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AN ABSTRACT SETTING FOR NONLINEAR DISTRIBUTED MULTIPASS PROCESSES

BY

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1. Introduction

In this paper we shall study the existence, uniqueness and stability of differential difference equations in a Banach space. The motivation for this work is to provide precise results in the theory of multipass processes. These are control processes defined over some region of space Ω where the control mechanism takes place successively over some time interval $[0,\tau]$. Each control strategy over Ω and time $[0,\tau]$ is called a 'pass' of the system and the measurements and controls on the i^{th} pass depend on the system states in the $(i-1)^{th}$ pass. (For more details on multipass processes, see, for example, Owens, [6],[7]).

We shall be concerned here with systems defined by distributed nonlinear equations and to give precise results it will be convenient to define the process as a differential difference equation in a Banach space. In section 2 we shall define the notation, for the convenience of the reader and in section 3 we shall discuss the general set-up of our system equations. In section 4 the existence and uniqueness of solutions will be considered and methods for obtaining a solution over the entire pass length developed. Lyapunov methods will be discussed in section 5 and a frequency domain criterion (circle theorem) developed in section 6. Finally in section 7 we shall give two examples to illustrate the theory.

2. Notation and Terminology

The notation used here will be standard Banach space notation; in particular, $X,Z,\bar{Z},V$ will denote Banach spaces and H will denote a Hilbert space. An operator A defined on any of these spaces will have domain (not necessarily equal to the whole space) denoted by $D(A)$. The resolvent operator $R(\lambda;A)$ is the operator

$$(\lambda I-A)^{-1}$$

defined for all $\lambda \notin \sigma(A)$ (the spectrum of A)

N^+ will denote the set of natural numbers (without 0) and $\mathbb{C}$ will denote the complex plane.

The standard L^p spaces will be used (e.g. we shall write $L^2[0,1]$ or $L_2[0,1]$ for the space of square integrable complex-valued functions). Also

$$L^p[0,\infty;X]$$

will denote the space of maps $f:[0,\infty)\to X$ such that

$$F = \int_0^\infty ||f(t)||_X^p \, dt < \infty,$$

with the norm of f defined as $F^{1/p}$. Also, the spaces ℓ^p of sequences $x=(x_i)$ which are p^{th} power summable are used,

i.e.

$$||x||_{\ell^p} = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} < \infty$$

and if the sequence has values in a Banach space X, we write $\ell^p(X)$ and define the norm of $x=(x_i)\in \ell^p(X)$ by

$$||x||_{\ell^p(X)} = (\sum_{i=1}^{\infty} ||x_i||_X^p)^{1/p}$$

3. Nonlinear Differential Difference Equations and Equilibri

A general nonlinear multipass process is described by the system of differential difference equations

$$(3.1) \quad \begin{cases} \dot{x}_i = f(x_i, x_{i-1}, \ldots, x_{i-\ell}, t) & i \geq 1, \quad t \in [0,\tau]. \\ x_i(0) = x_{io} \end{cases}$$

with the data

x_{io}, $i \geq 1$ given elements of some Banach space X. Suppose that the given states $x_o, x_{-1}, \ldots, x_{1-\ell} \in L^p(0,\tau;X)$. We shall seek solutions of the system in the space $L^p(0,\tau;X)$, and for this purpose we shall reformulate equations (3.1) as a difference equation in this space. This can be done simply by rewriting (3.1) in the form

(3.2) $$x_i(t) = x_{io} + \int_0^t f(x_i(s), x_{i-1}(s), \ldots, x_{i-\ell}(s), s)ds,$$

for $t \in [0,\tau]$.

Definition 3.1 By a (mild) solution of equation (3.1) we shall mean a function

$$x : N^+ \to L^p(0,\tau;X)$$

such that (3.2) is satisfied for almost every $t \in [0,\tau]$, where we denote $x(i) \in L^p(0,\tau;X)$ by x_i.

Let us denote the direct sum of m copies of $L^p(o,\tau;X)$ by $L^p_m(0,\tau;X)$; i.e. $L^p_m(o,\tau;X) = \bigoplus_{i=1}^{m} L^p(0,\tau;X)$. Then, defining the map $K: L^p_{\ell+1}(0,\tau;X) \to L^p(0,\tau;X)$ by

$$K(z_1,\ldots,z_{\ell+1})(t) = x_{io} + \int_0^t f(z_1(s),\ldots,z_{\ell+1}(s),s)ds,$$

we can rewrite equation (3.2) in the form

(3.3) $$x(i) = K(x(i), x(i-1), \ldots, x(i-\ell)) ,$$

which is, of course, a difference equation defined on $L^p(o,\tau;X)$.

Having derived a simple difference equation representation of (3.1) we can now define equilibrium points and Lyapunov stability in the usual way, hence

Definition 3.2 The point $z \in L^p(0,\tau;X)$ is called an equilibrium point of the equation (3.3) (and also of (3.1)) if

$$z = K(\underbrace{z,\ldots,z}_{\ell+1 \text{ times}}).$$

Definition 3.3 Let $z \in L^p(0,\tau;X)$ be an equilibrium point of the equation (3.3). Then z is stable if, for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\|x(i) - z\|_{L^p(0,\tau;X)} < \varepsilon , \quad i=1,2,\ldots$$

for all initial conditions $x_o, x_{-1}, \ldots, x_{1-\ell}$ such that

$$\| x_i - \bar{x} \|_{L^p(0,\tau;X)} < \delta, \quad 1-\ell \le i \le 0.$$

The point $\bar{x}$ is an <u>asymptotically stable equilibrium point</u> if $\exists\, \bar{\delta} > 0$ such that

$$\| x(i) - \bar{x} \|_{L^p(0,\tau;X)} \to 0 \quad \text{as} \quad i \to \infty$$

for all initial conditions $x_o, x_{-1}, \ldots, x_{1-\ell}$ such that

$$\| x_i - \bar{x} \|_{L^p(0,\tau;X)} \le \bar{\delta}, \quad 1-\ell \le i \le 0.$$

Finally, we shall define an input-output stability for the system (3.1)

<u>Definition 3.4</u> The system (3.1) is called (p,q) -input-output stable if, for any initial conditions $x_o, x_{-1}, \ldots x_{1-\ell} \in L^p(0,\tau;X)$ and for any initial values $x_{io} \in \ell^q(X)$, we have for any solution

$$x : N^+ \to L^p(0,\tau;X)$$

of (3.1) (or(3.3)), that

$$x \in \ell^q(L^p(0,\tau;X)) .$$

In this definition, the notation $\ell^q(Y)$, for any Banach space Y, denotes the space of maps $y : N^+ \to Y$ under the norm

$$\| y \|_{\ell^q(Y)} = \left(\sum_{i=1}^{\infty} \| y(i) \|^q \right)^{1/q} .$$

We note that (3.3) may be written directly in the form of an equation in $\ell^q(L^p(0,\tau;X))$ by introducing the nonlinear map

(3.4) $\underline{K} : \ell^q(L^p(0,\tau;X)) \to \ell^q(L^p(0,\tau;X))$

defined by

$$\underline{K}(x) = \{K(x(i), x(i-1), \ldots, x(i-\ell))\}_{1 \le i < \infty}$$

for $x \in \ell^q(L^p(0,\tau;X))$; $x(-1), \ldots, x(1-\ell)$ given in $L^p(0,\tau;X)$.

Hence (3.1) is (p,q) -input-output stable if the map $\underline{K}$ in (3.4) is well-defined and the solutions (if they exist) are given by

(3.5) $$x = \underline{K}(x) .$$

Next, consider the semilinear equation

(3.6) $$\dot{x}_i = Ax_i + g(x_i, x_{i-1}, \ldots, x_{i-\ell}, t)$$

where A is assumed to generate a strongly continuous semigroup on X. Then, by the variation of constants formula,

(3.7) $$x_i(t) = T(t)x_{io} + \int_o^t T(t-s)\, g(x_i(s), \ldots, x_{i-\ell}(s), s)ds$$

We could write this equation in the forms (3.3) or (3.5). However, it is convenient to define the operator G on $\ell^q(L^p(0,\tau;X)$ by the relation

$$(G(x))_i = g(x(i), \ldots, x(i-\ell)) \ , \ x \in \ell^q(L^p(0,\tau;X))$$

and the operator M on $\ell^q(L^p(0,\tau;X))$ by

$$(M(y))_i(t) = \int_o^t T(t-s)(y(i))(s)ds$$

for almost all $t \in [0,\tau]$ and all $y \in \ell^q(L^p(0,\tau;X))$. Then, if

$$h_i(t) = T(t)x_{io} ,$$

$h = \{h_i\}_{1 \leq i < \infty} \in \ell^q(L^p(0,\tau;X))$ and so equation (3.7) may be written in the form

(3.8) $$x = h + MGx , \quad x \in \ell^q(L^p(0,\tau;X)).$$

Let us note finally that if g is linear in $x_{i-1}, \ldots, x_{i-\ell}$ and independent of x_i, we can write

$$g(x(i-1), \ldots, x(i-\ell)) = L_1\, x(i-1) + \ldots + L_\ell x(i-\ell)$$

where each L_j is a linear operator on $L^p(0,\tau;X)$, and (3.7) may be written in the form

(3.9) $$\chi(i) = L\chi(i-1) + \zeta(i)$$

where

$$\mathcal{X}(i) = (x(i-\ell+1),\ldots, x(i)) \in L^p_\ell(0,\tau;X)$$

$$L = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & I \\ L_\ell & L_{\ell-1} & \cdots & \cdots & L_1 \end{pmatrix} \in \mathcal{L}(L^p_\ell(0,\tau;X)) .$$

(I,0 represent the identity and zero operators on $L^p(0,\tau;X)$ respectively) and

$$\zeta(i)(t) = (\underbrace{0,\ldots,0}_{\ell-1}, T(t)x_{io})^T .$$

Therefore, in the case where $\zeta(i) = 0$ for all i (zero initial conditions when t = 0), the equilibria of the system are given by elements of $\mathrm{Ker}(I-L) \subseteq L^p_\ell(0,\tau;X)$. Hence, for this system existence and stability questions are easily answered. The solution of equation (3.9) is given by

$$\mathcal{X}(i) = L^i\mathcal{X}(o) + \sum_{j=1}^{i} L^{i-j}\,\zeta(j)$$

and the solution is stable if the spectral radius r(L) satisfies

$$r(L) \leq 1 .$$

Having discussed the general forms of the system we shall now proceed to obtain existence results in the next section. The method we shall use will be a contraction mapping type argument applied for a small time $\tau_1 \in [0,\tau]$, giving a local (in time) solution for each i. We can then use stability arguments to extend the solution as far as possible. This, in general, will mean a trade-off between the size of the initial conditions and the length of time the solution can be shown to exist.

4. Existence and Uniqueness of Solutions

We shall consider first the general equation (3.1) which we have written in the form of a difference equation on $L^p(0,\tau;X)$; i.e.

(4.1) $$x(i) = K(x(i), x(i-1),\ldots,x(i-\ell)).$$

Theorem 4.1 Let $K : L^p_{\ell+1}(0,\bar{\tau};X) \to L^p(0,\bar{\tau};X)$ (for each $\bar{\tau}\in[0,\tau]$) be a (nonlinear) mapping such that

(i) $$\|K(v,z_1,\ldots z_\ell) - K(w,z_1,\ldots,z_\ell)\|$$
$$\leq k(\|v\|, \|w\|, \|z_1\|,\ldots, \|z_\ell\|)\ \|v-w\|$$

for some function $k : (\mathbb{R}^+)^{\ell+2} \to \mathbb{R}^+$, where norms are taken in $L^p(0,\bar{\tau}:X)$, for $\bar{\tau}\in[0,\tau]$.

(ii) $\exists\ \tau_1\in[0,\tau]$ and $a>o$, such that
$$\|K(z_1,\ldots,z_{\ell+1})\|_{L^p(0,\tau_1;X)} \leq a$$
for
$$\|z_i\|_{L^p(0,\tau_1;X)} \leq a \quad , \quad 1\leq i\leq \ell+1$$

and $k(\alpha_1,\ldots,\alpha_{\ell+2}) <1$, for all $\alpha_i\in\mathbb{R}^+$ such that

$\alpha_i \leq a$, $1\leq i\leq \ell+2$.

Then the difference equation (4.1) has a unique solution defined on $L^p(0,\tau_1;X)$, provided the initial values $x_o,x_{-1},\ldots,x_{1-\ell}$ satisfy the inequalities
$$\|x_i\|_{L^p(0,\tau_1;X)} \leq a, \qquad 1-\ell\leq i\leq 0$$

Proof. The proof is by induction. Consider the case i=1. Then we have the equation

(4.2) $$x(1) = K(x(1),x(0),\ldots,x(1-\ell)).$$

With $x(o),\ldots,x(1-\ell)$ fixed (in $B(o,a)$ = ball of radius a in $L^p(0,\tau_1;X)$), the mapping

$$K(\cdot,\ x(o),\ldots,x(1-\ell))\ :\ L^p(0,\tau_1;X)\to L^p(0,\tau_1;X)$$

maps the ball B(0,a) into itself by (ii). Moreover, using (i) and (ii) this map is clearly a contraction on B(o,a), and so there exists a unique fixed point $x(1)\in B(o,a)$; i.e. equation (4.2) is satisfied. Assuming now t at t e result is true for i=n, this same argument shows that the result is true for i=n+1. This completes the inductive argument. □

Im theorem 4.1 we have proved a local (in time) existence result for the difference equatiom (4.1). If τ_1 can be taken to be τ, then the conditions of the theorem are satisfied on the whole interval $[0,\tau]$ and so we then have a global existence theorem. It is difficult, however, to obtain more precise global results in the general case and so we shall now consider the semilinear equation,

(4.3) $$\dot{x}_i(t) = Ax_i(t) + f(x_i(t),\ x_{i-1}(t),\ldots,x_{i-\ell}(t)).$$

where, for simplicity, we shall assume that f is independent of t. In order to consider fairly general types of nonlinearities f it is necessary to assume that the semigroup generated by A has some 'smoothing' properties. (For example, the map $f(x) = x^2$ does not map $L^2(0,1)$ into itself. In fact,

$$f\ :\ L^2(0,1)\to L^1(0,1)\ .)$$

Hence, following Pritchard and Ichikawa ([4]), we shall assume that, for some Banach spaces $Z,V,\bar{Z}$ and some $\tau_1>0$,

(a). A generates a semigroup $T_t\in\mathcal{L}(Z,V)\cap\mathcal{L}(\bar{Z},V)$ for t>o

and

$$\left.\begin{aligned} \|T_t z\|_V &\le g(t)\,\|z\|_Z \\ \|T_t\bar{z}\|_V &\le \bar{g}(t)\,\|\bar{z}\|_{\bar{Z}} \end{aligned}\right\}\quad t>o,\ z\in Z,\ \bar{z}\in\bar{Z}$$

where $g\in L^p[0,\tau_1]$, $\bar{g}\in L^p[0,\tau_1]$, $\tau_1\in[0,\tau]$

(b) If $V^{\ell+1}$ denotes the direct sum of $\ell+1$ copies of V, then $f\ :\ V^{\ell+1}\to\bar{Z}$ and if $v_i\in L^r[0,\tau_1;V]$, $1\le i\le\ell+1$, with $\|v_i\|_{L^r[0,\tau_1;V]} \leq a$, there exists b (=b(a), depending on a)

such that

$$\| f(v_1,\ldots,v_{\ell+1})\|_{L^s[0,\tau_1;Z]} \leq b .$$

(c) $$\| f(v_1,w_1,\ldots w_\ell) - f(v_2,w_1,\ldots,w_\ell)\|_{L^s[0,\tau_1;Z]}$$

$$\leq \mathcal{K}(\| v_1\|_{L^r[0,\tau_1;V]}, \| v_2\|_{L^r[0,\tau_1;V]}, \| w_i\|_{L^r[0,\tau_1;V]}) \cdot$$

$$\cdot \| v_1-v_2\|_{L^r[0,\tau_1;V]}$$

where $\mathcal{K} : (\mathbb{R}^+)^{\ell+2} \to \mathbb{R}^+$.

(d) if $\| v_1\|_{L^r[0,\tau_1;V]} \leq a$, $\| v_2\|_{L^r[0,\tau_1;V]} \leq a$,

$\| w_i\|_{L^r[0,\tau_1;V]} \leq a$, $1\leq i\leq \ell$ then

$$\| \bar{g}\|_{L^q[0,\tau_1]} \mathcal{K}(\| v_1\|_{L^r[0,\tau_1;V]} , \| v_2\|_{L^r[0,\tau_1;V]} ,$$

$$\| w_i\|_{L^r[0,\tau_1;V]}) < 1.$$

(In the above conditions, $p,\bar{p},r,s,a,b$ are positive real numbers, and $p\geq r\geq 1, \bar{p}\geq q\geq 1$, $s\geq 1$, and $\frac{1}{r} = \frac{1}{q} + \frac{1}{s} - 1$.)

We can now prove

Theorem 4.2 Let conditions (a)-(d) above be satisfied, and suppose that the elements x_{io} belong to the set

(4.4) $$\{ z\in Z : \| g\|_{L^r[0,\tau_1]} \| z\|_Z + \| \bar{g}\|_{L^q[0,\tau_1]} \cdot b\leq a \}$$

for $i \geq 1$. Then the equation

(4.5) $$x_i(t) = T_t x_{io} + \int_0^t T_{t-s} f(x_i(s),\dots,x_{i-\ell}(s))ds$$

(the 'mild' form of (4.3)), has a unique solution in $L^r[0,\tau_1;V]$.

Proof First define

$$K(z_1,\dots,z_{\ell+1})(t) = T_t x_{io} + \int_0^t T_{t-s} f(z_1(s),\dots,z_{\ell+1}(s))ds$$

We shall show that with K so defined, the conditions of theorem 4.1 are satisfied. For (i), we have

$$\| K(v,z_1,\dots,z_\ell)(t) - K(w,z_1,\dots z_\ell)(t)\|_V$$

$$= \left\| \int_0^t T_{t-s}(f(v, z_1,\dots,z_\ell) - f(w,z_1,\dots,z_\ell))\right\|_V$$

$$\leq \int_0^t \bar{g}(t-s)\| f(v(s),z_1(s),\dots,z_\ell(s)) - f(w(s),z_1(s),\dots,z_\ell(s))\|_{\bar{Z}} ds$$

Since the RHS is a convolution,

$$\| K(v,z_1,\dots,z_\ell) - K(w,z_1,\dots,z_\ell)\|_{L^r|0,\tau_1;V|}$$

$$\leq \|\bar{g}\|_{L^q|0,\tau_1|} \| f(v,z_1,\dots,z_\ell) - f(w,z_1,\dots,z_\ell)\|_{L^s[0,\tau_1;\bar{Z}]}$$

and condition (i) of theorem 4.1 follows from (c).

Condition (ii) of theorem 4.1 follows from (a),(b) and (d) in a similar manner and so the result is a simple consequence of theorem 4.1. □

The following two corollaries follow as in Ickikawa and Pritchard ([4]).

Corollary 4.3 If we assume that $T_t \in \mathcal{L}(\bar{Z},Z)$ for $t>0$, and

$$\| T_t \bar{z}\|_Z \leq \bar{\bar{g}}(t)\ \|\bar{z}\|_Z,$$

where $\bar{\bar{g}} \in L^\rho[0,\tau_1]$, $\frac{1}{\rho} = 1 - \frac{1}{s}$, then the solution of equation (4.5) (which was shown to exist in $L^r[0,\tau_1;V]$) lies in $C[0,\tau_1;Z]$. □

Corollary 4.4. Let $\overline{V} \subset Z$ be a Banach space such that

(i) $T_t \in \mathcal{L}(Z,\overline{V})$, $t>o$, with $\| T_t z \|_{\overline{V}} \leq g_1(t) \| z \|_Z$, $t>o$,

where $g_1 \in L^{p_1}[\varepsilon,\tau_1] \cap C[\varepsilon,\tau_1]$ for some $\varepsilon>o$, $p_1 \geq 1$ and T_t can be extended to a strongly continuous semigroup on $\overline{Z}$.

(ii) $T_t \in \mathcal{L}(\overline{Z},\overline{V})$, $t>o$, with $\| T_t \bar{z} \|_{\overline{V}} \leq g_2(t) \| \bar{z} \|_{\overline{Z}}$, $t>o$,

and $g_2 \in L^{p_2}[0,\tau_1]$, $p_2 \geq 1$.

Let s be as in theorem 4.2 and suppose that $m \leq p_1$, $w \leq p_2$. Then, if $g_1 \in L^{p_1}[\varepsilon,\tau_1]$ and $\frac{1}{m} = \frac{1}{w} + \frac{1}{s} - 1$, the solution $z_n(\cdot)$ of (4.5) lies in $L^m[\varepsilon,\tau_1;\overline{V}]$, for all $n \geq 1$, and if $g_1 \in C[\varepsilon,\tau_1]$ and $\frac{1}{w} + \frac{1}{s} = 1$, then $z_n(\cdot) \in C[\varepsilon,\tau_1;\overline{V}]$ for all $n \geq 1$. □

We can now consider the extension of the solution beyond τ_1. Since T_t is a C_o-semigroup on Z, $\exists$ constants M, ω such that

$$\| T_t \|_{\mathcal{L}(Z)} \leq M e^{\omega t}$$

and so, if $0 \leq \lambda \leq 1$,

$$\| T_t \|_{\mathcal{L}(\overline{Z},V)} \leq \| T_{(1-\lambda)\frac{t}{2}} \|_{\mathcal{L}(Z,V)} \| T_{\lambda t} \|_{\mathcal{L}(Z)} \| T_{(1-\lambda)\frac{t}{2}} \|_{\mathcal{L}(\overline{Z},Z)}$$

$$\leq g((1-\lambda)\frac{t}{2}) M e^{\omega t} \bar{\bar{g}}((1-\lambda)\frac{t}{2})$$

$$= \gamma(t), \text{ say }.$$

Hence, from equation (4.5), we obtain

$$\| x_i(t) \|_V \leq \| T(t) \|_{\mathcal{L}(\overline{Z},V)} \| x_{io} \|_{\overline{Z}} + \int_o^t \| T(t-s) \|_{\mathcal{L}(\overline{Z},V)} \cdot$$

$$\cdot \| f(x_i(s),\ldots,x_{i-\ell}(s)) \|_V \, ds .$$

$$\leq \gamma(t) \| x_{io} \|_{\bar{Z}} + \int_0^t \gamma(t-s) \| f(x_i(s),\ldots,x_{i-\ell}(s)) \|_V \, ds$$

Let $\chi_i(t) = \| x_i \|_{L^r[0,t;V]}$. Then ,

(4.6) $\chi_i(t) \leq \| \gamma \|_{L^r[0,t]} \| x_{io} \|_{\bar{Z}} + \| \gamma \|_{L^q[0,t]} \mathcal{K}(\chi_i(t),0,\chi_{i-1}(t),\ldots,\chi_{i-\ell}(t))\chi_i(t)$

where we have assumed that $\gamma \in L^q_{\ell oc} \cap L^r_{\ell oc}$ and

$$f(0,\omega_1,\ldots,\omega_\ell) = 0 \, , \; \forall \, \omega_i \in V.$$

Suppose now that $x_{i-1},\ldots,x_{i-\ell}$ have been extended beyond the time τ_1, say to $\tau_2 > \tau_1$, and that

$$\| x_j \|_{L^r[0,\tau_2;V]} \leq a \quad , \quad i-\ell \leq j \leq i-1 \, .$$

Then by (4.6), if

(4.7) $$\| \gamma \|_{L^q[0,t]} \mathcal{K}(a,o,a,\ldots,a) \leq u(t) < 1$$

we have

$$\chi_i(t) \leq \| \gamma \|_{L^r[0,t]} \| x_{io} \|_{\bar{Z}} + \chi_i(t)u(t), \qquad 0 \leq t \leq \tau_1$$

and so

$$\chi_i(t) \leq \frac{1}{1-u(t)} \{ \| \gamma \|_{L^r[0,t]} \| x_{io} \|_{\bar{Z}} \}.$$

Hence, if

(4.8) $$\frac{1}{1-u(t)} \{ \| \gamma \|_{L^r[0,t]} \| x_{io} \|_{\bar{Z}} \} < a \, , \qquad o \leq t \leq \tau_1$$

then the solution x_i can be extended beyond τ_1. The condition (4.8) shows the way that the maximum time interval of definition of a solution depends on $\| \gamma \|_{L^q[0,t]}$, $\mathcal{K}$, a and the initial value $\| x_{io} \|_{\bar{Z}}$. In the case of a stable semigroup, we have $\omega < 0$, and so it may be that

$$\|\gamma\|_{L^r[0,\infty]} < \infty$$

and so if a is small enough so that (4.7) is satisfied for $t = \infty$, then a global solution exists if

$$\|x_{io}\|_{\overline{Z}} < a(1-\varkappa(a,o,a,\ldots,a)\|\gamma\|_{L^q[0,\infty]})\|\gamma\|^{-1}_{L^q[0,\infty]}$$

Note that if we estimate $\|T(t)x_{io}\|_V$ by $g(t)\|x_{io}\|_{\overline{Z}}$,

then we can replace condition (4.8) by the condition

$$(4.9) \qquad \frac{1}{1-u(t)}\{\|g\|_{L^r[0,t]}\|x_{io}\|_{\overline{Z}}\} < a \quad ,$$

and a solution will exist for as long as (4.9) holds.

5. <u>Lyapunov Function Method</u>

In the above local existence and extension results we have shown that, under the conditions (4.8) or (4.9), the solution $x_i(t)$ which starts in the ball of radius

$$a_1 = a(1-u(t))\|\gamma\|^{-1}_{L^r[0,t]}$$

or $$a_2 = a(1-u(t))\|g\|^{-1}_{L^r[0,t]}$$

will remain in the ball of radius a in $L^r[0,t;V]$. If t can be taken to be greater than or equal to τ(the pass length) then we have a kind of stability for each pass (cf. Owens, [6]). Another approach to obtaining such a stability along the pass is by the use of a Lyapunov function which will be

$$V(t) = \langle x_i(t), x_i(t)\rangle_{Z}$$

where we shall now assume that all spaces $Z,\overline{Z},V$ etc under consideration are Hilbert spaces . We assume that

$$D(A) \subseteq \overline{V} \subseteq (\overline{Z})^* \subseteq Z \subseteq \overline{Z} \subseteq \overline{V}^*$$

where the injections are continuous and each space is dense in its successor, and that for $v \in D(A)$, $\exists\ B \in \mathcal{L}(Z)$, $D \in \mathcal{L}(\overline{V}, Z)$ such that

$$\langle Av, v\rangle_Z + \langle v, Av\rangle_Z \leq -\| Dv \|_Z^2 + \langle Bv, v\rangle_Z \quad .$$

Then, again following Ichikawa and Pritchard ([4]), we have

Theorem 5.1 Assume that the conditions of corollaries 4.3, 4.4 hold, with $m = 2$, so that $x_i \in L^2[\varepsilon, \tau_1; \overline{V}]$, $i \geq 1$. If $x_i \in L^{s'}[\varepsilon, \tau_1; \overline{Z}^*]$ where $1/s + 1/s' = 1$, and s is as in theorem 4.2, then

$$(5.1) \quad \| x_i(\tau_1) \|_Z^2 - \| x_i(\varepsilon) \|_Z^2 \leq - \int_\varepsilon^{\tau_1} \Big[\| Dx_i(s) \|_Z^2 - \langle Bx_i(s), x_i(s)\rangle_Z$$

$$-2\langle x_i(s), f(x_i(s), x_{i-1}(s), \ldots x_{i-\ell}(s))\rangle_{\overline{Z}^*, \overline{Z}} \Big] ds$$

This result enables us to obtain estimates of $\| x_i(\tau_1) \|_Z$ in terms of $\| x_{io} \|_Z$ by taking ε small and noting that $x_i \in C[0, \varepsilon; Z]$. By summing inequalities (5.1) over i, we obtain the following simple corollary.

Corollary 5.2 Under the assumptions of theorem 5.1, we have

$$(5.2) \quad \| x(\tau_1) \|_{\ell^2(Z)}^2 - \| x(\varepsilon) \|_{\ell^2(Z)}^2 \leq - \int_\varepsilon^{\tau_1} \Big[\| \underline{D}x(s) \|_{\ell^2(Z)}^2$$

$$- \langle \underline{B}x(s), x(s)\rangle_{\ell^2(Z)}$$

$$-2\langle x(s), \underline{f}(x(s))\rangle_{\ell^2(\overline{Z}^*), \ell^2(\overline{Z})} \Big] ds$$

where $x(t) = \{x_i(t)\}_{i \geq 1}$, and

$$\underline{D} = \text{diag}\,\{D, D, \ldots\} \in \mathcal{L}\Big[\bigoplus_{i=1}^{\infty} Z, \bigoplus_{i=1}^{\infty} Z\Big]$$

$$\underline{B} = \text{diag}\,\{B, B, \ldots\}$$

(5.3) $$\underline{f}(x(s)) = \{f(x_i(s), x_{i-1}(s),\dots,x_{i-\ell}(s))\}_{\underline{i \geq 1}} \quad . \square$$

6. Frequency Domain Conditions - The Circle Theorem

Consider again the semilinear equation

(6.1) $$x_i(t) = T(t)x_{io} + \int_o^t T(t-s)f(x_i(s),\dots,x_{i-\ell}(s))ds$$

and write it in the form

(6.2) $$x(t) = \underline{T}(t)x_o + \int_o^t \underline{T}(t-s)\underline{f}(x(s))ds$$

where

$$\underline{T}(t) = \text{diag}\{T(t),T(t),\dots\} \in \mathcal{L}[\overset{\infty}{\underset{i=1}{\oplus}} H, \overset{\infty}{\underset{i=1}{\oplus}} H],$$

for each $t \geq 0$, and $\underline{f}$ is defined by (5.3). (We are now restricting our systems to be defined on a separable Hilbert space H. The introduciton of the spaces $Z,V,\overline{Z}$ etc. above was necessary to account for a large class of nonlinear systems. The cost of obtaining a circle theorem will be the considerable restriction on the type of nonlinearity which can be handled.)

We shall consider the system (6.2) as being defined on the space $\underline{H} = \overset{\infty}{\underset{i=1}{\oplus}} H$ and for the purposes of this discussion we shall let the pass length $\tau_1 \to \infty$, and search for conditions which guarantee that the solution of (6.2) belongs to $\ell^2[0,\infty;\underline{H}]$ for any $x_o \in \underline{H}$. How $\underline{H}$ is a separable Hilbert space with inner product defined by

$$\langle \underline{h},\underline{h} \rangle_{\underline{H}} = \sum_{i=1}^{\infty} \langle h_i,h_i \rangle_H \quad , \quad \underline{h} = (h_i)_{\underline{i \geq 1}} \in \underline{H} \quad ,$$

and $\underline{T}$ is a semigroup on $\underline{H}$ with generator

$$\underline{A} = \text{diag}\{A,A,\dots\} : \overset{\infty}{\underset{i=1}{\oplus}} D(A) \to \underline{H}$$

Clearly, $\sigma(A) = \sigma(\underline{A})$ and so

$$\underline{R(s;A)} = R(s;\underline{A})$$

(where R denotes the resolvent operator.). We can therefore use the results of ([2]; see also [3], [8]). Let

$$\xi_s : \mathbb{C}/\{s\} \longrightarrow \mathbb{C}$$

be the complex-valued function defined by

$$\xi_s(\lambda) = \frac{1}{s-\lambda}, \quad \text{for all } s \in \mathbb{C}^+ .$$

Consider the following two conditions:

(C1) The region traced out by the set valued map $\omega \mapsto \xi_{(i\omega)}(\sigma(A))$ does not contain a curve which encircles or passes through the point $[-2(a+b)^{-1}, 0]$.

(C2) The region traced out by the set valued map $\omega \mapsto \xi_{(i\omega)}(\sigma(A))$ does not intersect the region $R(a) \subseteq \mathbb{C}$ for $-\infty<\omega<\infty$, where

(a) $R(a)$ = disc of radius $\frac{1}{2}(a^{-1}-b^{-1})$ with centre $[-\frac{1}{2}(a^{-1}+b^{-1}), 0]$ if $a>0$

(b) $R(a)$ = half plane $\text{Re}\, s \leq -b^{-1}$ if $a=0$

(c) $R(a)$ = exterior of the disc in (a) if $a<0$.

We then have the following result:

Theorem 6.1 Suppose that the operator A satisfies conditions (C1), and (C2) and that $\underline{f}$ satisfies

$$\langle \underline{f}(\underline{h})+a\underline{h}, \underline{f}(\underline{h})+b\underline{h} \rangle_{\underline{H}} \geq 0, \quad \forall \underline{h} \in \underline{H} .$$

Then, for any solution $x(t)$ of (6.2), we have

$$x \in L^2[0,\infty;\underline{H}] . \quad \square$$

Corollary 6.2 If A satisfies (C1), (C2) and function f satisfies

$$\langle f(h_1,\ldots,h_{\ell+1}) + ah_1, f(h_1,\ldots,h_{\ell+1}) + bh_1 \rangle_H \geq 0,$$

$\forall\, h_1,\ldots,h_{\ell+1} \in H$, then the solution $x(t)$ of (6.1) satisfies

$$x \in L^2[0,\infty;\underline{H}] . \quad \square$$

7. Examples

In this section we shall give some examples of the types of systems to which our results apply. We shall consider first the case of a diffusion equation which, for simplicity will be taken as the heat equation.

Example 7.1 (Heat diffusion equation).

Consider the equation

$$(7.1)\qquad z^i_t = z^i_{xx} + z^i z^{i-1}$$

(where we have now denoted the pass number by a superscript), subject to the boundary and initial conditions:

$$z^i(x,0) = z^i_o(x) \ , \ z^i(0,t) = z^i(1,t) = 0 \ ,$$

and we shall assume that the equation is defined on the spacial domain $[0,1]$. Then we take $Z = L^2[0,1]$ and the semigroup $T(t)$ generated by the operator A defined by

$$Az = z_{xx} \ , \ z \in D(A) \ ,$$

$$D(A) = H^2[0,1] \cap H^1_o[0,1] \ .$$

We shall define $V = L^{2\alpha}[0,1]$, $\overline{Z} = L^{\alpha}[0,1]$, $\alpha \geq 1$. From the Sobolev embedding theorem [1]

$$H^{\delta}[0,1] \subseteq L^{2\alpha}[0,1] \text{ for } \delta \geq \tfrac{1}{2}-\tfrac{1}{2\alpha}$$

we have the following inequalities for the semigroup

$$(T(t)z)(x) = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \sin n\pi x \int_0^1 \sin(n\pi p) z(p)dp \ :$$

$$(i)\qquad \| T(t)z \|_V \leq \frac{C}{t^{1/4-1/4\alpha}} \| z \|_Z = g(t) \| z \|_Z$$

$$(ii)\qquad \| T(t)\overline{z} \|_V \leq \frac{C}{t^{1/2-1/4\alpha}} \| \overline{z} \|_{\overline{Z}} = \overline{g}(t) \| \overline{z} \|_{\overline{Z}} \ , \qquad 1 \leq \alpha < 2$$

$$\| T(t)\overline{z} \|_V \leq \frac{C}{t^{1/4-1/6\alpha}} \| \overline{z} \|_{\overline{Z}} = \overline{g}(t) \| \overline{z} \|_{\overline{Z}} \ , \qquad \alpha \geq 2 \ ,$$

where C is a generic constant.

We also have, for the nonlinearity, the inequalities

(iii) $\| v\,z \|_{\bar{Z}} \leq \| v \|_V \| z \|_V$

(iv) $\| v_1 z - v_2 z \|_{\bar{Z}} \leq \| v_1 - v_2 \|_V \, \| z \|_V$

and finally,

(v) $$\| T(t)\bar{z} \|_Z \leq \frac{C}{t^{1/4}} \| \bar{z} \|_{\bar{Z}} = \bar{\bar{g}}(t) \| \bar{z} \|_{\bar{Z}} , \quad 1 \leq \alpha < 2$$

$$\leq C \| \bar{z} \|_{\bar{Z}} = \bar{\bar{g}}(t) \| \bar{z} \|_{\bar{Z}} , \quad \alpha \geq 2.$$

The consitions (d) (before theorem 4.1) and (4.4) now become

(7.2) $$\| \bar{g} \|_{L^q[0,\tau_1]} \cdot a < 1$$

and

(7.3) $$\| g \|_{L^r[0,\tau_1]} \| z_o^i \|_Z + \| \bar{g} \|_{L^q[0,\tau_1]} a^2 \leq a$$

(since we may take $b = a^2$)

Hence, we see that a solution exists for each finite τ_1 and sufficiently small initial values. Letting $\tau_1 = \tau$ (pass length) we see from (7.2) that

$$\| \bar{g} \|_{L^q[0,\tau]} a^2 < a$$

and so if

(7.4) $$\| z_o^i \|_Z \leq (a - a^2 \| \bar{g} \|_{L^q[0,\tau]}) / \| g \|_{L^r[0,\tau]}$$

and

(7.5) $$\| z^o \|_{L^r[0,\tau;V]} \leq a < \frac{1}{\| \bar{g} \|_{L^q[0,\tau]}}$$

then a solution exists for the whole pass length.

Note that (7.4) and (7.5) represent a trade-off between the initial time conditions x_{io} and the difference equation initial conditions x_j. Thus, if the x_j are taken to have the largest value $\| \bar{g} \|^{-1}_{L^q[0,\tau]}$, then the x_{io} must be zero.

Thus, for example (choosing appropriate r,q,s etc.), we find that, under conditions (7.4), (7.5), a solution exists in

$$L^{8-\varepsilon}[0,\tau;L^4[0,1]] \cap C[0,\tau;L^2[0,1]],$$

where ε is arbitrarily small. One could also show that the solution lies in other L^p spaces by using interpolation results (cf. [1,4,5]).

In order to use the Lyapunov theory, note that

$$\int_0^1 z_x^2(x)dx \geq 4\|z\|^2_{C[0,1]} ,$$

and so, by theorem 5.1 ,

$$(7.6) \quad \|z^i(t)\|^2_Z - \|z^i(\varepsilon)\|^2_Z \leq -2\int_\varepsilon^t \|z^i(s)\|^2_{C[0,1]}(4-\|z^{i-1}(s)\|^2_Z)\, ds$$

If we assume that $\|z^o(t)\|_Z \leq 2$ for all $t \geq o$, and that $\|z_o^i\|_Z \leq 2$, then it follows by induction (letting $\varepsilon \to 0$ in the above expression) that the solution is defined for all t and i and we have

$$(7.7) \qquad \|z^i(t)\|^2_Z \leq 2 , \quad \forall\, t \geq 0,\ i \geq 0.$$

Moreover, from (7.6), we obtain

$$\|z(t)\|^2_{\ell^2(Z)} - \|z(\varepsilon)\|^2_{\ell^2(Z)} \leq -2\int_\varepsilon^t 4\|z(s)\|^2_{\ell^2(C[0,1])} - \sum_{i=1}^{\infty}(\|z^i(s)\|^2_{C[0,1]}\|z^{i-1}(s)\|^2_Z)\, ds$$

But, if $\|z^o(t)\|_Z \leq 2$ for all $t \geq o$, we obtain from (7.7) that

$$\|z(t)\|^2_{\ell^2(Z)} \leq \|z(\varepsilon)\|^2_{\ell^2(Z)}$$

for all $\varepsilon > 0$. Hence, if $\|z(0)\|^2_{\ell^2(Z)} = \sum_{i=1}^{\infty}\|z_o^i\|^2_Z < \infty$

then $\|z(t)\|^2_{\ell^2(Z)} < \infty$. We have therefore obtained the following input-output stability for this system; namely, if $\|z^o(t)\|^2_Z \leq 2$, $\forall\, t \geq o$, and the initial values z_o^i satisfy $\|z_o^i\|_Z \leq 2$, $\forall\, i \geq o$ and $\{z_o^i\}_{i\geq 1}$ belongs to $\ell^2(Z)$, then $z(t) = (z^i(t))_{i\geq 1} \in \ell^2(Z)$.

Example 7.2 In this example we shall consider the parabolic system

(7.8) $\dot{z}_i = Az_i + f(z_i, z_{i-1}, \ldots, z_{i-\ell})$, $z_i \in H$ (Hilbert space) where A generates an analytic semigroup, and the spectrum of A is contained in the sector

$$S_{d,\phi} = \{\lambda : \pi - \phi < |\arg(\lambda - d)| < \pi, \ \lambda \neq d\}.$$

We shall also assume that $f : H^{\ell+1} \to H$ and satisfies

(7.9) $\langle f(z, h_1, \ldots, h_\ell) + az, \ f(z, h_1, \ldots, h_\ell) + bz \rangle_H \geq 0$

$z, h_1, \ldots, h_\ell$ H and for some $b > a > o$.

If we assume that f is Lipschitz then we can apply the existence and uniqueness theory here as in example 7.1. We shall now show that the theory of section 6 applies and that we can obtain frequency domain criteria for the stability of (7.8). The map $\xi_{(i\omega)}$ in section 6 is easily shown to map the sector $S_{d,\phi}$ into the circle of radius 1/2d and centre (1/2d,0). Hence assumption (7.9) means that conditions (C1),(C2) are trivially satisfied and so by corollary (6.2), the solution $z = (z_i)_{i \geq 1}$ of equation (7.8) belongs to $L^2[0,\infty;H]$.

A specific example of an equation of the form (7.8) is the system defined by

$$\frac{\partial z_i(t,x)}{\partial t} = \frac{\partial^2 z_i(t,x)}{\partial x^2} + pz_i(t,x) + z_i(t,x)|z_{i-1}(t,x)|$$

where $p > o$ and $\frac{\partial z}{\partial x} = 0$ when $x = 0,1$.

Then the operator A defined by

$$Az = \frac{\partial^2 z}{\partial x^2} + pz$$

with domain

$$D(A) = \{z \in L_2(0,1) : \frac{\partial^2 z}{\partial x^2} \in L_2(0,1) \ , \ \frac{\partial z}{\partial x} = 0 \ \text{ at } x = 0,1\}$$

generates an analytic semigroup and A has the spectrum

$$\lambda_j = -p - (j-1)^2 \pi^2 \ , \quad j \geq 1.$$

Also, for any o<a<b ,

$$\int_o^1 z_i^2(x)(|z_{i-1}(x)|+ a)(|z_{i-1}(x)| + b)\ dx \geq 0$$

and so the general theory applies in this case.

8. Conclusions

In this paper, we have discussed the existence, uniqueness and stability of nonlinear multipass processes which are modelled by partial differential equations. We have shown that, under fairly mild conditions, a local (in time) solution exists and by placing other restrictions on the system we may extend the solution for a finite time. The length of the time interval over which we are able to extend the solution is dependent on the 'size' of the initial values - generally, the smaller we take the initial values the longer the solution can be shown to exist.

The circle theorem has also been generalized to the case of nonlinear differential difference equations in a Hilbert Space and we have given a frequency domain criterion in terms of the spectrum of the linear part of the system. However, as is the case with the circle criterion, one must impose much stronger conditions on the nonlinearity than with the Lyapunov methods of section 5.

Finally, we have given two examples to illustrate the theory. Both examples are of the diffusion type, but it should be noted that the method works for hyperbolic systems such as the wave equation.

9. References

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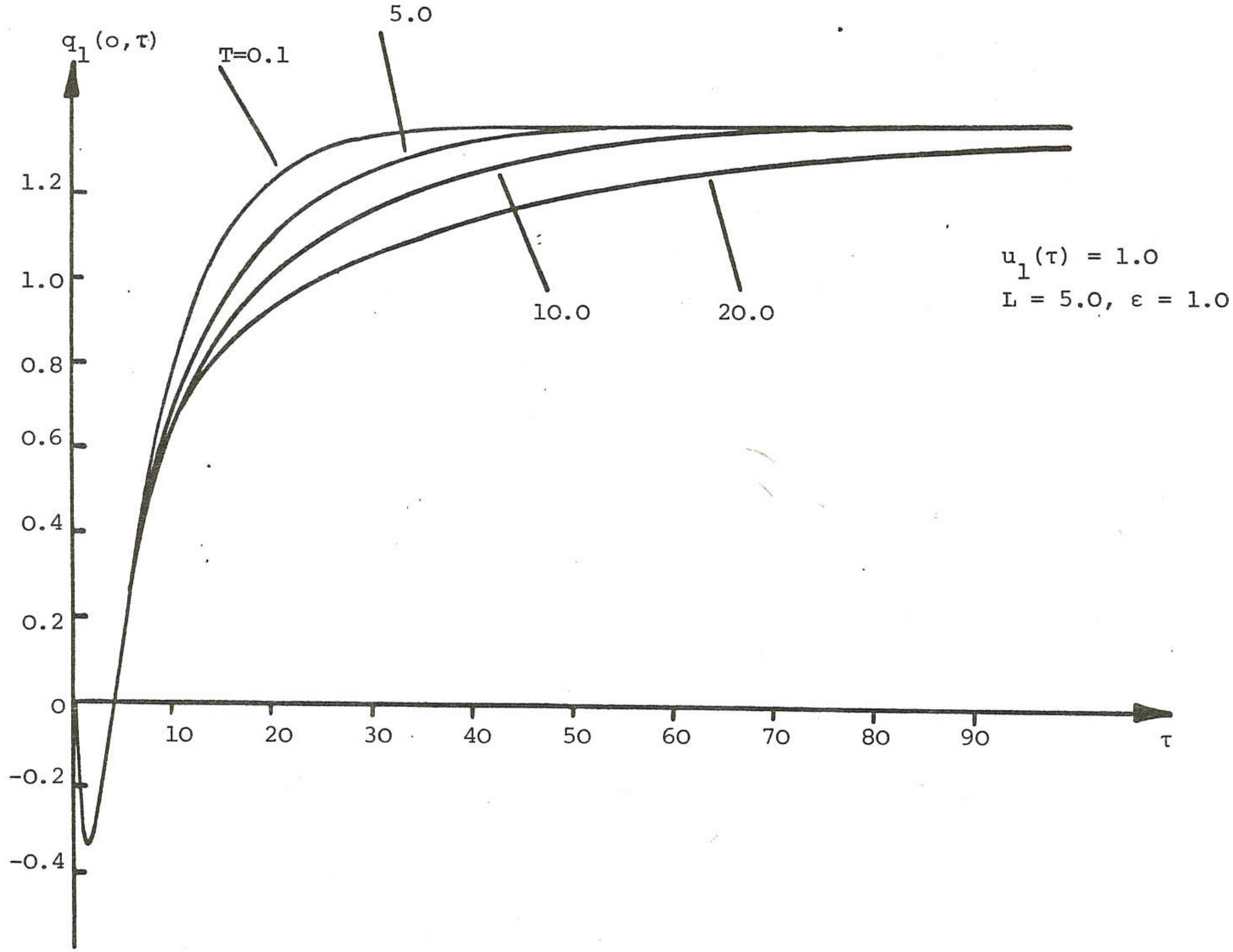

Fig. 8 Unit step responses for various end-vessel capacitances