper system when $u^{(o)}(t) \equiv u_A(t)$ if the demand $r(t)$ is the response of H_o from zero initial conditions to the step input $\hat{r}(t) = \beta$, $t > o$.

The first choice is the simpler but, as $u^{(o)}(t)$ is the first guess at $u(t)$ in the successive approximation scheme, (ii) probably provides a more accurate estimate. In both cases, all required responses, W_A, ξ and $\tilde{\varepsilon}$ are computed from known data with no need to use any available model of the plant G.

Finally, we state the following simple alternative to (60) that avoids the inversions required for $\tilde{\varepsilon}(t)$.

Corollary 3.2: Under the conditions of theorem 3, suppose also that $||N_\infty^P(W_A)||_m < 1$, then, defining $\lambda_2(t) = ||N^{P,\mu}(W_A)||_m$, for any μ,

$$|u_j(t) - u_j^{(1)}(t)| \leq \frac{\lambda_2(t)}{1 - \lambda_2(t)} \max_{o \leq t' \leq t} ||u^{(1)}(t') - u^{(o)}(t')||_m$$

$$1 \leq j \leq \ell$$

3.4 A Note on Robustness of the Design

The frequency domain result on robustness outlined in section 2.3 has the following time-domain analogue (see Appendix 10 for the proof):

Proposition 2: If the conditions of theorem 3 hold with (58) replaced by $||N_\infty^P(W_A)|| < 1$, then the closed-loop system of Fig. 1(a) will remain stable with G replaced by $\tilde{G}$ if $\tilde{G}KF$ is controllable and observable, $G-\tilde{G}$ is stable and

$$||N_\infty^P(\tilde{Y}-Y)||_m < \frac{1 - ||N_\infty^P(W_A)||_m}{||N_\infty^P(z_A)||_m} \qquad (63)$$

where z_A is the step response matrix of $(I + KFG_A)^{-1}KF$

3.5 Stability and Measurement Nonlinearities

In reference (1) it was shown that the effect of the inclusion of measurement nonlinearities on stability can be assessed from simulation data. The extension of these ideas to the multivariable case suffers again from the noncommutation of multivariable convolution operators. It has not been possible to see how theorem 2 can easily be generalized to the non-linear case but the following analysis indicates that theorem 3 is capable of generalization. Attention is focussed on the problem of stability assessment only for the purposes of brevity.

Suppose that F is a nonsingular scalar mxm gain matrix used in the approximating feedback scheme of Fig. 1(b) but that the real feedback scheme takes the form of Fig. 2 where the memoryless nonlinearity N has the structure

$$N(y) = Fy + N^{(1)}(y) + N^{(2)}(y) \qquad (64)$$

where $N^{(1)}$ is an m-vector valued nonlinearity of finite incremental gain in the multivariable sense that, for $1 \leq j \leq m$ and all $y, y' \in R^m$,

$$|N_j^{(1)}(y) - N_j^{(1)}(y')| \leq \sum_{k=1}^{m} \nu_{jk} |y_k - y_k'| \tag{65}$$

for suitable choice of positive constants ν_{jk}. The nonlinearity $N^{(2)}$ is assumed to be bounded in the sense that, for all $y \in R^m$ and some choice of constants $\{q_j\}$,

$$|N_j^{(2)}(y)| \leq \frac{q_j}{2}, \quad 1 \leq j \leq m \tag{66}$$

More compactly, defining

$$\nu = \begin{bmatrix} \nu_{11} & \cdots & \nu_{1m} \\ \vdots & & \vdots \\ \nu_{m1} & \cdots & \nu_{mm} \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \tag{67}$$

then (65) and (66) can be written as

$$||N^{(1)}(y) - N^{(1)}(y')||_P \leq \nu ||y - y'||_P \tag{68}$$

$$||N^{(2)}(y)||_P \leq \frac{1}{2} q \tag{69}$$

respectively.

The following result is proved in section 11:

Theorem 4: Suppose that the conditions of theorem 3 are satisfied, then the controller K will input/output stabilize the nonlinear system of Fig. 2 if G is stable and

$$r((I_\ell - N_\infty^P(W_A))^{-1} N_\infty^P(z_A F^{-1}) \ \nu \ N_\infty^P(Y)) < 1 \tag{70}$$

where z_A is as given in Proposition 2.

Note as in all previous results that all elements of data in (70) can be computed from the data G_A,K,F,E and ν. Note also that input/output stability is independent of $N^{(2)}$ and is guaranteed for all 'small enough' gain matrices ν.

4. Controller Design for Discrete Plant using Approximate Models

It has been seen[1] that an approximation theory for continuous scalar systems has a natural extension to discrete systems with synchronous sampling simply by replacing transfer function matrices by z-transfer function matrices, the D-contour by the two contours $z = 1$ and $z = R$ (R 'large') and a suitable definition of the error measure $N_T(E)$. The continuous multivariable theory described in this paper relies explicitly on the underlying scalar definitions. It is clear therefore that all of the results described here carry over to the case of discrete synchronous systems with the same modifications. The details are omitted for brevity.

5. Conclusions

The paper has presented a theoretical generalization of the results of the companion paper[1] to the multivariable case by providing a computationally feasible set of techniques for incoporating the observed differences between the step response data obtained from an uncertain or unknown plant and that of an approximate model into stability and performance assessment predictions based on the approximate model. The frequency response analysis of section two has a natural graphical interpretation that is well-suited to computer-aided-design and, in one special case (Corollary 1.2), is similar to the Gershgorin based procedures of the inverse Nyquist array[5,6] and dyadic expansion[6] methods. The major problem with this analysis is that of all frequency-domain analyses i.e. it is only possible to make precise statements about stability. The techniques of section 3 do make possible the calculation of transient bounds on the implemented feedback scheme in terms of transient responses deduced from the approximating feedback scheme (Theorem 2 and 3) but the techniques necessarily base both their stability and performance assessment on time domain analysis only. The use of transient data in closed-loop stability assessment is not usual in control design but the benefits in both performance assessment and inclusion of nonlinearities (theorem 4) are self-evident and have been amply demonstrated in the scalar case[1].

A vital aspect of all the analysis is that stability and performance assessment is possible using only simple data and without the need to know or have available a detailed plant model. In this sense the work is in the same spirit as that of Davison[15], Porter[16], Koivo[17], Åstrom[2] and Owens[1,3,4] but, in contrast to much of this work, stability is guaranteed over a computable gain range and the approximate model enables the designer to have the choice of using a fairly accurate (and normally high order) model to reduce the uncertainty at the expense of increased design complexity or of using a rough-and-ready model to simplify the design exercise at the expense of producing a conservative design with large uncertainty in transient performance. In both cases the results of the paper guarantee stability!

When compared with the scalar case[1], the multivariable analysis in this paper shows an overall structural similarity but requires (i) a more sophisticated mathematical approach to avoid non-commutation problems and to enable performance assessment in each output separately and (ii) a slight increase in computational complexity to evaluate convolutions of plant responses. The multivariable case is certainly not as straightforward as the scalar case. It opens up a wide variety of computational routes to the solution of the problem e.g. the spectral radius conditions may be more assessable to graphical analysis if an upper bound is used. There are an infinity of upper bounds and it is natural to search for the most convenient for the problem at hand. This will be the subject of further study.

6. Acknowledgements

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Appendixes

8. Proof of Theorem 2: We regard the stability problem as an input-output stability[8] problem in $L_{\infty}^{m}(o, +\infty)$ (the m^{th} Cartesian product of $L_{\infty}(o, +\infty)$) and denote by $L_{\infty e}^{m}(o, +\infty)$ the extended space[8] of $L_{\infty}^{m}(o, +\infty)$). The natural projection of $f \in L_{\infty e}^{m}$ into $L_{\infty}^{m}(O,T)$ (regarded as a subspace of $L_{\infty}^{m}(o, +\infty)$) is denoted $P_T f$. If $L : L_{\infty}^{d_2}(o, +\infty) \to L_{\infty}^{d_1}(o, +\infty)$ is defined by the relation $y = L u$ with $y_i = \sum_j L_{ij} u_j$ and

$$y(t) = F_o u(t) + \int_o^t F_1(t')u(t-t')dt' \tag{71}$$

then, $P_T L_{ij}$ has induced norm

$$||P_T L_{ij}||_{\infty} = |(F_o)_{ij}| + \int_o^T |(F_1(t))_{ij}| dt \tag{72}$$

It is clear from (43) and (25) that we then have

$$||P_T L||_P = ||F_o||_P + \int_o^T ||F_1(t)||_P \, dt \tag{73}$$

$$= N_T^P(\breve{F})$$

where $\breve{F}$ is the step response matrix of L given by (24).

Assuming zero initial conditions, the feedback system of Fig. 1(a) can be regarded as the equation

$$y = GKr - GKF\, y \tag{74}$$

in $L_{\infty e}^{m}$. Applying the truncation operator P_t, invoking causality and, after a little rearrangement, noting that the stability of Fig. 1(b) ensures the invertibility of $(I + G_A KF)$, indicates that (74) has the form, for any $t \geq o$,

$$P_t y = P_t(I + G_A KF)^{-1} GKP_t r - P_t(I + G_A KF)^{-1}(G-G_A)KFP_t y \tag{75}$$

The system is input/output stable if, and only if, this equation with $t = +\infty$ has a solution $y \in L_{\infty}^{m}(o, +\infty)$ whenever $r \in L_{\infty}^{m}(o, +\infty)$. Applying lemma 3 regarding (75) as an equation of the form $y = W_t y$, we see that W_t maps $L^{m}(o,t)$ into itself as condition (c) and the stability of Fig. 1(b) ensure that both $(I + G_A KF)^{-1}(G-G_A)KF$ and $(I + G_A KF)^{-1}GK = (I + G_A KF)^{-1}$ $(G_A + G - G_A)K$ are bounded. Clearly stability is ensured if W_∞ is a P-contraction. But $||W_t(x) - W_t(y)||_P = ||P_t(I + G_A KF)^{-1}(G-G_A)KFP_t(x-y)||_P \leq ||P_t(I + G_A KF)^{-1}(G-G_A)KF||_P \; ||P_t(x-y)||_P$ so that W_t is a P-contraction if

$$r(||P_t(I + G_A KF)^{-1}(G-G_A)KF||_P) < 1 \tag{76}$$

Using (73), $||P_t(I + G_A KF)^{-1}(G-G_A)KF||_P$ can be replaced by $N_t^P(V_A)$ where $V_A(t)$ is the step response matrix of $(I + G_A KF)^{-1}(G-G_A)KF$. Our proof of stability concludes with the observations that suitable controllability and observability assumptions convert input/output stability predictions into asymptotic stability predictions and that, using Laplace transforms,

$$\begin{aligned} V_A(t) &= \mathcal{L}^{-1}((I_m + G_A(s)K(s)F(s))^{-1}(G(s)-G_A(s))K(s)F(s)\,\tfrac{1}{s}) \\ &= \mathcal{L}^{-1}(\{(I_m + G_A(s)K(s)F(s))^{-1}E(s)\}K(s)F(s)) \end{aligned} \tag{77}$$

where $E(s) = (G(s)-G_A(s))\frac{1}{s}$ is the Laplace transform of $E(t)$. $V_A(t)$ can hence be computed from (47).

As $N_t^P(V_A) \leq N_\infty^P(V_A)$ for all $t \geq o$, Fact 5 indicates that W_t is a P-contraction if W_∞ is a P-contraction and hence $P_t y$ can be obtained by successive approximation $P_t y^{(k+1)} = W_t P_t y^{(k)}$ for any choice of $y^{(o)} \in L_\infty^m(o, +\infty)$. Using (75), the equation $P_t y^{(1)} = W_t P_t y^{(o)}$ takes the form, after a little rearrangement,

$$\begin{aligned} P_t y^{(1)} &= P_t(I + G_A KF)^{-1} G_A K\, r \\ &\quad + P_t(I + G_A KF)^{-1}(G-G_A)(K H_o \alpha - KFH_1 \beta) \\ &= P_t\, y_A + P_t \eta \end{aligned} \tag{78}$$

where, for all $t \geq o$,

$$\eta(t) = ((I + G_A KF)^{-1}(G-G_A)(KH_o\alpha - KFH_1\beta))(t)$$

$$= \mathcal{L}^{-1}((I + G_A(s)K(s)F(s))^{-1}(G(s) - G_A(s))(K(s)H_o(s)\alpha$$

$$- K(s)F(s)H_1(s)\ \beta)\ \frac{1}{s})$$

$$= \int_o^t H_E(t-t')\ \{H_{KH_o}(t')\alpha - H_{KFH_1}(t')\beta\}dt' \qquad (79)$$

by an argument similar to that used in (77). Using lemma (3) we obtain the error estimate

$$||P_t y - P_t y^{(1)}||_P \leq (I_m - N_t^P(V_A))^{-1} N_t^P(V_A)\,||P_t y^{(1)} - P_t y^{(o)}||_P \qquad (80)$$

which implies (51) by the definition of $||.||_P$ and the norm in $L_\infty(o,t)$.

The final observations follow from the observation that $N_t^{P,\mu}(V_A) \geq N_t^P(V_A)$ for all choices of μ_{ij} (including $\mu_{ij}(t) = +\infty$) and that $N_t^{P,\mu}(V_A) \leq N_\infty^P(V_A)$ when it follows from the series expansion of the inverse that $(I - N_t^P(V_A))^{-1}N_t^P(V_A) \leq (I - N_t^{P,\mu}(V_A))^{-1}N_t^{P,\mu}(V_A)$.

9. Proof of Theorem 3 and Corollary 3.1: The stability problem is regarded as in appendix 8 but the closed-loop relations are written in terms of the input

$$u = Kr - KFG\ u \qquad (81)$$

After a little rearrangement this becomes

$$P_t u = P_t(I + KFG_A)^{-1}KP_t r - P_t(I + KFG_A)^{-1}KF(G - G_A)P_t\ u$$

$$= P_t\ u_A - P_t(I + KFG_A)^{-1}KF(G - G_A)P_t\ u \qquad (82)$$

Regarding this as a relation $P_t\ u = W_t P_t u$ in $L_\infty^\ell(o,t)$, the assumptions ensure that W_t maps $L_\infty^\ell(o,t)$ into itself and, noting that

$$0 \leq N_t^P(W_A) = ||P_t(I + KFG_A)^{-1}KF(G - G_A)||_P$$

$$\leq ||(I + KFG_A)^{-1}KF(G - G_A)||_P$$

$$= N_\infty^P(W_A) \qquad (83)$$

where $W_A(t)$ is the step response matrix of $(I + KFG_A)^{-1}KF(G - G_A)$, it is seen that W_t is a P-contraction for all $t \geq o$ (including $t = +\infty$) if $r(N_\infty^P(W_A)) < 1$. Input output stability follows as in the proof of theorem 2, noting that W_A can be computed in the manner indicated in theorem 3. Asymptotic stability then follows from condition (b).

Given the above conditions, u can be obtained by successive approximation with initial guess $u^{(o)}$. More precisely, lemma 3 indicates that, if

$$P_t u^{(1)} = P_t u_A - P_t(I + KFG_A)^{-1}KF(G - G_A)P_t u^{(o)}$$

$$= P_t u_A - P_t(I + KFG_A)^{-1}KF(G - G_A)P_t H_2\beta \qquad (84)$$

then, for all $t \geq o$,

$$||P_t u - P_t u^{(1)}||_P \leq (I_\ell - N_t^P(W_A))^{-1}N_t^P(W_A)||P_t u^{(1)} - P_t u^{(o)}||_P \qquad (85)$$

which implies (60) as required. The form of $u^{(1)}$ is $u^{(1)} = u_A + \xi$ where

$$\xi(t) = -((I_\ell + KFG_A)^{-1}KF(G - G_A)H_2\beta)(t)$$

$$= -\mathcal{L}^{-1}((I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)(G(s) - G_A(s))H_2(s)\beta\tfrac{1}{s})$$

$$= -\mathcal{L}^{-1}((I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)E(s)H_2(s))\beta$$

$$= -\mathcal{L}^{-1}(W_A(s)H_2(s))\beta \qquad (86)$$

which is simply (59).

Finally, Corollary 3.1 follows by writing

$$||y(t) - y^{(1)}(t)||_P \leq ||P_t(y-y^{(1)})||_P = ||P_t(Gu - G_A u^{(1)})||_P$$

?

$$\leq ||P_t G P_t (u - u^{(1)})||_P + ||P_t (G - G_A) P_t u^{(1)}||_P$$

$$\leq ||P_t G||_P \cdot ||P_t (u-u^{(1)})||_P + ||P_t (G - G_A)||_P \cdot ||P_t u^{(1)}||_P \quad (87)$$

which is simply (62) as $||P_t G||_P = N_t^P(Y)$ and $||P_t (G - G_A)||_P = N_t^P(E)$ and $||P_t (u - u^{(1)})||_P \leq \tilde{\varepsilon}(t)$.

10. Proof of Proposition 2:

Let $\tilde{W}_A$ be the 'W_A- matrix' generated by $\tilde{G}$, then, by considering elements, note that $N_\infty^P(\tilde{W}_A) \leq N_\infty^P(W_A) + N_\infty^P(\tilde{W}_A - W_A)$ and that

$$||(I+KFG_A)^{-1}KF(\tilde{G}-G_A) - (I+KFG_A)^{-1}KF(G-G_A)||_P \leq ||(I+KFG_A)^{-1}KF||_P ||\tilde{G}-G||_P \quad (88)$$

indicates that

$$N_\infty^P(\tilde{W}_A - W_A) \leq N_\infty^P(z_A)\ N_\infty^P(\tilde{Y} - Y) \quad (89)$$

It follows that $||N_\infty^P(\tilde{W}_A - W_A)||_m \leq ||N_\infty^P(z_A)||_m \cdot ||N_\infty^P(\tilde{Y}-Y)||_m$

and hence that

$$||N_\infty^P(\tilde{W}_A)||_m \leq ||N_\infty^P(W_A)||_m + ||N_\infty^P(\tilde{W}_A - W_A)||_m$$

$$\leq ||N_\infty^P(W_A)||_m + ||N_\infty^P(z_A)||_m \cdot ||N_\infty^P(\tilde{Y}-Y)||_m$$

$$< 1 \quad (90)$$

by (63). Equation (90) implies that $r\ (N_\infty^P(\tilde{W}_A)) < 1$ and hence, with the other assummptions, that the perturbed feedback system is stable.

11. Proof of Theorem 4: The feedback system of Fig. 2 is characterized by the equation

$$u_{n\ell} = Kr - K(F + N^{(1)} + N^{(2)})G\ u_{n\ell} \quad (91)$$

or, after a little rearrangement

$$u_{n\ell} = L_c^u((r - N^{(2)}G\ u_{n\ell}) - N^{(1)}Gu_{n\ell}) \quad (92)$$

where L_c^u is the map $r \rightarrow u$ defined by Fig. 1(a). The conditions of theorem 3 ensure that L_c^u is bounded in $L_\infty^\ell(o,+\infty)$. Regarding $r - N^{(2)}G\ u_{n\ell} \in L_\infty^\ell(o,\infty)$ as known, (92) can be written as $u_{n\ell} = W_{n\ell}(u_{n\ell})$ where $W_{n\ell}$ maps $L_\infty^\ell(o,+\infty)$

into itself and satisfies

$$||W_{n\ell}(u') - W_{n\ell}(u'')||_P = ||L_C^u(N^{(1)}Gu' - N^{(1)}Gu'')||_P$$

$$\leq ||L_C^u||_P \, ||N^{(1)}G\, u' - N^{(1)}Gu''||_P$$

$$\leq ||L_C^u||_P \, \nu \, ||Gu' - Gu''||_P$$

$$\leq ||L_C^u||_P \, \nu \, ||G||_P \, ||u'-u''||_P \tag{93}$$

$W_{n\ell}$ is hence a P-contraction if $r(||L_C^u||_P \, \nu \, ||G||_P) < 1$ and the nonlinear map $r \to u_{n\ell}$ defined by (91) is then bounded. The stability of G then implies the boundedness of $y_{n\ell} = Gu_{n\ell}$. The proof of the theorem follows by noting that $||G||_P = N_\infty^P(Y)$ and that $||L_C^u||_P \leq (I_\ell - N_\infty^P(W_A))^{-1} N_\infty^P(z_A F^{-1})$. To prove the second inequality, use $H_2 = 0$ in theorem 3 to yield $\xi = 0$, $u^{(o)} = 0$ and $u^{(1)} = u_A$ when the bound (60) takes the form

$$||u - u_A||_P \leq (I_\ell - N_\infty^P(W_A))^{-1} N_\infty^P(W_A) \, ||u_A||_P \tag{94}$$

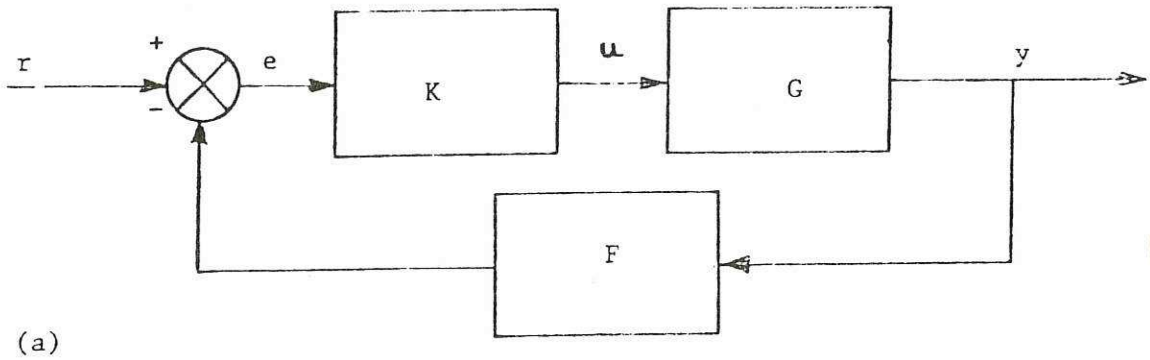
Clearly $||u||_P \leq ||u - u_A||_P + ||u_A||_P$ and hence, using (94)

$$||u||_P \leq (I - N_\infty^P(W_A))^{-1} \, ||u_A||_P \tag{95}$$

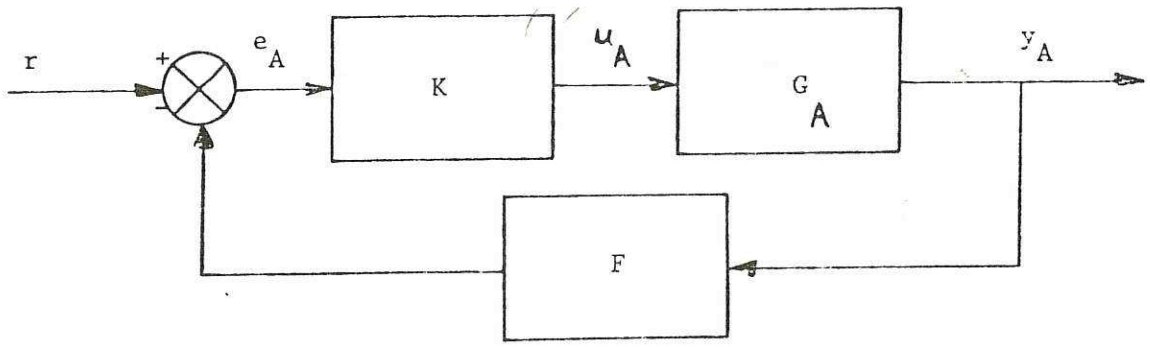
but $||u_A||_P \leq ||(I_\ell + KFG_A)^{-1}K||_P \, ||r||_P$ and $||(I_\ell + KFG_A)^{-1}K||_P = N_\infty^P(z_A F^{-1})$ from the definition of z_A. The required bound on $||L_C^u||_P$ is hence obtained by substitution into (95) to obtain

$$||u||_P \leq (I - N_\infty^P(W_A))^{-1} N_\infty^P(z_A F^{-1}) \, ||r||_P \tag{96}$$

and comparing with the inequality $||u||_P \leq ||L_C^u||_P \, ||r||_P$.



(a)



(b)

Fig. 1 (a) Real and (b) Approxmating Feedback Systems

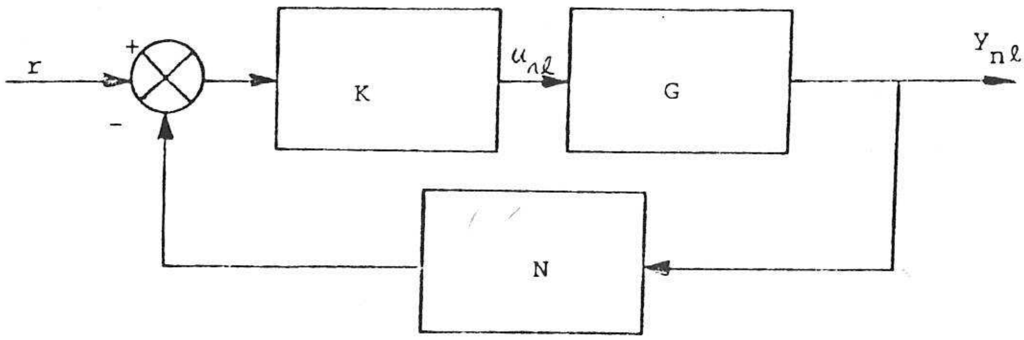


Fig. 2 Nonlinear Feedback Scheme