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A MULTIVARIABLE FEEDBACK CONTROL PROBLEM WITH APPLICATION TO STRIP SHAPE CONTROL FOR SENDZIMIR MILLS

by

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ABSTRACT

Consideration is given to the design of feedback controllers for a plant with transfer function matrix $G(s) = \mathrm{diag}\{g_i(s)\}G_m$ where G_m is a square and constant matrix and $g_i(s)$ is a scalar transfer function, $1 \leq i \leq m$. Particular emphasis is placed on the case when $g_i(s) = g(s)$, $1 \leq i \leq m$, and G_m is singular or 'almost' singular and the robustness of the design with respect to errors in G_m is represented in terms of a system of strict inequalities. An application to strip shape control for Sendzimir mills is indicated.

1. Introduction

We consider an m-input/m-output system with transfer function matrix (TFM)

$$G(s) = \text{diag}\{g_1(s),\ldots,g_m(s)\}G_m \qquad \ldots(1)$$

where G_m is a real, constant mxm matrix and g(s) is a strictly proper, stable transfer function (TF). The stability assumption is not necessary but is motivated by the objective of applying the analysis to strip shape control for Sendizimir mills which can be approximated[1] by a TFM of the form of (1). The objective of the analysis is to design a unity negative feedback system[2] with forward path controller K(s) as illustrated in Fig.1 to ensure the stability and satisfactory transient performance characteristics required. In the following analysis we will distinguish between the cases when G_m is nonsingular and when G_m is singular or almost singular. In both cases it is shown that the robustness of the design can be represented by a system of strict inequalities.

2. The Case of $|G_m| \neq 0$

It is natural to set

$$K(s) = G_m^{-1}\,\text{diag}\{k_1(s),\ldots,k_m(s)\} \qquad \ldots(2)$$

where $k_i(s)$ is a proper scalar TF, $1 \leq i \leq m$. It is immediately verified that

$$|I_m + G(s)K(s)| = \prod_{i=1}^{m} (1+g_i(s)k_i(s)) \qquad \ldots(3)$$

and that the closed-loop TFM

$$H_c(s) = (I_m+G(s)K(s))^{-1}G(s)K(s) = \mathrm{diag}\{h_i(s)\}_{1\leq i\leq m}$$

$$h_i(s) = g_i(s)k_i(s)/(1+g_i(s)k_i(s)) \quad , \quad 1\leq i\leq m \qquad \ldots(4)$$

indicating that the closed-loop multivariable system is non-interacting and also stable if, and only if, the scalar feedback systems $h_i(s)$ shown in Fig.2 are stable.

The simplicity of the above analysis is deceptive as it relies crucially upon the invertibility of G_m. If G_m is singular it is certainly necessary to modify the approach. It is also necessary to change the approach is G_m is 'almost singular' as small errors in estimation of elements of G_m could then lead to large errors in elements of G_m^{-1}. The resulting control system is hence very sensitive to such modelling errors and possibly unstable!

3. The Case of G_m 'Almost Singular'

In the remainder of the paper we assume that $g_i(s) = g(s)$, $1\leq i\leq m$, when

$$G(s) = g(s)G_m \qquad \ldots(5)$$

Suppose that G_m is diagonalizable by the mxm nonsingular transformation T to give

$$T^{-1}G_mT = \mathrm{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_m\} \qquad \ldots(6)$$

and that the eigenvalues satisfy the separation condition

$$\mu_2 \triangleq \min_{1\leq i\leq \ell} |\lambda_i| >> \max_{\ell+1\leq i\leq m} |\lambda_i| \triangleq \mu_1 \qquad \ldots(7)$$

for some ℓ. It is trivially verified that $\{\lambda_i\}_{1\leq i\leq \ell}$ is invariant under complex conjugation. Intuitively the eigenvalues $\lambda_{\ell+1},\ldots,\lambda_m$ can be identified with the zero eigenvalues and the 'small' eigenvalues that are sensitive to modelling errors. Write also

$$T = [T_1, T_2] \quad , \quad T^{-1} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \qquad \ldots(8)$$

where T_1 is the $m \times \ell$ matrix of eigenvectors of T corresponding to the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_\ell$ and V_1 is the $\ell \times m$ matrix of 'dual' eigenvectors corresponding to these eigenvalues. Note that $T^{-1}T = I_m$ indicates that

$$V_1T_1 = I_\ell \ , \ V_2T_2 = I_{m-\ell} \ , \ V_1T_2 = 0 \ , \ V_2T_1 = 0 \qquad \ldots(9)$$

The physical interpretation of this decomposition is seen by writing

$$G(s) = T \text{ diag } \{\lambda_i\}_{1\leq i\leq m} T^{-1}g(s)$$

$$= T_1 \left[g(s)\text{diag}\{\lambda_i\}_{1\leq i\leq \ell}\right]V_1$$

$$+ T_2 \left[g(s)\text{diag}\{\lambda_i\}_{\ell+1\leq i\leq m}\right] V_2 \qquad \ldots(10)$$

or, with the obvious identification of G_1 and G_2,

$$G(s) = T_1G_1(s)V_1 + T_2G_2(s)V_2 \qquad \ldots(11)$$

which is illustrated in block form in Fig.3. Clearly the 'G_1 loop' corresponding to the large 'insensitive' eigenvalues dominates the response characteristics of the system as, roughly speaking,

$$y = Gu = y_1 + y_2 \qquad \ldots(12)$$

where $y_1 = T_1G_1V_1u = O(\mu_2) >> O(\mu_1) = T_2G_2V_2u = y_2$. As G_2 is stable it is tempting therefore to ignore this loop in design. This notion can be formalized by choosing

$$K(s) = T_1K_1(s)V_1 \quad , \quad K_1(s) = \text{diag}\{k_i(s)\}_{1\leq i\leq \ell} \qquad ...(13)$$

when it is easily verified that

$$T^{-1}K(s)T = \begin{pmatrix} K_1(s) & 0 \\ 0 & 0 \end{pmatrix} \qquad ...(14)$$

and hence that the return-difference takes the form

$$|I_m+G(s)K(s)| \equiv |I_\ell+G_1(s)K_1(s)| \equiv \prod_{i=1}^{\ell} (1+g(s)\lambda_ik_i(s)) \qquad ...(15)$$

and that the closed-loop TFM

$$H_c(s) = (I_m+GK)^{-1}GK$$

$$= T_1(I_\ell+G_1(s)K_1(s))^{-1}G_1(s)K_1(s)V_1 \qquad ...(16)$$

Remembering that G_2 is stable it is clear that K stabilizes G if, and only if, K_1 stabilizes G_1 in the configuration of Fig.4. This configuration has closed-loop TFM

$$\tilde{H}_c = (I_\ell+G_1K_1)^{-1}G_1K_1 = \text{diag}\{\frac{g\lambda_ik_i}{1+g\lambda_ik_i}\}_{1\leq i\leq \ell} \qquad ...(17)$$

and (16) can be written as

$$H_c(s) = T_1\tilde{H}_c(s)V_1 \qquad ...(18)$$

A particularly simple form is obtained by choosing

$$\lambda_ik_i(s) = k(s) \quad , \quad 1\leq i\leq \ell \qquad ...(19)$$

when

$$\tilde{H}_c(s) = \frac{g(s)k(s)}{1+g(s)k(s)}I_\ell \quad , \quad H_c(s) = \frac{g(s)k(s)}{1+g(s)k(s)}T_1V_1 \qquad ...(20)$$

Finally, although the sensitivity of the design to the small eigenvalues (and hence the 'near' singularity of G_m) has been reduced, we have lost some performance as illustrated by noting that the choice of reference demand r(s) of the form

$$r(s) = T_2\hat{r}(s) \qquad \ldots(21)$$

yields the output from zero initial conditions

$$y = H_c r = T_1\tilde{H}_c V_1 T_2 \hat{r} \equiv 0 \qquad \text{(by (9))} \qquad \ldots(22)$$

ie the closed-loop system does not respond to demands in the subspace spanned by the eigenvectors corresponding to the 'small, sensitive' eigenvalues. In contrast the response to the demand $r = T_1\hat{r}$ in the subspace spanned by the eigenvectors of the 'large, insensitive' eigenvalues is simply (using (9))

$$y = H_c r = T_1\tilde{H}_c V_1 T_1 \hat{r} = T_1 \tilde{H}_c \hat{r} = T_1 \hat{y} \not\equiv 0 \qquad \ldots(23)$$

where $\hat{y}$ is the response of the $\ell \times \ell$ feedback system of Fig.4 to the demand $\hat{r}$. In fact, if $\hat{y}$ is the 'deal' response $\hat{y} = \hat{r}$, it is seen that the response y to r is the ideal response y = r ie the demands in the subspace spanned by the eigenvectors corresponding to the 'large' eigenvalues can be made to be arbitrarily good.

This analysis can clearly be continued to examine in more detail such concepts as steady-state response, interaction etc. As this note is essentially a theoretical preparation for application to the strip shape control problem(1), such considerations are not pursued here.

4. Robustness of the Design with respect to Errors in G_m

Given the design technique of section 3 a stable feedback regulator can be simply designed. If however (as in ref (1)) G_m is known to be inaccurate it is important to produce a means of predicting the maximum errors in elements of G_m that can be tolerated without spoiling feedback stability ie just how robust is the final design? Suppose therefore that K has been designed for $G = G_m g$ but that G_m is subjected to a 'matrix error' Δ. In such a situation the stability of the implemented feedback scheme is described by the return-difference

$$
\begin{aligned}
&|I_m + (G_m+\Delta) g(s) K| \\
&\quad \equiv |I_m + Kg(G_m+\Delta)| \\
&\quad \equiv |I_m + KG + Kg\Delta| \\
&\quad \equiv |I_m + KG| \cdot |I_m + (I+KG)^{-1} Kg\Delta| \\
&\quad \equiv |I_m + GK| \cdot |I_m + (I+KG)^{-1} Kg\Delta| \qquad \ldots(24)
\end{aligned}
$$

A necessary and sufficient condition for Δ to retain stability is hence that

$$|I_m + (I+KG)^{-1} Kg\Delta| \neq 0 \qquad \forall \quad \text{Re } s \geq 0 \qquad \ldots(25)$$

A simple calculation using (9) and (11) yields

$$(I+KG)^{-1} Kg = T_1 (I_\ell + K_1 G_1)^{-1} K_1 g V_1 \qquad \ldots(26)$$

and hence, using the identity $|I_m+AB| = |I_\ell+BA|$ valid[2] for any $m \times \ell$ matrix A and $\ell \times m$ matrix B, equation (25) can be replaced to the condition

$$|I_\ell + (I_\ell + K_1 G_1)^{-1} K_1 g V_1 \Delta T_1| \neq 0 \qquad \forall \quad \text{Re } s \geq 0 \qquad \ldots(27)$$

This expression is rather complicated but it can be replaced by a sufficient condition based upon the observation that a diagonally (row) dominant matrix is nonsingular. More precisely, a sufficient condition for the error Δ to be such that stability is retained is that

$$1 > \sum_{j=1}^{\ell} |F_{rj}(s)|$$

$$1 \le r \le \ell \quad , \quad \forall \quad s \in \mathcal{D} \triangleq \{s : \text{Re } s \ge 0\} \qquad \ldots (28)$$

where the $\ell x \ell$ TFM F(s) is defined by

$$F(s) \triangleq (I_\ell + K_1(s)G_1(s))^{-1} K_1(s)g(s)V_1 \Delta T_1 \qquad \ldots (29)$$

The frequency dependent condition (28) can be replaced by the frequency independent condition

$$1 > \sum_{j=1}^{\ell} \sup_{\text{Re} s \ge 0} |F_{rj}(s)| \quad , \quad 1 \le r \le \ell \qquad \ldots (30)$$

Noting that F is strictly proper and analytic and bounded in the interior of D the suprema are achieved on the imaginary axis ie equation (28) is valid if

$$1 > \sum_{j=1}^{\ell} \sup_{\omega \ge 0} |F_{rj}(i\omega)| \quad , \quad 1 \le r \le \ell \qquad \ldots (31)$$

This is the basic robustness relation used in the following development where it is shown to generate a set of strict linear inequalities describing the magnitude of errors Δ that retain stability.

Although linear in the perturbation Δ, F is a fairly complex function in general. It can however be written in the element form

$$F_{rj}(s) = \sum_{p=1}^{m} \sum_{q=1}^{m} f_{rjpq}(s) \, \Delta_{pq} \qquad \ldots (32)$$

for suitable choice of $f_{rjpq}(s)$ and clearly

$$\sup_{\omega \ge 0} |F_{rj}(i\omega)| \le \sum_{p=1}^{m} \sum_{q=1}^{m} \sup_{\omega \ge 0} |f_{rjpq}(i\omega)| \, |\Delta_{pq}| \qquad \ldots (33)$$

It follows that (31) is satisfied if

$$1 > \sum_{j=1}^{\ell} \sum_{p=1}^{m} \sum_{q=1}^{m} \sup_{\omega \geq 0} |f_{rjpq}(i\omega)| \, |\Delta_{pq}| \quad , \quad 1 \leq r \leq \ell \qquad \ldots(34)$$

or, equivalently, if

$$1 > \sum_{p=1}^{m} \sum_{q=1}^{m} c_{rpq} \, |\Delta_{pq}| \quad , \quad 1 \leq r \leq \ell \qquad \ldots(35)$$

where the scalars

$$c_{rpq} \triangleq \sum_{j=1}^{\ell} \sup_{\omega \geq 0} |f_{rjpq}(i\omega)| \geq 0$$

$$1 \leq r \leq \ell \quad , \quad 1 \leq p \leq m \quad , \quad 1 \leq q \leq m \qquad \ldots(36)$$

Equations (35) and (36) describe a computable class of perturbations or errors Δ that guarantee the retention of closed-loop stability. They are expressed in terms of ℓ linear inequalities in m^2 variables which could conceivably cause problems if m is large. We can however derive a more conservative estimate by noting that (35) is valid if

$$\max_{\substack{1 \leq p \leq m \\ 1 \leq q \leq m}} |\Delta_{pq}| < 1 \Big/ \max_{1 \leq r \leq \ell} \sum_{p=1}^{m} \sum_{q=1}^{m} c_{rpq} \qquad \ldots(37)$$

the RHS being easily computed.

Finally we note one particular case when calculating can be simplified. Consider the choice of K(s) via (13) and (19) when it is trivially verified that

$$(I_\ell + K_1 G_1)^{-1} K_1 g = \frac{g(s)k(s)}{1+g(s)k(s)} \operatorname{diag}\{\lambda_1^{-1}, \ldots, \lambda_\ell^{-1}\} \qquad \ldots(38)$$

and hence that

$$F_{rj}(s) = \frac{g(s)k(s)}{1+g(s)k(s)} \lambda_r^{-1} \sum_{p=1}^{m} \sum_{q=1}^{m} (V_1)_{rp} \Delta_{pq} (T_1)_{qj}$$

$$1 \le r \le \ell \quad , \quad 1 \le j \le \ell \qquad \ldots(39)$$

and

$$f_{rjpq}(s) = \frac{g(s)k(s)}{1+g(s)k(s)} \lambda_r^{-1} (V_1)_{rp} (T_1)_{qj} \qquad \ldots(40)$$

yielding

$$c_{rpq} = \sup_{\omega \ge 0} \left| \frac{g(i\omega)k(i\omega)}{1 + g(i\omega)k(i\omega)} \right| d_{rpq} \qquad \ldots(41)$$

where

$$d_{rpq} = \sum_{j=1}^{\ell} |\lambda_r^{-1}| \, |(V_1)_{rp} (T_1)_{qj}| \qquad \ldots(42)$$

Note that c_{rpq} is deduced easily from V_1, T_1, $\{\lambda_r\}_{1 \le r \le \ell}$ and the single TF gk/(1+gk). Note also the following observations

(i) the coefficient c_{rpq} is proportional to λ_r^{-1} indicating that large (resp. small) eigenvalues lead to small (resp. large) values of c_{rpq}. Small eigenvalues tend hence to increase the sensitivity of the control system by reducing the permissible perturbations Δ.

(ii) the properties of the single TF gk/(1+gk) clearly affect stability. If, for example, it possesses a strong resonance, all c_{rpq} will be large hence increasing sensitivity to the perturbation Δ.

5. Illustrative Example

Take

$$G(s) = \tfrac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{s+1} \qquad \ldots(43)$$

from which

$$G_m = \tfrac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} , \qquad g(s) = \frac{1}{s+1} \qquad \ldots(44)$$

and $\lambda_1 = 1$, $\lambda_2 = 0$. Take $\ell = 1$, whence

$$T_1 = \begin{pmatrix}1\\ 1\end{pmatrix} , \qquad V_1 = \tfrac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix} \qquad \ldots(45)$$

Considering, for simplicity, the case of proportional control, we have

$$K(s) = \begin{pmatrix}1\\ 1\end{pmatrix}\tfrac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix} k \qquad \ldots(46)$$

where $k_1(s) = k$ is constant. Equation (19) is trivially satisfied with $k = k_1(s) = k(s)$ and hence the closed-loop system is stable if

$$k+1 > 0 \qquad \ldots(47)$$

with closed-loop TFM

$$H_c(s) = \begin{pmatrix}1\\ 1\end{pmatrix}\tfrac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\frac{k}{s+1+k} \qquad \ldots(48)$$

We obtain from (41) and (42) that

$$c_{111} = c_{112} = c_{121} = c_{122} = \tfrac{1}{2}\frac{k}{1+k} \qquad \ldots(49)$$

and hence that stability is retained under all perturbations Δ to G_m satisfying

$$1 > \tfrac{1}{2}\frac{k}{1+k}\{|\Delta_{11}| + |\Delta_{12}| + |\Delta_{21}| + |\Delta_{22}|\} \qquad \ldots(50)$$

Alternatively

$$\max_{\substack{1\leq p\leq 2\\ 1\leq q\leq 2}} |\Delta_{pq}| < \tfrac{1}{2}\,(1+k^{-1}) \qquad \ldots(51)$$

Note that the use of low gains k tends to reduce sensitivity.

6. Conclusions

The special case considered has been shown to be amenable to analysis both in the standard, nonstandard and robustness sense. The approach taken is nonunique and could possibly be improved. This may become apparent when it is applied to the strip shape control problem.

References

(1) M.J.Grimble, J.Fotakis: 'The design of strip shape control systems for Sendzimir mills', 19th IEEE Conference on Decision and Control, Albequeque, 1980.

(2) D.H.Owens: 'Feedback and multivariable systems', Peter Peregrinus, 1978.

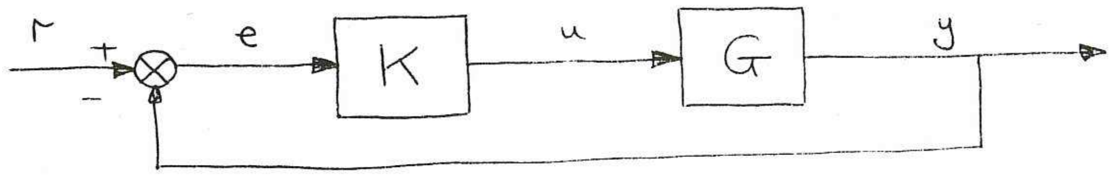

Fig. 1.

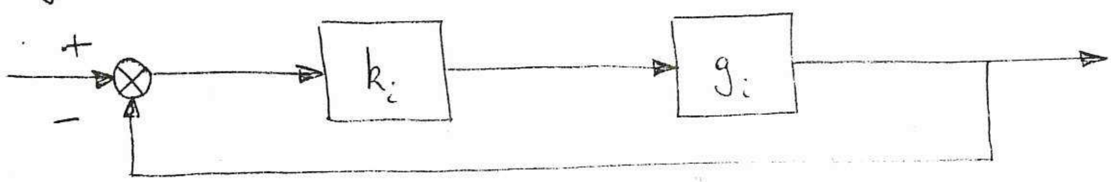

Fig 2.

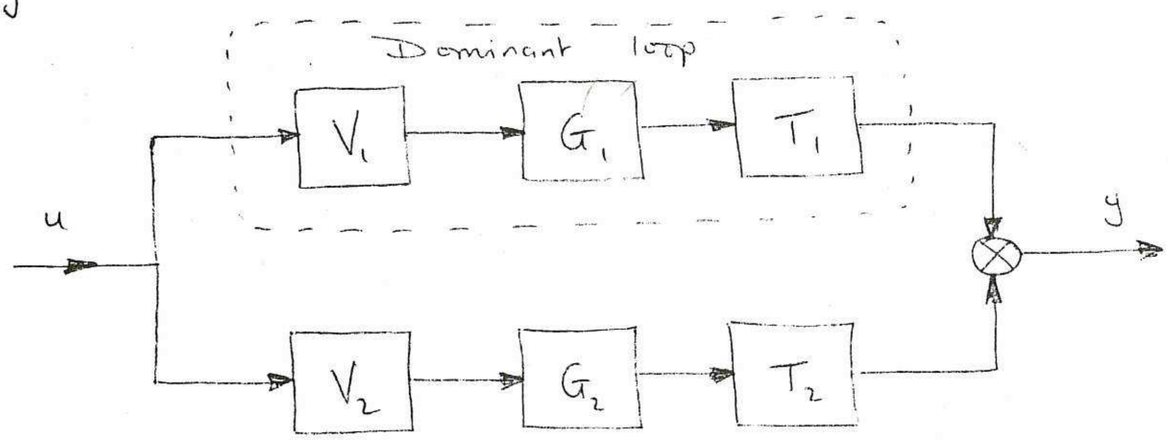

Fig. 3.

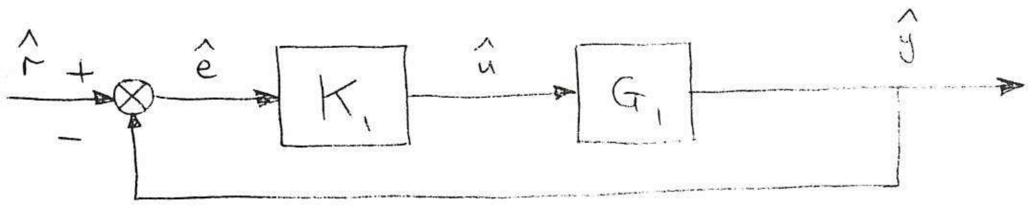

Fig. 4.