White Rose Research Online
eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk
Universities of Leeds, Sheffield and York

Deposited via The University of Sheffield.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/id/eprint/76043/

Monograph:
Owens, d (1981) Sufficient Conditions for Noncontrolability of Homogeneous Bilinear Systems. Research Report. ACSE Report 137 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

Reuse
Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown
If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

UNIVERSITY OF LEEDS
University of Sheffield
UNIVERSITY of York

629. 8 (S)

[redacted]

SUFFICIENT CONDITIONS FOR NONCONTROLLABILITY OF HOMOGENEOUS BILINEAR SYSTEMS

by

D. H. Owens, BSc,ARCS,PhD,AFIMA,CEng,MIEE

Department of Control Engineering
University of Sheffield
Mappin Street, Sheffield S1 3JD

Research Report No. 137

January 1981

629.8 (S)

5 067223 01
SHEFFIELD UNIV.
APPLIED SCIENCE
LIBRARY
2.11.81

Abstract

A simple condition for the noncontrollability of homogeneous bilinear systems is derived in terms of the impulse response of the associated linear feedback system. The result can frequently be checked simply by inspection of the transfer function poles and zeros.

The last decade has seen considerable progress in the understanding of the structure and properties of nonlinear systems (see, for example, references (1)-(6)) with particular interest being focussed in the study of internally bilinear systems (see, for example references (3), (4) and (6)) with emphasis on the important problems of controllability and characterizations of the reachable set from a given point in finite time (see, for example, references (2), (3), (4), (6) and (7)) in terms of Lie algebraic properties of the defining state equations. It is unfortunately true however that we are still at the stage when the available analytic results are difficult to check computationally and it is consequently of great importance in practice to obtain 'easily checkable' (necessary <u>or</u> sufficient) conditions that can be investigated to provide some indication of the possibility of controllability <u>or its absence</u>. It is the second problem that is of interest here.

In this note we derive conditions for the homogeneous (in the state) ℓ-input bilinear system

$$\dot{x}(t) = (A + B(u(t)))x(t)$$

$$B(u) = \sum_{i=1}^{\ell} u_i B_i C_i \qquad \ldots(1)$$

<u>not</u> to be controllable in $R^n-\{0\}$ in terms of the properties of the

impulse response matrix

$$H(t,k) = C\, e^{(A-kBC)t}\, B \qquad \ldots(2)$$

of the linear system S(A,B,C) with

$$B = [B_1, B_2, \ldots, B_\ell] \quad , \qquad C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_\ell \end{pmatrix} \qquad \ldots(3)$$

and subjected to unit negative feedback with scalar gain k. In certain special cases, the required properties can also be identified in terms of the pole-zero structure of the associated 'open-loop' transfer function matrix

$$G(s) = C(sI_n - A)^{-1}B \qquad \ldots(4)$$

The system will be termed controllable in $R^n-\{0\}$ if, and only if, for each pair of points x_o and x_f in $R^n-\{0\}$, there exists a finite time $t_f>0$ and piecewise continuous control inputs $u_i(t)$, $1\leq i\leq \ell$, defined on $[0,t_f]$ such that the unique solution of (1) originating at x_o at time $t = 0$ is driven to the state x_f at $t = t_f$. All matrices A, B_i, C_i $(1\leq i\leq \ell)$ are assumed to be real and constant of dimensions $n\times n$, $n\times r_i$ and $r_i\times n$ $(1\leq i\leq \ell)$ respectively and, without loss of generality, we will take rank B_i = rank $C_i = r_i$, $1\leq i\leq \ell$. We will also require the partial ordering in $L(R^p,R^q)$ defined by the relation

$$D \geq 0 \quad \text{iff} \quad D_{ij} \geq 0 \quad \text{for} \quad 1\leq i\leq q \;,\; 1\leq j\leq p \qquad \ldots(5a)$$

$$D > 0 \quad \text{if} \quad D \geq 0 \text{ and } D_{ij} > 0 \text{ for some } i \text{ and } j \qquad \ldots(5b)$$

Our main result can now be stated:

<u>Theorem</u>: If there exists $k^* \in R$ such that $H(t,k)$ satisfies the relation

$$H(t,k) \geq 0 \qquad \forall \quad t \geq 0 \qquad \forall \quad k \in [k^*, \infty[\qquad \ldots(6)$$

then the bilinear system (1) is <u>not</u> controllable in $R^n - \{0\}$. Moreover, under these conditions, all trajectories originating at time $t = 0$ in the cone $P_B \triangleq \{x \in R^n : x = B\alpha \text{ for some } \alpha \geq 0\}$ generated by taking positive linear combinations of the columns of the B lie in the cone $P_C \triangleq \{x \in R^n : Cx \geq 0\}$.

<u>Proof</u>: Write the model (1) in the form

$$\dot{x}(t) = (A + B(u_o)) + B(u(t) - u_o))x(t) \qquad \ldots(7)$$

where $u_o^T = (-k, -k, \ldots, -k) \in R^\ell$ and k is arbitrary. The solution of this equation is also the solution of the integral equation

$$x(t) = e^{(A+B(u_o))t} x(o) + \int_o^t e^{(A+B(u_o))(t-s)} B(u(s) - u_o) x(s) ds \qquad \ldots(8)$$

Noting that $B(u_o) = \sum_{i=1}^{\ell} (-k) B_i C_i = -kBC$ and that $B(u(s) - u_o) = \sum_{i=1}^{\ell} (u_i(s) + k) B_i C_i = B \text{ block diag}\{(u_i(s)+k) I_{r_i}\}_{1 \leq i \leq \ell} C$, it is seen that, after multiplying by C and choosing the initial condition $x(o) = B\alpha \neq 0$ with $\alpha \geq 0$, we obtain

$$Cx(t) = H(t,k)\alpha + \int_o^t H(t-s,k) \text{block diag}\{(u_i(s)+k) I_{r_i}\}_{1 \leq i \leq \ell} Cx(s) ds \qquad \ldots(9)$$

Consider now control on the time interval $[0, t_f]$ with t_f arbitrary and choose $k \geq \max\{k^*, \sup_{0 \leq t \leq t_f} \max_{1 \leq i \leq \ell} |u_i(t)|\}$. Clearly $H(t,k)\alpha \geq 0$,

$0 \leq t \leq t_f$, and $H(t-s,k) \text{block diag}\{(u_i(s)+k)I_{r_i}\}_{1 \leq i \leq \ell} \geq 0$, $0 \leq s \leq t \leq t_f$. Noting also that $Cx(s)\big|_{s=0} = CB\alpha = H(0)\alpha \geq 0$, a simple argument using (9) indicates that $Cx(t) \geq 0$ for all $t \in [0,t_f]$, and hence that the state trajectory does not leave P_c. The result follows as both control input and t_f were arbitrary.

Remark 1: The identification of (6) with the monotonicity of the system step response matrix immediately yields a simple corollary to this result. This is omitted.

Remark 2: The factorization of $B(u)$ defined by (1) is non-unique as can be seen by the identity $B_iC_i = (B_iN_i^{-1})(N_iC_i)$ valid for any real nonsingular $r_i x r_i$ matrix N_i. Suitable choice of the N_i may possibly help in satisfying (6). Also transformation of the control variable to the new input vector $v(t) = Tu(t)$ could help.

The sufficient conditions for noncontrollability have the advantage of being expressed in terms of a quantity (the 'impulse response') familiar to practicing control engineers. The natural engineering approach to checking the validity of (6) is to use Laplace transforms and the identity

$$\begin{aligned} H(t,k) &= \mathcal{L}^{-1}\{C(sI_n - A + kBC)^{-1}B\} \\ &= \mathcal{L}^{-1}\{(I_m + kG(s))^{-1}G(s)\} \end{aligned} \qquad \ldots(10)$$

(where $m = r_1+r_2+..+r_\ell$) to compute $H(t,k)$. For example, consider the two-input system

$$\dot{x}(t) = \begin{pmatrix} -1+u_1(t) & 1 \\ 1 & -1+u_2(t) \end{pmatrix} x(t) \qquad \ldots(11)$$

with the data $\ell = 2$, $r_1 = r_2 = 1$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad B_1 = C_1^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad B_2 = C_2^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \ldots (12)$$

and hence $B = C = I_2$. Using (10) it is easily verified that

$$H(t,k) = \mathcal{L}^{-1}\left\{ \frac{1}{(s+k)(s+k+2)} \begin{pmatrix} s+1+k & 1 \\ 1 & s+1+k \end{pmatrix} \right\}$$

$$= \tfrac{1}{2} \begin{pmatrix} e^{-kt}+e^{-(k+2)t} & , & e^{-kt}-e^{-(k+2)t} \\ e^{-kt}-e^{-(k+2)t} & , & e^{-kt}+e^{-(k+2)t} \end{pmatrix}$$

$$\geq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \forall\, t \geq 0 \quad \forall\, k \in R \qquad \ldots (13)$$

and hence that the system (11) is not controllable.

To conclude, the following proposition identifies one particular situation when condition (6) can be easily investigated by 'visual' inspection of G(s) only:

<u>Proposition:</u> Condition (6) holds if $\ell = 1$, $r_1 = 1$ and the <u>transfer function</u> G(s) has the form

$$G(s) = \frac{g_\infty (s-z_1)(s-z_2)\ldots.(s-z_{n-1})}{(s-p_1)(s-p_2)\ldots\ldots(s-p_n)} \qquad \ldots (14)$$

where the poles $\{p_j\}_{1\leq j\leq n}$ and zeros $\{z_j\}_{1\leq j\leq n-1}$ are all real and satisfy the 'interlacing condition'

$$p_1 < z_1 < p_2 < z_2 \quad \ldots\ldots < z_{n-1} < p_n \qquad \ldots (15)$$

and the gain $g_\infty \geq 0$.

Proof: The interlacing property ensures that the transfer function

$$G(s) = \sum_{j=1}^{n} R_j/(s-p_j) \qquad \ldots(16)$$

where the residues $\{R_j\}$ are either all positive or all negative. In fact $R_j \geq 0$, $1 \leq j \leq n$, as $\lim_{|s|\to\infty} sG(s) = \sum_{j=1}^{n} R_j = g_\infty \geq 0$ by assumption and hence

$$H(t,0) = \mathcal{L}^{-1}\{G(s)\} = \sum_{j=1}^{n} R_j e^{p_j t} \geq 0 \quad \forall\ t \geq 0 \qquad \ldots(17)$$

The result now follows by applying a similar argument to $(1+kG(s))^{-1}G(s)$ noting, from elementary root-locus arguments, that this transfer function has real, interlaced poles and zeros and gain equal to g_∞ for all choices of k.

Remark 2: The conditions $\ell = 1$ and $r_1 = 1$ indicate that the system has only one input and that rank $B(u) = 1$. Equivalently the bilinear system is generated from the single-input/single-output system $y(t) = Cx(t)$, $\dot{x}(t) = Ax(t)+Be(t)$ with input $e(t)$ defined by the parametric feedback law $e(t) = u(t)y(t)$. Clearly controllability is impossible if the linear system is 'interlaced' (the gain condition is irrelevant here as, if $g_\infty<0$, introduce the new control variable $v(t) = -u(t)$). This is a nontrivial point as many physical systems possess the interlacing property.

Remark 3: The computational problems arising in the verification of the conditions of the proposition are relatively minor, even for high order systems.

SHEFFIELD UNIV.
APPLIED SCIENCE
LIBRARY

REFERENCES

(1) W.A.Porter: 'An Overview of Polynomic System Theory', Proc.IEEE, 64, 1976, 18-23.

(2) R.W.Brockett: 'Nonlinear Systems and Differential Geometry', ibid, 61-72.

(3) C.Bruni, G.D.Pillo, G.Koch: 'Bilinear Systems: An appealling class of 'nearly linear' systems in theory and practice', IEEE Trans, 1974, AC-19, 334-348.

(4) R.R.Mohler: 'Bilinear Control Processes', Academic Press, New York and London, 1973.

(5) P.Crouch: 'Realization Theory for Dynamical Systems', Proc.IEE, 126, 1979, 605-615.

(6) P.Crouch: 'Polynomic Systems Theory: A Review', ibid, Pt.D, 127, 220-228.

(7) R.M.Hirschorn: 'Global Controllability of Nonlinear Systems', SIAM J. Control and Optimization, 14, 1976, 700-711.