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VARIABLE STRUCTURE SYSTEMS AND SYSTEM ZEROS

by

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Summary

The analysis of variable structure systems in the sliding mode yields the concept of equivalent control which leads naturally to a new method for determining the zeros and zero directions of linear multivariable systems. The analysis presented is conceptually easy and computationally attractive.

1. Introduction

Numerous methods have been proposed for calculating system zeros. The concepts underlying such calculations range from the use of high gain output feedback[1,2,3] to geometric formulations[4,5] and the use of generalised inverses[6]. Other algorithms obtain the zeros as the poles of the system minimal order right or left inverse[7] or by pole-zero cancellation techniques[8].

In the present study the relationship between variable structure systems (VSS) and the zeros of the linear time-invariant multivariable system S(A, B, C)

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + B\underset{\sim}{u}; \quad \underset{\sim}{x} \in R^n, \ \underset{\sim}{u} \in R^m \tag{1}$$

$$\underset{\sim}{y} = C\underset{\sim}{x} \qquad \underset{\sim}{y} \in R^m, \ m < n \tag{2}$$

are investigated. It is assumed throughout that B and C have full rank and $|CB| \neq 0$. (The case where CB is singular can be considered with suitable modifications and will be reported elsewhere.)

A new method of computing the zeros of S(A, B, C) is derived by considering the theory of variable structure systems. It is shown that the system zeros are the eigenvalues of a reduced order matrix which arises naturally in variable structure systems design. The algorithm is computationally simple and yields insight into the operation of variable structure systems in the sliding mode.

The paper begins with a brief overview of variable structure systems design. The application of this theory to the computation of system zeros and zero directions is then investigated and the physical interpretation of the method discussed. Additional properties of the proposed algorithm are investigated by decomposing the state-space, and worked examples are included to illustrate the validity of the method.

2. Variable Structure Systems in the sliding mode

Variable structure systems[9,10] are characterized by a discontinuous control action which changes structure upon reaching a set of switching surfaces. The control has the form

$$u_i = \begin{cases} u_i^+(\underset{\sim}{x}) & s_i(\underset{\sim}{x}) > 0 \\ u_i^-(\underset{\sim}{x}) & s_i(\underset{\sim}{x}) < 0 \end{cases} \qquad (3)$$

where u_i is the ith component of $\underset{\sim}{u}$ and $s_i(\underset{\sim}{x})$ is the ith of the m switching hyperplanes which satisfy

$$\underset{\sim}{s}(x) = C\underset{\sim}{x} = \underset{\sim}{0} \qquad \underset{\sim}{s}(\underset{\sim}{x}) \in R^m. \qquad (4)$$

The above system with discontinuous control is termed a variable structure system (VSS) since the effect of the switching hyperplanes is to alter the feedback structure of the system.

Sliding motion occurs, if at a point on a switching surface, $s_i(\underset{\sim}{x}) = 0$, the directions of motion along the state trajectories on either side of the surface are not away from the switching surface. The state then slides and remains for some finite time on the surface $s_i(\underset{\sim}{x}) = 0$[9,10]. The conditions for sliding motion to occur on the ith hyperplane may be stated in numerous ways. We need

$$\lim_{s_i \to 0^+} \dot{s}_i \le 0 \quad \text{and} \quad \lim_{s_i \to 0^-} \dot{s}_i \ge 0 \qquad (5)$$

or equivalently

$$s_i \dot{s}_i \le 0 \qquad (6)$$

in the neighbourhood of $s_i(\underset{\sim}{x}) = 0$. In the sliding mode the system satisfies the equations

$$s_i(\underset{\sim}{x}) = 0 \quad \text{and} \quad \dot{s}_i(\underset{\sim}{x}) = 0 \qquad (7)$$

and the system has invariance properties, yielding motion which is independent of certain system parameters and disturbances. Thus variable structure systems are usefully employed in systems with uncertain and time-varying parameters.

Consider the behaviour of the system dynamics during sliding when the sliding mode exists on all the hyperplanes assuming that the non-unique control $\underset{\sim}{u}$ has been suitably chosen. During sliding equation (4) and its derivative

$$\dot{\underset{\sim}{s}} = C\dot{\underset{\sim}{x}} = \underset{\sim}{0} \qquad (8)$$

hold and the equations governing the system dynamics may be obtained by substituting an equivalent control $\underset{\sim}{u}_{eq}$ for the original control $\underset{\sim}{u}$. From (1) and (8) the linear equivalent control is

$$\underset{\sim}{u}_{eq} = -(CB)^{-1}\, CA\underset{\sim}{x} \qquad (9)$$

and substituting in eqn (1) yields the equations governing the system dynamics in the sliding mode

$$\dot{\underset{\sim}{x}} = [I - B(CB)^{-1}C]A\underset{\sim}{x} = A_{eq}\underset{\sim}{x}. \qquad (10)$$

Notice that during sliding m state variables can be expressed in terms of the remaining (n-m) state variables from (4). This allows a reduction in the order of the system matrix (see section 5).

3. System zeros

The use of variable structure systems theory in calculating the system zeros is motivated by the observation that variable structure systems in the sliding mode are very closely related to the output zeroing problem.

3.1 The output zeroing problem

MacFarlane and Karcanias[11] state that a necessary and sufficient condition for an input

$$\underset{\sim}{u}(t) = \underset{\sim}{g}\, \exp(zt)\, 1(t) \qquad (11)$$

to yield rectilinear motion in the state space of the form

$$\underset{\sim}{x}(t) = \underset{\sim}{x}_o\, \exp(zt)\, 1(t) \qquad (12)$$

such that

$$\underset{\sim}{y}(t) \equiv 0 \quad \text{for} \quad t \geq 0 \qquad (13)$$

is that

$$\begin{bmatrix} zI-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_o \\ \underset{\sim}{g} \end{bmatrix} = \underset{\sim}{0} = P(z) \begin{bmatrix} \underset{\sim}{x}_o \\ \underset{\sim}{g} \end{bmatrix} \qquad (14)$$

where z is a system zero, $\underset{\sim}{x}_o$ the related state zero direction, $\underset{\sim}{g}$ the input zero direction and 1(t) denotes the Heaviside unit step function. At the complex frequency s = z the state ($\underset{\sim}{x}_o$) and input ($\underset{\sim}{g}$) zero directions satisfy

$$\begin{pmatrix} \underset{\sim}{x}_o \\ \underset{\sim}{g} \end{pmatrix} \in N\{P(z)\} \qquad (15)$$

where N{P(z)} is the null space or kernel of P(z)

3.2 Zero calculation using VSS theory

Considering the switching functions $\underset{\sim}{s}$ to be the system outputs $\underset{\sim}{y}$, VSS and the output zeroing problem reduce to the selection of a control $\underset{\sim}{u}$ and a state vector $\underset{\sim}{x}$ such that the output $\underset{\sim}{y}(t) \equiv \underset{\sim}{0}$ for $t \geq 0$. Calculation of the system zeros using VSS theory exploits the fact that if $\underset{\sim}{y}(t) = 0$ for $t \geq 0$ then $\dot{\underset{\sim}{y}}(t) = 0$ for $t \geq 0$. The algorithm which consists of determining the eigenvalues of the matrix A_{eq} which arises when the feedback control yielding $\dot{\underset{\sim}{y}}(t) = \underset{\sim}{0}$ is applied to S(A, B, C), is summarised below.

(i) Calculate $\underset{\sim}{u}$ using $\dot{\underset{\sim}{y}} = C\dot{\underset{\sim}{x}} = \underset{\sim}{0}$. This yields

$$C(A\underset{\sim}{x} + B\underset{\sim}{u}) = \underset{\sim}{0}$$

and

$$\underset{\sim}{u} = -(CB)^{-1}CA\underset{\sim}{x} \qquad (16)$$

which is the equivalent control $\underset{\sim}{u}_{eq}$ of VSS theory.

(ii) Substitute in (1) to yield

$$\dot{\underset{\sim}{x}} = [I - B(CB)^{-1}C]A\underset{\sim}{x} = A_{eq}\underset{\sim}{x}. \qquad (17)$$

(iii) Determine the eigenvalues and eigenvectors of A_{eq}.

(iv) Any eigenvector $\underset{\sim}{x}_o^i$ satisfying $C\underset{\sim}{x}_o^i = \underset{\sim}{0}$ is a state zero direction and the corresponding eigenvalue is a system zero.

(v) The corresponding input zero directions are given by

$$\underset{\sim}{g}_i = \underset{\sim}{u}_{eq}\Big|_{\underset{\sim}{x}=\underset{\sim}{x}_o^i} = -(CB)^{-1}CA\underset{\sim}{x}_o^i. \qquad (18)$$

In practice we need consider only the (n-m) eigenvalues $\lambda_i \in \sigma(A_{eq}) - \{0\}^m$ in steps (iii) and (iv). This becomes clear in section 4 (see equation (33)) since it is evident that the eigenvectors associated with the m zero-valued eigenvalues of A_{eq} do not lie in the null space of C, i.e. $C\,\underset{\sim}{x}_o^{\,i} \neq \underset{\sim}{0}$.

Consider an example to illustrate the procedure.

Example 1:

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \underset{\sim}{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\underset{\sim}{y} = [20 \quad 9 \quad 1] \underset{\sim}{x} \; .$$

The eigenvalues of

$$A_{eq} = [I - B(CB)^{-1}C]A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -9 \end{bmatrix}$$

are 0, -4, -5 with corresponding eigenvectors

$$\underset{\sim}{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underset{\sim}{e}_2 = \begin{bmatrix} 1 \\ -4 \\ 16 \end{bmatrix}, \quad \text{and} \quad \underset{\sim}{e}_3 = \begin{bmatrix} 1 \\ -5 \\ 25 \end{bmatrix} .$$

Since $C\underset{\sim}{e}_1 \neq \underset{\sim}{0}$, $C\underset{\sim}{e}_2 = \underset{\sim}{0}$ and $C\underset{\sim}{e}_3 = \underset{\sim}{0}$, the zeros of the system are -4 and -5 with state zero directions

$$\begin{bmatrix} 1 \\ -4 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -5 \\ 25 \end{bmatrix} .$$

The respective (scalar) input zero directions are from (18)

$$g = [6 \quad -9 \quad -3] \begin{bmatrix} 1 \\ -4 \\ 16 \end{bmatrix} = -6$$

and

$$g = [6 \quad -9 \quad -3] \begin{bmatrix} 1 \\ -5 \\ 25 \end{bmatrix} = -24 \; .$$

Note that the zero directions are obtained without resorting to the higher dimensional (n+m) system matrix P(s) (15).

3.3 Physical Interpretation

The (VSS) equivalent control is such that it assigns some of the closed loop eigenvalues and the corresponding eigenvectors to coincide with the system zeros and state zero directions, thus driving the system to be unobservable. Consider the observability matrix

$$O(C, A_{eq}) = \left[C, CA_{eq}, CA_{eq}^2, \ldots\ldots, CA_{eq}^{n-1}\right]^T . \qquad (19)$$

Since $$CA_{eq} = C\left[I - B(CB)^{-1}C\right]A = 0 \qquad (20)$$

it follows that

$$CA_{eq}^{\,i} = 0 \qquad i = 1, 2, 3, \ldots n-1$$

and

$$\text{rank}[O(C, A_{eq})] = \text{rank}\ (C) = m < n.$$

The action of the equivalent control $\underset{\sim}{u}_{eq}$ is therefore to drive the system to be unobservable.

4. Decomposition of the state space

The calculation of the system zeros using the proposed method relies on pole-zero cancellation through appropriate state feedback using the equivalent control $\underset{\sim}{u}_{eq}$. The zeros of the system are given by the eigenvalues of a certain matrix associated with the (n-m) dimensional unobservable subspace. To find this matrix an unobservability decomposition is employed[12].

Introduce the similarity transformation

$$\bar{\underset{\sim}{x}} = T\underset{\sim}{x} \qquad (21)$$

where

$$T = \begin{bmatrix} C \\ \hline P \end{bmatrix} \begin{matrix} \}m \\ \}n-m \end{matrix} \qquad (22)$$

C is the output matrix which spans the observable subspace, and P is a matrix chosen to ensure that T is nonsingular. Therefore

$$\dot{\bar{x}} = TA_{eq} T^{-1}\bar{x} \tag{23}$$

$$y = CT^{-1} \bar{x} . \tag{24}$$

Define the partitioned inverse of T as

$$T^{-1} = [\, \underbrace{V}_{m} : \underbrace{W}_{n-m} \,]\}n \tag{25}$$

and recall that a generalised inverse of a $(k\times\ell)$-matrix S is a matrix S^g satisfying[13]

$$SS^gS = S . \tag{26}$$

Using these definitions and exploiting the identity $TT^{-1} = I_n$ it can be readily shown that

$$CV = I_m \tag{27}$$

$$PW = I_{n-m} \tag{28}$$

$$PV = 0 \tag{29}$$

$$CW = 0 . \tag{30}$$

Therefore V and W can be taken to be generalized inverses of C and P such that conditions (29) and (30) are satisfied, i.e. $PC^g = 0$ and $CP^g = 0$. From equations (17) and (23)

$$\dot{\bar{x}} = \left[\begin{array}{c} C \\ \hline P \end{array}\right] A_{eq}[C^g \;\; P^g]\bar{x} = \begin{bmatrix} CA_{eq}C^g & CA_{eq}P^g \\ PA_{eq}C^g & PA_{eq}P^g \end{bmatrix} \bar{x} \tag{31}$$

and

$$y = C[C^g : P^g]\bar{x} . \tag{32}$$

From (20), $CA_{eq} = 0$ and therefore

$$\dot{\bar{x}} = \begin{bmatrix} 0_m & 0 \\ PA_{eq}C^g & PA_{eq}P^g \end{bmatrix} \bar{x} \tag{33}$$

$$y = [I_m : 0]\bar{x} . \tag{34}$$

This is the standard unobservability decomposition, and the zeros are therefore given by the (n-m) eigenvalues of $PA_{eq}P^g$.

The calculation of the inverse of T yields V and W. However, if the computation of the matrix inverse is to be avoided, the matrix P should be chosen such that equations (29) and (30) are satisfied. Furthermore, since $P^g \in N\{C\}$, a possible choice for P^g is $M = N\{C\}$. P is then equal to M^g subject to the condition $M^gM = I_m$. The zeros of the system are therefore equal to the eigenvalues of $M^gA_{eq}M$ where M is a basis matrix for the null space of C.

Since the state eigenvectors $\underset{\sim}{\omega}_i$ (or state zero directions) lie in the null space of C, they can be expressed as a linear combination of the basis vectors of M,

$$\underset{\sim}{\omega}_i = M\underset{\sim}{\alpha}_i \qquad \underset{\sim}{\alpha}_i \neq \underset{\sim}{0} \,. \qquad i=1,2,\ldots,n-m \tag{35}$$

Thus

$$A_{eq}\underset{\sim}{\omega}_i = A_{eq}M\underset{\sim}{\alpha}_i = z_iM\underset{\sim}{\alpha}_i$$

where z_i is the zero associated with $\underset{\sim}{\omega}_i$ and

$$M^gA_{eq}M\underset{\sim}{\alpha}_i = z_iM^gM\underset{\sim}{\alpha}_i = z_i\underset{\sim}{\alpha}_i \,. \tag{36}$$

Therefore $\underset{\sim}{\alpha}_i$ is an eigenvector of $M^gA_{eq}M$ corresponding to the zero z_i.

To calculate the state zero directions $\underset{\sim}{\omega}_i$ the eigenvectors $\underset{\sim}{\alpha}_i$ of the (n-m) th order matrix $M^gA_{eq}M$ should be determined and substituted in (35). The input zero directions $\underset{\sim}{g}_i$ are given by replacing $\underset{\sim}{x}_o$ by $M\underset{\sim}{\alpha}_i$ in equation (12) to yield

$$\underset{\sim}{g}_i = -(CB)^{-1}CAM\underset{\sim}{\alpha}_i \,. \tag{37}$$

The algorithm presented above resembles that of the NAM method of MacFarlane and Kouvaritakis[4]. In both methods the zeros are determined as the eigenvalues of an (n-m)-dimensional matrix. It can be easily shown that the matrix $N_e = M^g[I - B(CB)^{-1}C]$ qualifies for the matrix N in the NAM algorithm. This is because

$$N_e B = 0$$

and

$$N_e M = I_{n-m}$$

where the matrix M is the same in both methods. However, the approach proposed in this paper offers the advantage of calculating the state and input zero directions without resorting to the null space of the (n+m)-dimensional system matrix P(s). Our technique involves only certain matrix multiplications in the calculation of the zero directions.

Example 2: Consider the system[4]

$$A = \begin{bmatrix} -1 & 1 & 3 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -3 & -1 \\ 0 & 3 & -1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & -1 & -3 & 2 \end{bmatrix}$$

From (10)

$$A_{eq} = \frac{1}{3} \begin{bmatrix} -2 & -6 & 6 & 4 \\ 1 & 12 & 42 & -29 \\ 1 & 6 & 12 & -11 \\ 2 & 15 & 39 & -31 \end{bmatrix}$$

Also

$$P^g = M = N\{C\} = \begin{bmatrix} 2 & -6 \\ 2 & 0 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} .$$

From (36) the zeros are given by the eigenvalues of

$$M^g A_{eq} M = \begin{bmatrix} -0.5 & -1.5 \\ 0.5 & -2.5 \end{bmatrix} .$$

Therefore, the system has two zeros at -1 and -2.

The eigenvectors $\underset{\sim}{\alpha}_i$ of $M^g A_{eq} M$ are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The state zero directions are given by $\underset{\sim}{\omega}_i = M\underset{\sim}{\alpha}_i$ (36)

and hence $\underset{\sim}{\omega}_1 = [0 \quad 6 \quad 2 \quad 6]^T$ and $\underset{\sim}{\omega}_2 = [-4 \quad 2 \quad 2 \quad 4]^T$.

The input zero directions (37) are

$$\underset{\sim}{g}_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{g}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} .$$

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6. Conclusions

It has been shown that the study of a reduced order system which arises in the design of variable structure systems leads to a new method of calculating multivariable system zeros. The resulting algorithm is computationally fast and simple to implement. It has particular advantages when determining the state and input zero directions since the calculation of the (n+m)-dimensional nullspace of the system matrix P(s) is avoided.

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THE MODELLING OF SEMIFLEXIBLE CONVEYOR STRUCTURES FOR COAL-FACE STEERING INVESTIGATIONS

PART 1: SPATIALLY DISCRETE MODELS

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