Random Processes in Nonlinear Systems

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1. Introduction

Most systems encountered in practice are nonlinear to some extent due to inherent distortion introduced by the components of the system such as saturation or because they include deliberately introduced nonlinear effects (e.g. bang-bang controllers). Any system for which the superposition principle does not hold is defined to be nonlinear. Nonlinear systems exhibit phenomena like jumps, limit cycles, hysteresis and chaotic motions which are not possible in linear systems. It is these characteristics which often dictate that the study of nonlinear systems is restricted to specific system structures.

The statistical analysis of nonlinear systems is in general an extremely difficult problem and a unified theory applicable to a broad class of systems does not exist. Systems which contain two or more single-valued nonlinear elements, multivalued nonlinearities or nonlinear functions of two or more system variables are particularly difficult to analyse and recourse is often made to either simulation or piece-wise linear analysis.

The present study briefly reviews some of the methods which are available for the statistical analysis of static and dynamic nonlinear systems, linearisation methods and identification algorithms. Because of space limitations only a few of the established approaches to the problem are presented and throughout the reader is referred to the references for further details.

2. Static Nonlinear Systems

Consider the system illustrated in Fig.1 where u(t) is applied as an input to a single-valued instantaneous nonlinear element N(•) to produce an output y(t).

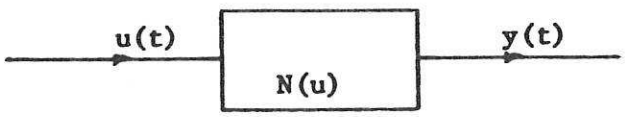

Fig.1

If the input is stationary in the strict sense the k'th order probability density function of y can be obtained from $f_u(u_1,u_2,\ldots u_k; t_1,t_2..t_k)$ the density of u [1]. To determine $f_y(y_1..y_k; t_1,..t_k)$ solve the system of equations $y_1 = N(u_1),\ldots y_k = N(u_k)$ for $u_1,\ldots u_k$ and assuming a unique solution

$$f_y(y_1,..y_k; t_1,..t_k) = \frac{f_u(u_1,\ldots u_k; t_1,\ldots t_k)}{|N'(u_1)|\ldots|N'(u_k)|} \qquad \ldots(1)$$

For example consider the evaluation of the density $f_y(y;t)$ when $y(t) = u^2(t)$. When $y>0$, the eqn $y = x^2$ has two solutions $u_1 = \sqrt{y}$ and $u_2 = -\sqrt{y}$. Further since $|dy/du| = 2\sqrt{y}$, then from eqn (1)

$$f_y(y;t) = \frac{f_u(\sqrt{y};t) + f_u(-\sqrt{y};t)}{2\sqrt{y}}$$

If $y<0$, then $f_y(y) = 0$.

The moments of the output y, Fig.1, can however be expressed directly in terms of the probability density function of the input. The auto-correlation function of the output is for example given by

$$R_{yy}(\tau) = \int_{-\infty}^{\infty}\int y_1 y_2 \, f(y_1,y_2;\tau)\,dy_1\,dy_2 \qquad \ldots(2)$$

or $$R_{yy}(\tau) = \iint N(u_1)N(u_2)f(u_1,u_2;\tau)\,du_1\,du_2 \qquad \ldots(3)$$

An analytical expansion of this integral can be obtained when u is stationary and normally distributed

$$f(u_1,u_2;\tau) = \frac{1}{2\pi\sigma^2(1-\rho^2)^{\frac{1}{2}}} \; \text{Exp}\left\{-\left(\frac{u_1^2+u_2^2-2\rho u_1 u_2}{2\sigma^2(1-\rho^2)}\right)\right\} \qquad \ldots(4)$$

where
$$\begin{aligned} &E[u(t)] = 0 \\ &E[u^2(t)] = \sigma^2 \\ &E[u(t)u(t+\tau)] = \sigma^2\rho(\tau) \end{aligned} \qquad \ldots(5)$$

and $\rho(\tau)$ is the normalised covariance function of the input. Using Mercer's formula [2] eqn (4) can be expanded as

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \; \exp\left\{-\left(\frac{a_1^2+a_2^2-2\rho a_1 a_2}{2(1-\rho^2)}\right)\right\}$$

$$= \left\{ \exp\left\{ -\left(\frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right\} / 2\pi \right\} \sum_{n=0}^{\infty} \rho^n Q_n(a_1) Q_n(a_2) \qquad \ldots(6)$$

where $Q_n(a)$ is the n'th order Hermite polynomial

$$Q_n(a) = \exp\left(\frac{a^2}{2}\right) \cdot \frac{(-1)^n}{\sqrt{n!}} \frac{d^n}{da^n} \left\{ \exp\left(-\frac{a^2}{2}\right) \right\} \qquad \ldots(7)$$

and $a_i = u_i/\sigma$.

Combining eqn's (3), (4) and (6) yields

$$R_{yy}(\tau) = \sum_{n=0}^{\infty} \rho^n k_i^2 \qquad \ldots(8)$$

where $k_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(\sigma a) Q_n(a) \exp\left(-\frac{a^2}{2}\right) da \qquad \ldots(9)$

The autocorrelation function of the output eqn (8) is given therefore as a power series of the normalised autocorrelation function of the input where the coefficients k_i, eqn (9), depend on the form of nonlinearity.

For symmetrical non-linearities, $N(u) = -N(-u)$, all even coefficients k_{2j} vanish, and for nonlinearities which can be expressed as truncated power series with zero coefficients for power greater than j, then $k_i = 0$ for $i>j$. As an example, the output autocorrelation function for the bang-bang nonlinearity

$y = 1 \quad , \quad u>0$

$y = -1 \quad , \quad u<0$

is given by

$$R_{yy}(\tau) = \frac{2}{\pi} \operatorname{Sin}^{-1} \rho(\tau) \qquad \ldots(10)$$

Although the input-output cross-correlation function $R_{uy}(\tau)$ can be determined by following a similar procedure to the above a slightly more general result can be obtained by introducing separable processes [3].

Let $f(u_1, u_2; \tau)$ be the second order probability density function of the stationary process u(t) in Fig.1 and define

$$g(u_2, \tau) = \int_{-\infty}^{\infty} u_1 f(u_1, u_2; \tau) du_1 \qquad \ldots(11)$$

If the g-function separates as

$$g(u_2,\tau) = g_1(u_2)g_2(\tau) \quad \forall\, u,\tau \qquad \ldots(12)$$

then u(t) is said to be a separable process, where

$$g_1(u_2) = \frac{u_2 f(u_2)}{g_2(0)} \qquad \ldots(13)$$

$$g_2(\tau) = \frac{g_2(0)R_{uu}(\tau)}{R_{uu}(0)} \qquad \ldots(14)$$

The separable class of random processes is fairly wide and includes the Gaussian process, sine wave process, pseudo-random-binary-sequences etc.

Define the cross-correlation function

$$R_{uy}(\tau) = \int_{-\infty}^{\infty}\int u_1 N(u_2) f(u_1,u_2;\tau)\,du_1 du_2 \qquad \ldots(15)$$

substituting from eqn's (12) through (14) yields

$$\begin{aligned} R_{uy}(\tau) &= \int N(u_2) g_1(u_2) g_2(\tau)\,du_2 \\ &= \frac{R_{uu}(\tau)}{R_{uu}(0)} \int N(u_2) u_2 f(u_2)\,du_2 \\ &= C_F R_{uu}(\tau) \qquad \ldots(16) \end{aligned}$$

where C_F is a constant scale factor.

Equation (16) which is known as the invariance property, shows that $R_{uy}(\tau)$ is directly proportional to $R_{uu}(\tau)$ for any static nonlinear characteristic $N(\cdot)$ providing u(t) is separable.

3. Nonlinear Systems with Dynamics

3.1 Functional Series Methods

A functional representation of nonlinear systems which is a generalisation of the linear convolution integral was first studied by Volterra early in the twentieth century. Volterra investigated analytic functionals and introduced the representation [4]

$$y(t) = \sum_{n=1}^{\infty} \int_{\Omega} \ldots \int h_n(\tau_1, \ldots \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i$$

$$= \sum_{n=1}^{\infty} y_n(t) \qquad \ldots(17)$$

which has become known as the Volterra series. The functions $h_i(\tau_1, \ldots \tau_i)$ in eqn (17) are referred to as Volterra kernels. The kernels are bounded and continuous in each τ_j, symmetric functions of their arguments, and for causal systems $h_i(\tau_1, \ldots \tau_i) = 0$ for any $\tau_j < 0$. Systems which contain nonlinear memory elements such as hysteresis or backlash are excluded from the description of eqn (17).

Consider the Volterra series representation of the system illustrated in Fig.2.

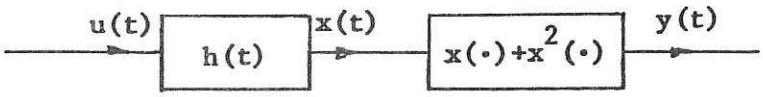

Fig.2

From the convolution integral

$$x(t) = \int h(\tau) u(t-\tau) d\tau$$

But $y(t) = x(t) + x^2(t)$ and therefore

$$y(t) = \int h(\tau_1) u(t-\tau_1) d\tau_1 + \int\int h(\tau_2) h(\tau_3) u(t-\tau_2) u(t-\tau_3) d\tau_2 d\tau_3 \qquad \ldots(18)$$

is the Volterra series representation.

Taking the Fourier Transform of the n'th order kernel in eqn (17) yields the n'th order kernel transform [4]

$$H_n(j\omega_1, \ldots j\omega_n) = \int_{-\infty}^{\infty} \ldots \int h_n(\tau_1, \ldots \tau_n) \exp\{-j(\omega_1\tau_1 + \ldots + \omega_n\tau_n)\} d\tau_1 \ldots d\tau_n \qquad \ldots(19)$$

Similarly, considering just the n'th order component of the output $y_n(t)$ and taking Fourier transforms relates the multispectral density

$$Y_n(j\omega_1, \ldots j\omega_n) = H_n(j\omega_1, \ldots j\omega_n) U(j\omega_1) \ldots U(j\omega_n) \qquad \ldots(20)$$

to the input spectrum $U(j\omega)$, from which the output spectrum can be

evaluated as

$$Y_n(j\omega) = \frac{1}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \cdots \int Y_n(j\omega - j\mu_1, j\mu_1 - j\mu_2, \cdots j\mu_{n-1}) d\mu_1 \cdots d\mu_{n-1} \quad \ldots(21)$$

The statistical analysis of systems described by a Volterra series when the kernels are known has been studied extensively particularly with reference to communication systems [5-8] and Gaussian, sine wave plus Gaussian or random pulse train inputs. An excellent review of the use of functionals in the analysis of nonlinear systems is given by Barnett [9].

Consider the evaluation of the autocorrelation of the output, input-output cross-correlation and associated spectral densities for a system with known Volterra kernels assuming the output is strict-sense stationary. Define the correlation functions

$$R_{uy}(\tau) = E[y(t)u(t-\tau)]$$

$$= \sum_{n=1}^{\infty} R_{uy_n}(\tau) \quad \ldots(22)$$

$$R_{yy}(\tau) = E[y(t)y(t-\tau)]$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} R_{y_n y_m}(\tau) \quad \ldots(23)$$

To evaluate the above expressions it is necessary to determine the partial correlation functions $R_{uy_n}(\tau)$, $R_{y_n y_m}(\tau)$. Rather than evaluating the general expressions which are given in the literature [5-8] consider the system illustrated in Fig.2 to illustrate the procedure.

Thus from eqn (18)

$$R_{uy}(\tau) = \int h(\tau_1) E\{u(t-\tau_1)u(t-\tau)\} d\tau_1$$

$$+ \int\int h(\tau_2)h(\tau_3) E\{u(t-\tau_2)u(t-\tau_3)u(t-\tau)\} d\tau_2 d\tau_3 \quad \ldots(24)$$

$$R_{yy}(\tau) = \int\int h(\tau_1)h(\tau_2) E\{u(t-\tau_1)u(t-\tau-\tau_2)\} d\tau_1 d\tau_2$$

$$+ \int\int\int h(\tau_1)h(\tau_2)h(\tau_3) E\{u(t-\tau_1)u(t-\tau-\tau_2)u(t-\tau-\tau_3)\} d\tau_1 d\tau_2 d\tau_3$$

$$+ \iiint h(\tau_1)h(\tau_2)h(\tau_3)E\{u(t-\tau_1-\tau)u(t-\tau_2)u(t-\tau_3)\}d\tau_1 d\tau_2 d\tau_3$$

$$+ \iiiint h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)E\{u(t-\tau_1)u(t-\tau_2)u(t-\tau-\tau_3)u(t-\tau-\tau_4)d\tau_1 d\tau_2 d\tau_3 d\tau_4$$

...(25)

In general all the moments of the input process up to order n must be known before eqn's (24) and (25) can be evaluated. When the input is a zero mean white Gaussian process where

$$E\{u(t_1)u(t_2)\ldots u(t_i)\} = 0 \quad i \text{ odd} \qquad \ldots(26)$$
$$= \sum_{i} \prod_{n\neq m} \delta(t_n - t_m) \quad i \text{ even}$$

and the summation is over all ways of dividing i objects into pairs, eqn's (24) and (25) reduce to

$$R_{uy}(\tau) = h(\tau) \qquad \ldots(27)$$

$$R_{yy}(\tau) = \int h(\tau+\tau_2)h(\tau_2)d\tau_2$$

$$+ 2\iint h(\tau+\tau_3)h(\tau+\tau_4)h(\tau_3)h(\tau_4)d\tau_3 d\tau_4 \qquad \ldots(28)$$

Spectral densities are computed by taking Fourier Transforms of the associated correlation functions and using eqn's (19)-(21). An algebra of nonlinear systems based on the Volterra series has been developed by George [10] and this simplifies the notation considerably in many problems.

3.2 The Fokker-Planck-Kolmogorov Equation

Consider the class of dynamic systems which can be represented by the stochastic vector differential equations

$$d\underset{\sim}{x}/dt = A(\underset{\sim}{x},t) + C(\underset{\sim}{x},t)V(t) \qquad \ldots(29)$$

where $\underset{\sim}{x} = \{x_i\}$ are the n-state variables, $A(x,t) = \{a_i\}$ and $C = \{c_{ij}\}$ are coefficient matrices and $\underset{\sim}{V}(t)$ is an m-dimensional Gaussian white noise vector with the properties $E[\underset{\sim}{V}(t)] = 0$, $E[\underset{\sim}{V}(t)\underset{\sim}{V}(s)^T] = \underset{\sim}{Q}\delta(t-s)$, $\underset{\sim}{Q} = \text{diag}\{\sigma_{ii}^2\}$. $\underset{\sim}{V}(t)$ can be used to represent random external disturbances, modelling discrepancies and random parametric perturbations.

Because Gaussian white noise is not mathematically meaningful, rewrite eqn (29) in terms of the incremental Wiener process $d\underset{\sim}{W}(t) = \underset{\sim}{V}(t)dt$ to

yield the Itô [11,12] stochastic differential equation

$$d\underset{\sim}{x} = \underset{\sim}{A}(\underset{\sim}{x},t)dt + \underset{\sim}{C}(\underset{\sim}{x},t)d\underset{\sim}{W}(t) \qquad \ldots(30)$$

where

$$E\{d\underset{\sim}{W}\} = 0 \ , \ E\{d\underset{\sim}{W}(t)d\underset{\sim}{W}(s)^T\} = \underset{\sim}{Q}\delta(t-s)dt$$

Equation (30) generates a Markov process $\underset{\sim}{x}(t)$ since $\{\underset{\sim}{x}(t)\}$ and $\underset{\sim}{w}(t)$ are independent with independent increments. Consequently, the solution of eqn (30) is completely characterized by the first order probability density function $f(\underset{\sim}{x},t)$ and the transitional probability density function $f(\underset{\sim}{x},t_2/\underset{\sim}{x},t_1)$ for $t_2>t_1$, both of which can be shown to satisfy the Fokker-Planck-Kolmogorov equation [1,11,12,13]

$$\frac{\partial f(\cdot)}{\partial t} = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(a_i(\underset{\sim}{x},t)f(\cdot)) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}((\underset{\sim}{C}\underset{\sim}{Q}\underset{\sim}{C}^T)_{ij}f(\cdot)) \qquad \ldots(31)$$

In general it is of more practical value to determine the moments $m_n = E[x_i^n]$ of the system states rather than the probability density function $f(\underset{\sim}{x},t)$. Using either the method of moments or Itô's fundamental lemma it can be shown, for example, that the first two moments are given by the solution of the ordinary differential equations [13,14]

$$\frac{dE(x_i)}{dt} = E(a_i(\underset{\sim}{x},t)) \qquad \ldots(32)$$

$$\frac{dE(x_i x_j)}{dt} = E(a_i x_i + a_j x_j) + E((\underset{\sim}{C}\underset{\sim}{Q}\underset{\sim}{C}^T)_{ij}) \qquad \ldots(33)$$

for $(i,j) = 1,2\ldots n$, and given initial conditions $\underset{\sim}{x}(t_o)$.

To illustrate the procedure consider a linear first order system with transfer function $\frac{1}{1+ST}$ driven by unity variance Gaussian white noise. The system model in state-space form is

$$Tdx = -xdx + dw \qquad \ldots(34)$$

and from eqn (31) its associated Fokker-Planck-Kolmogorov equation is

$$\frac{\partial f}{\partial t} = \frac{1}{T}\frac{\partial(xf)}{\partial x} + \frac{1}{2T}\frac{\partial^2 f}{\partial x^2} \qquad \ldots(35)$$

The first two moments are, from eqn's (32) and (33)

$$\frac{dm_1}{dt} = -\frac{1}{T} m_1 \qquad \ldots(36)$$

$$\frac{dm_2}{dt} = -\frac{2}{T} m_2 + \frac{1}{T^2} \qquad \ldots(37)$$

with solutions

$$m_1(t) = m_1(0)e^{-t/T} \qquad \ldots(38)$$

$$m_2(t) = m_2(0)e^{-2t/T} + \frac{1}{2T}(1-e^{-2t/T}) \qquad \ldots(39)$$

where $m_1(0)$ and $m_2(0)$ are initial conditions.

If the system equations (29) are nonlinear in the states the lower order moment eqn's (32) and (33) are in general functions of higher order moments and closed analytic solutions are not possible. Similarly the probability density function $f(\underset{\sim}{x},t)$ which satisfies the Fokker-Planck-Kolmogorov equation cannot be found except by linearization or approximation methods. An alternative is to simulate the stochastic difference eqn (29) and evaluate the required moments by averaging over the realisations [13-15].

4. Linearisation Methods

The relative simplicity of the methods of statistical analysis for linear systems compared with the inherent complexity of the analysis outlined above has led to the development of approximation methods based on linearisation techniques [16].

The simplest form of linearisation is based upon the expansion of the nonlinear function in a Taylor series about some operating point and retaining only the linear terms in the analysis. Thus the nonlinear function $g(x_1,\ldots x_n)$ is replaced by the approximate expression

$$g(x_1,\ldots x_n) \approx g(\bar{x}_1,\ldots,\bar{x}_n) + \sum_{i=1}^{n} g'_{x_i}(\bar{x}_1,\ldots,\bar{x}_n)(x_i-\bar{x}_i) \qquad \ldots(40)$$

where $\bar{x}_i$ is the mean of x_i and $g'_{x_i} = \partial f/\partial x_i$.

Although eqn (40) is linear with respect to fluctuations it is nonlinear with respect to expectations.

Equation (40) is valid only for continuous functions with continuous first derivatives, and cannot therefore be used to study the characteristics of discontinuous components such as relays or limiters. To linearise such characteristics the method of statistical linearisation was developed.

The earliest method of statistical linearisation was developed by Booton [17] for static nonlinearities and stochastic inputs with zero mean. Booton's method consists of replacing the nonlinearity $N(\cdot)$ by an equivalent gain which is selected so as to minimise the mean square of the difference between the output of the devices.

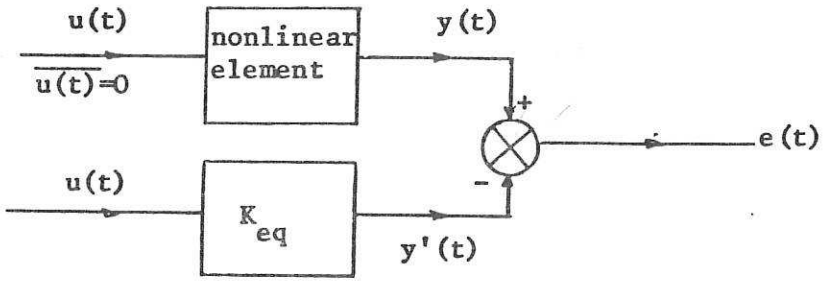

Fig.3

Consider the system illustrated in Fig.3 where

$$e(t) = y(t) - K_{eq}u(t)$$

$$\overline{e^2(t)} = \overline{y^2(t)} - 2K_{eq}\overline{u(t)y(t)} + K^2_{eq}\overline{u^2(t)} \qquad \ldots(41)$$

Selecting K_{eq} so as to minimise $\overline{e^2(t)}$ yields

$$K_{eq} = \frac{\overline{u(t)y(t)}}{\overline{u^2(t)}} = \frac{\int yuf(u)\,du}{\int u^2 f(u)\,du} \qquad \ldots(42)$$

When the input is Gaussian white $\int u^2 f(u)\,du = \sigma^2$ and hence

$$K_{eq} = \frac{1}{\sigma^2}\int yuf(u)\,du \qquad \ldots(43)$$

It can readily be shown [18] that K_{eq} in eqn (43) is equivalent to the first term in the Wiener series representation eqn (53) of a nonlinear zero memory system.

Somerville and Atherton [19] extended Booton's method to include cases of non-zero mean input signals as illustrated in Fig.4 to yield

$$K_{dc} = \frac{\overline{y(t)}}{\overline{u(t)}}$$

$$K_{eq} = \frac{\overline{u(t)y(t)} - \overline{u(t)}\cdot\overline{y(t)}}{\overline{u^2(t)} - \overline{u(t)}^2} \qquad \ldots(44)$$

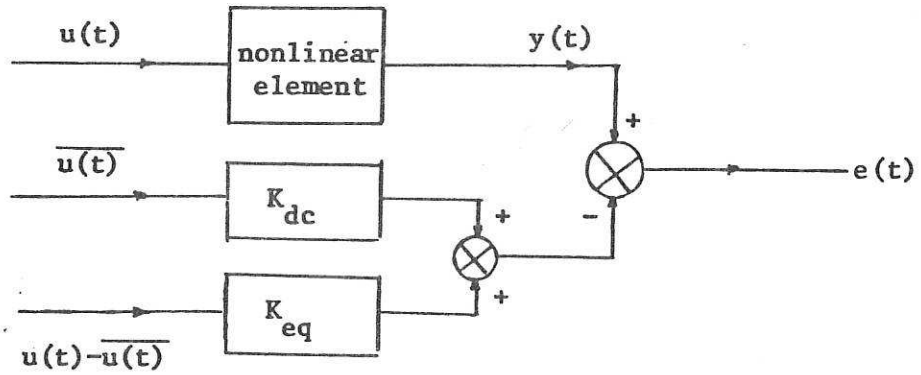

Fig.4

To conserve the spectrum of the output Pupkov [20] proposed replacing the zero memory nonlinear component N(·) by a dynamic stationary linear system

$$N(u(t)) = K_{dc}\overline{u(t)} + \int_{-\infty}^{t} h(t_1)\,(u(t-t_1)-\overline{u(t-t_1)})\,dt_1 \qquad \ldots(45)$$

where h(t) is determined to ensure the autocorrelation equivalence of the left and right hand sides of eqn (45)

$$\int_{-\infty}^{\infty}\int h(t_1)h(t_2)R_{uu}(\tau-t_1-t_2)\,dt_1\,dt_2 = R_{yy}(\tau) \qquad \ldots(46)$$

and K_{dc} is given by eqn (44).

If h(t) in eqn (45) ($\overline{u(t)} = 0$) is selected to minimise the mean squared error this leads to the Wiener-Hopf equation

$$\int h(t_1)R_{uu}(t_1-\tau)\,d\tau = R_{uy}(\tau) \qquad \ldots(47)$$

Although this equation is in general difficult to solve for h(t), when the input is a separable process [3] from eqn (16)

$$R_{uy}(\tau) = C_F R_{uu}(\tau)$$

and the optimum linear approximation to the nonlinear element is Booton's equivalent gain $C_F = K_{eq}$.

Harmonic linearisation, which yields the describing function [21] consists in the simplest case of replacing the nonlinear element by a linear one whose transfer function is equal to the complex ratio of the fundamental component of the output to the sinusoidal input. The method has been used extensively to obtain mathematical descriptions of a large number of nonlinear systems and several modifications have been developed including describing functions which relate all the output harmonics to the input fundamentals.

5. Identification

Identification algorithms for nonlinear systems can be categorised as functional series methods, algorithms for block oriented systems and parameter estimation techniques [22].

Identification using the Volterra series representation eqn (17) involves the measurement of the Volterra kernels. To illustrate the approach consider the identification of a system which can be described by just the first two Volterra kernels

$$y(t) = \int_0^\infty h_1(\tau_1)u(t-\tau_1)d\tau_1 + \int_0^\infty\int h_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2 \quad \ldots(48)$$

Defining the mean squared error as $E\{(z(t)-y(t))^2\}$ where z(t) is the measured output and applying calculus of variations yields

$$E\{z(t)\} = \int_0^\infty h_1(\tau_1)E\{u(t-\tau_1)\}d\tau_1 + \int_0^\infty\int h_2(\tau_1,\tau_2)E\{u(t-\tau_1)u(t-\tau_2)\}\, d\tau_1,d\tau_2 \quad \ldots(49)$$

$$E\{z(t)u(t-\sigma)\} = \int_0^\infty h_1(\tau_1)E\{u(t-\tau_1)u(t-\sigma)\}d\tau_1 + \int_0^\infty\int h_2(\tau_1,\tau_2)E\{u(t-\tau_1)u(t-\tau_2)u(t-\sigma)\}d\tau_1 d\tau_2 \quad \ldots(50)$$

$$E\{z(t)u(t-\sigma_1)u(t-\sigma_2)\} = \int_0^\infty h_1(\tau_1)E\{u(t-\tau_1)u(t-\sigma_1)u(t-\sigma_2)\}d\tau_1$$

$$+ \iint_0^\infty h_2(\tau_1,\tau_2)E\{u(t-\tau_1)u(t-\tau_2)u(t-\sigma_1)u(t-\sigma_2)\}d\tau_1 d\tau_2 \quad \ldots(51)$$

The solution of this set of equations for a general stochastic input is extremely difficult. However, if the system input is white Gaussian then eqn's (49)-(51) reduce to

$$\overline{z(t)} = \int_0^\infty h_2(\tau,\tau)d\tau$$

$$R_{uz}(\sigma_1) = h_1(\sigma_1) \quad \ldots(52)$$

$$R_{uuz}(\sigma_1,\sigma_2) = \bar{z}\delta(\sigma_1-\sigma_2) + 2h_2(\sigma_1,\sigma_2)$$

and the solution for $h_1(t)$ and $h_2(t_1,t_2)$ is direct providing the mean level $\bar{z}$ is removed. Identification of systems which contain higher than second order kernels is very difficult using this approach. Alternative schemes involve approximating the kernels by an expansion of orthogonal functions and estimating the coefficients [22].

Wiener used a Gram-Schmidt orthogonalisation procedure to construct a new functional series where the functionals $\{G_n\}$ are orthogonal for a Gaussian white stimulus. The first two terms in the Wiener series are

$$G_1[k_1,u(t)] = \int_{-\infty}^\infty k_1(\tau)u(t-\tau)dt \quad \ldots(53)$$

$$G_2[k_2,u(t)] = \iint_{-\infty}^\infty k_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$$

$$- P\int_{-\infty}^\infty k_2(\tau_1,\tau_2)d\tau_1 \quad \ldots(54)$$

where P is the power spectral density of the white noise input. In general the Wiener kernels are not equal to the Volterra kernels. Numerous methods have been developed to identify the kernels in Wiener's series [22] the most popular being a correlation method by Lee and Schetzen [4]. The procedure consists of computing multidimensional correlation functions between the white Gaussian input and the system output to yield

$$k_n(\tau_1,\ldots\tau_n) = \frac{1}{n!P}\overline{\{y(t) - \sum_{m=o}^{n-1} G_m[k_m,u(t)]\}u(t-\tau_1)}$$

$$\overline{\ldots u(t-\tau_n)} \quad \forall\ \tau_1,\tau_2..\tau_n \qquad \ldots(55)$$

In an attempt to reduce the computational burden associated with the functional-series methods various authors have considered the identification of block oriented systems [22] which can be represented by interconnections of linear dynamic systems and static nonlinear elements. Consider the system illustrated in Fig.5 to illustrate the approach.

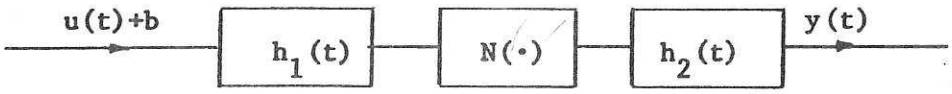

Fig.5

By extending the theory of separable processes [22-24] it can readily be shown that for a Gaussian white input with mean level b

$$R_{uy''}(\sigma) = C_{FG}\int h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 \qquad \ldots(56)$$

$$R_{u^2y''}(\sigma) = C_{FFG}\int h_1^2(\sigma-\tau_1)h_2(\tau_1)d\tau_1 \qquad \ldots(57)$$

where providing $h_1(t)$ is stable bounded-inputs bounded outputs C_{FG} and C_{FFG} are constants and the superscript " indicates a zero mean process. Estimates of the individual linear subsystems $h_1(t)$ and $h_2(t)$ can be obtained by decomposing eqn's (56) and (57) [24] and the nonlinearity can then be determined. Similar results, which provide estimates of the individual component subsystems, are available for feedback, feedforward and multiplicative block oriented systems [24].

Parameter estimation methods for nonlinear systems where the structural form of the describing differential equations are known are fairly well established [22]. When little a priori information is available and the process is treated as a black-box, the usual approach is to expand the input/output using a suitable model representation, which is usually

selected to be nonlinear in the input and output variables but linear in the parameters. For example, by considering an observability condition it can be shown that [25] the model

$$y(m+1) = q\{y(m-1),\ldots y(m-N),u(m-k),\ldots u(m-k-N)\} \qquad \ldots(58)$$

is a general representation for a wide class of nonlinear systems when $q\{\cdot\}$ is a suitable nonlinear operator. Parameter estimation algorithms developed for linear systems cannot in general be applied to estimate the parameters in the model of eqn (58) because even additive measurement noise introduces multiplicative and bilinear noise terms which produce bias in the estimates. Algorithms which overcome this difficulty are however available [25].

6. Conclusions

The statistical analysis of nonlinear systems is in general a difficult task. Whilst some of the techniques currently available have been briefly described above details of other alternative approaches are readily available in the literature.

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