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**PAM BOX**

PROJECTED MARKOV PARAMETERS AND  
MULTIVARIABLE ROOT-LOCI

BY

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ABSTRACT

An inductive proof of the relationship between the asymptotic behaviour of multivariable root-loci and the projected Markov parameters introduced by Kouvaritakis and Edmunds is provided.

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Work on the generalization of the root-loci methodology to multivariable systems is now well advanced with contributions in the areas of algebraic function theory [1]-[2], numerical methods [3]-[5] and compensation theory [5]-[7]. A brief review of the state of the art to 1980 can be found in [8] and to 1978 in [9].

The purpose of this note is to provide a concise inductive proof of the basic results provided by Kouvaritakis and Edmunds [5] in a generalized form that relates to the techniques of dynamic transformation published by Owens [3].

Consider the m-input/m-output, linear, time-invariant system  $S(A,B,C)$  in  $R^n$  subjected to unity negative feedback with scalar gain  $p$  and, in particular, consider the unbounded solutions of the resulting return-difference relation

$$|I_m + p Q(s)| = 0 \quad (1)$$

where  $Q(s) = C(sI_n - A)^{-1}B$  is the system transfer function matrix. Note that

$$Q(s) = s^{-1}Q_1 + s^{-2}Q_2 + \dots \quad (2)$$

for 'large enough' values of  $s$  where  $\{Q_j\}$  are the system Markov parameter matrices and

$$Q_j = C A^{j-1} B, \quad j \geq 1 \quad (3)$$

Note also [3], [4], [6] that  $S(A,B,C)$  is said to have uniform rank  $k$  if

$$Q_j = 0 \quad (j < k), \quad |Q_k| \neq 0 \quad (4)$$

or, equivalently, if  $\lim_{s \rightarrow \infty} s^k Q(s)$  is finite and non-zero

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Proposition 1: If  $Q^{(1)}(s) \stackrel{\Delta}{=} Q(s) \neq 0$ , there exists an integer  $k_1 \geq 1$  such that the limit  $\lim_{|s| \rightarrow \infty} s^{k_1} Q^{(1)}(s)$  exists and is non-zero (i.e.  $k_1$  is the index of the first non-zero Markov parameter of  $Q^{(1)}$ ). Moreover,  $Q^{(1)}$  has no infinite zeros of order  $< k_1$ .

Proof: If  $ps^{-k_1} \rightarrow 0$  as  $p \rightarrow \infty$  then  $pQ^{(1)}(s) \rightarrow 0$  and equation (1) cannot be satisfied.

The next step in the analysis is to embed the unbounded solutions of (1) in the unbounded solutions of the sequence of extended relations

$$|I_{\eta_j} + p Q^{(j)}(s) + \Omega_j(s,p)| = 0 \quad (5)$$

where  $\eta_j = m - \sum_{i=1}^{j-1} d_i > 0$ ,  $Q^{(j)}$  is strictly proper and  $\Omega_j(s,p)$  is a  $m \times m$  matrix function of  $s$  and  $p$ . We clearly retrieve (1) by setting  $j=1$  and  $\Omega_1(s,p) \equiv 0$

The basic propositions are as follows:

Proposition 2: If  $Q^{(j)}(s) \neq 0$ , there exists an integer  $k_j \geq 1$  such that the limit

$$\lim_{|s| \rightarrow +\infty} s^{k_j} Q^{(j)}(s) \stackrel{\Delta}{=} Q_{k_j}^{(j)} \quad (6)$$

exists and is non-zero. Assume that

$$(a) \lim_{p \rightarrow \infty} s \Omega_j(s,p) = 0 \quad (7)$$

on all unbounded branches of the root-locus satisfying  $\lim_{p \rightarrow \infty} p^{-1} s^{k_j-1} = 0$ , and

(b)  $Q_{k_j}^{(j)}$  has 'simple null structure' in the sense [5] that we can construct a nonsingular eigenvector matrix  $W_j$  of  $Q_{k_j}^{(j)}$  such that

$$W_j^{-1} Q_{k_j}^{(j)} W_j = \begin{pmatrix} \Lambda_j & 0 \\ 0 & 0 \end{pmatrix} \quad (8)$$

where the matrix  $\Lambda_j$  is nonsingular of dimension  $d_j \times d_j$  and  $d_j = \text{Rank } Q_{k_j}^{(j)}$ .

Then equation (5) has  $k_j d_j$   $k_j^{\text{th}}$  order infinite zeros whose asymptotic directions and pivots are identical to those of the uniform rank  $k_j$  system

$$G_j(s) \triangleq V_j Q^{(j)}(s) U_j \quad (9)$$

where  $V_j, U_j$  are  $d_j \times \eta_j$  and  $\eta_j \times d_j$  matrices respectively obtained by partitioning of  $W_j^{-1}$  and  $W_j$  as follows

$$W = \begin{bmatrix} U_j & M_j \end{bmatrix}, \quad W_j^{-1} = \begin{bmatrix} V_j \\ - & - \\ N_j \end{bmatrix} \quad (10)$$

(Note: (i) The assumptions on  $\Omega_j$  are trivially verified if  $\Omega_j = 0$ .

Under the stated conditions, however, note that the number, asymptotic directions and pivots of the  $k_j^{\text{th}}$  order infinite zeros are independent of  $\Omega_j$ .

(ii) The matrices  $M_j$  and  $N_j$  are clearly related to the annihilators described in [5]. In fact they are identical if  $j=1$  and  $k_j=1$ ).

Proof of Proposition 2: The proof is a straightforward application of previous techniques and is outlined below.

We consider only unbounded branches of the root-locus satisfying  $\lim_{p \rightarrow \infty} p^{-1} s^{k_j-1} = 0$  and note that  $p^{-1} s^{k_j}$  is bounded on all other branches. To prove this, write (5) in the form

$$\left| I_{\eta_j} + \frac{p}{s^{k_j}} \{s^{k_j} Q^{(j)}(s)\} + \Omega_j(s,p) \right| = 0 \quad (11)$$

and note from (6) and (7) that the left-hand-side approaches 1 if  $p^{-1} s^{k_j}$  is unbounded. This is clearly impossible! Next write (5) in the form

$$\begin{aligned} 0 &= \left| I_{\eta_j} + W_j^{-1} \{p Q^{(j)}(s) + \Omega_j(s,p)\} W_j \right| \\ &\equiv \left| \begin{array}{c} I_{d_j} + p G_j(s) + V_j \Omega_j(s,p) U_j, \quad p V_j Q^{(j)}(s) M_j + V_j \Omega_j(s,p) M_j \\ p N_j Q^{(j)}(s) U_j + N_j \Omega_j(s,p) U_j, \quad I_{\eta_j - d_j} + p N_j Q^{(j)}(s) M_j + N_j \Omega_j(s,p) M_j \end{array} \right| \\ &\equiv \left| I_{\eta_j - d_j} + p N_j Q^{(j)}(s) M_j + N_j \Omega_j(s,p) M_j \right| \times \\ &\quad \left| I_{d_j} + p G_j(s) + p \psi_1^{(j)}(s,p) \right| \end{aligned} \quad (12)$$

where, from Schurs' formula,

$$\begin{aligned} \psi_1^{(j)}(s,p) &\triangleq -p^{-1}(pV_j Q^{(j)}(s)M_j + V_j \Omega_j(s,p)M_j)(I_{\eta_j} - d_j) \\ &+ p N_j Q^{(j)}(s)M_j + N_j \Omega_j(s,p)M_j^{-1}(p N_j Q^{(j)}(s) U_j \\ &+ N_j \Omega_j(s,p)U_j) + p^{-1}V_j \Omega_j(s,p)U_j \end{aligned} \quad (13)$$

Note that  $G_j(s)$  is clearly of uniform rank  $k_j$  as  $\lim_{s \rightarrow \infty} s^{k_j} G_j(s) = V_j Q^{(j)} U_j = \Lambda_j$  which is nonsingular by assumption. Note also that

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{k_j} V_j Q^{(j)}(s) M_j &= \lim_{s \rightarrow \infty} s^{k_j} N_j Q^{(j)}(s) M_j \\ &= \lim_{s \rightarrow \infty} s^{k_j} N_j Q^{(j)}(s) U_j = 0 \end{aligned} \quad (14)$$

We now focus attention on those branches of the root-locus where  $p^{-1} s^{k_j-1} \rightarrow 0$  and  $p^{-1} s^{k_j}$  has only non-zero cluster points as  $p \rightarrow +\infty$  (i.e. only  $k_j^{\text{th}}$  order infinite zeros). On such branches it follows from (7) and (14) that

$$\lim_{p \rightarrow \infty} s^{k_j+1} \psi_1^{(j)}(s,p) = 0 \quad (15)$$

and hence that (12) is dominated by  $G_j(s)$  and, in particular, the  $k_j^{\text{th}}$  and  $(k_j+1)^{\text{th}}$  Markov parameter matrices of  $G_j(s)$ . The result now follows using identical reasoning used in the case of uniform rank systems in references [3] and [6].

Proposition 3: With the notation of Proposition 2, suppose that  $d_j < \eta_j$  and consider only those branches of the root-locus where  $\lim_{p \rightarrow \infty} p^{-1} s^{k_j} = 0$ . On such branches, equation (5) can be replaced by

$$\left| I_{\eta_{j+1}} + p Q^{(j+1)}(s) + \Omega_{j+1}(s,p) \right| = 0 \quad (16)$$

where

$$Q^{(j+1)}(s) = N_j (I_{\eta_j} + P_j(s) s^{k_j} G_j^+)^{-1} P_j(s) M_j \quad (17)$$

$$\lim_{|s| \rightarrow +\infty} s^{k_j} Q^{(j+1)}(s) = 0 \quad (18)$$

$$G_j^+ = U_j \Lambda_j^{-1} V_j \quad (19)$$

and

$$P_j(s) = Q^{(j)}(s) - s^{-k_j} Q_{k_j}^{(j)} = \sum_{i=k_j+1}^{\infty} s^{-i} Q_i^{(j)} \quad (20)$$

If  $Q^{(j+1)}(s) \not\equiv 0$ , there exists an integer  $k_{j+1} > k_j$  such that the limit

$$\lim_{|s| \rightarrow \infty} s^{k_{j+1}} Q^{(j+1)}(s) \triangleq Q_{k_{j+1}}^{(j+1)} \quad (21)$$

exists and is non-zero (i.e.  $k_{j+1}$  is the index of the first non-zero Markov parameter of  $Q^{(j+1)}$ ). Moreover, equation (5) has no infinite zero of order  $\nu$  in the range  $k_j < \nu \leq k_{j+1} - 1$  and

$$\lim_{p \rightarrow \infty} s \Omega_{j+1}(s, p) = 0 \quad (22)$$

on all unbounded branches of the root-locus satisfying  $\lim_{p \rightarrow \infty} p^{-1} s^{k_{j+1}-1} = 0$ .

Proof of Proposition 3: Using Schur's formula, definition (20) and the identities (14), write (12) in the form

$$\begin{aligned} 0 &= |I_{n_j} + W_j^{-1} \{p Q^{(j)}(s) + \Omega_j(s, p)\} W_j| \\ &= |I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j| \cdot |I_{n_j} + \begin{pmatrix} (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} & 0 \\ 0 & I_{n_{j+1}} \end{pmatrix} W_j^{-1} \{ \\ &\quad p P_j(s) + \Omega_j(s, p)\} W_j| \\ &= |I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j| \cdot |I_{d_j} + (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} \{p V_j P_j(s) U_j \\ &\quad + V_j \Omega_j(s, p) U_j\}| \cdot |I_{n_{j+1}} + N_j \Gamma_j(s, p) M_j| \quad (23) \end{aligned}$$

where

$$\begin{aligned} \Gamma_j(s, p) &= \{p P_j(s) + \Omega_j(s, p)\} - \{p P_j(s) + \Omega_j(s, p)\} U_j (I_{d_j} + \\ &\quad (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j \{p P_j(s) + \Omega_j(s, p)\} U_j)^{-1} (I_{d_j} + \\ &\quad \frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j \{p P_j(s) + \Omega_j(s, p)\} \quad (24) \end{aligned}$$

The nonsingularity of  $\Lambda_j$  ensures that the first determinant in (23) is never zero on unbounded branches of the root-locus such that  $p^{-1} s^{k_j} \rightarrow 0$  as  $p \rightarrow \infty$ .

A similar result also holds for the second determinant as

$$\begin{aligned} & \left( I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j \right)^{-1} \{ p V_j P_j(s) U_j + V_j \Omega_j(s,p) U_j \} \\ & \equiv \left( \Lambda_j + \frac{s^{k_j}}{p} \right)^{-1} \{ V_j s^{k_j} P_j(s) U_j + V_j \frac{s^{k_j}}{p} \Omega_j(s,p) U_j \} \\ & \rightarrow 0 \quad (\text{as } p \rightarrow \infty) \end{aligned} \tag{25}$$

due to the definition of  $P_j$ , property (7) of  $\Omega_j$  and the fact that we are focussing attention only on those unbounded parts of the root-locus where  $p^{-1} s^{k_j}$  (and hence  $p^{-1} s^{k_j - 1}$ ) tend to zero as  $p \rightarrow \infty$ . Equation (23) therefore reduces to

$$0 = \left| I_{n_j+1} + N_j \Gamma_j(s,p) M_j \right| \tag{26}$$

Our proof of (16) proceeds by analysis of  $\Gamma_j(s,p)$ . First, we use the matrix identity

$$(I + XY)^{-1} X = X(I + YX)^{-1} \tag{27}$$

to write  $\Gamma_j$  in the form

$$\begin{aligned} \Gamma_j(s,p) &= \{ p P_j + \Omega_j \} - \{ p P_j + \Omega_j \} U_j \left( I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j \right)^{-1} \\ & \quad \cdot V_j \left( I_{n_j} + \{ p P_j + \Omega_j \} U_j \left( I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j \right)^{-1} V_j \right)^{-1} \\ & \quad \{ p P_j + \Omega_j \} \end{aligned} \tag{28}$$

Next note that

$$X - XY(I + XY)^{-1} X = (I + XY)^{-1} X \tag{29}$$

and hence that we can write

$$\begin{aligned} \Gamma_j(s,p) &= \left( I_{n_j} + \{ p P_j(s) + \Omega_j(s,p) \} U_j \left( I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j \right)^{-1} V_j \right)^{-1} \\ & \quad \{ p P_j(s) + \Omega_j(s,p) \} \end{aligned} \tag{30}$$

The identity

$$\begin{aligned} & (I + XY)^{-1} - (I + XZ)^{-1} \\ &= (I + XY)^{-1} X(Z - Y) (I + XZ)^{-1} \end{aligned} \quad (31)$$

then leads to

$$\begin{aligned} \Gamma_j(s,p) &= (I_{n_j} + \{p P_j(s) + \Omega_j(s,p)\} \frac{s^k_j}{p} U_j \Lambda_j^{-1} V_j)^{-1} \\ & \{p P_j(s) + \Omega_j(s,p)\} + \psi_2^{(j)}(s,p) \end{aligned} \quad (32)$$

where

$$\begin{aligned} \psi_2^{(j)}(s,p) &= (I_{n_j} + \{p P_j(s) + \Omega_j(s,p)\} U_j (I_{d_j} + \frac{p}{s^k_j} \Lambda_j)^{-1} \{p P_j(s) + \\ & \Omega_j(s,p)\} (\frac{s^k_j}{p} U_j \Lambda_j^{-1} V_j - \\ & U_j (I_{d_j} + \frac{p}{s^k_j} \Lambda_j)^{-1} V_j) (I_{n_j} + \{p P_j(s) + \\ & \Omega_j(s,p)\} \frac{s^k_j}{p} U_j \Lambda_j^{-1} V_j)^{-1} \{p P_j(s) + \Omega_j(s,p)\} \end{aligned} \quad (33)$$

Finally, we use the matrix identity

$$\begin{aligned} & (I + XY)^{-1} X - (I + ZY)^{-1} Z \\ &= (I + XY)^{-1} X - Z(I + YZ)^{-1} \quad (\text{using (27)}) \\ &= (I + XY)^{-1} (X-Z) (I + YZ)^{-1} \end{aligned} \quad (34)$$

to write

$$\begin{aligned} \Gamma_j(s,p) &= p(I_{n_j} + P_j(s) \frac{s^k_j}{p} U_j \Lambda_j^{-1} V_j)^{-1} P_j(s) \\ & + \psi_2^{(j)}(s,p) + \psi_3^{(j)}(s,p) \end{aligned} \quad (35)$$

where

$$\begin{aligned} \psi_3^{(j)}(s,p) &= (I_{n_j} + \{p P_j(s) + \Omega_j(s,p)\} \frac{s^k_j}{p} U_j \Lambda_j^{-1} V_j)^{-1} \\ & x \Omega_j(s,p) (I_{n_j} + \frac{s^k_j}{p} U_j \Lambda_j^{-1} V_j p P_j(s))^{-1} \end{aligned} \quad (36)$$

Using these relations and (17) and (19) we see that (26) becomes

$$0 = |I_{n_{j+1}} + p Q^{(j+1)}(s) + N_j \psi_2^{(j)}(s,p) M_j + N_j \psi_3^{(j)}(s,p) M_j| \quad (37)$$

which is just (16) if we set

$$\Omega_{j+1}(s,p) = N_j(\psi_2^{(j)}(s,p) + \psi_3^{(j)}(s,p))M_j \quad (38)$$

Equation (18) follows trivially from (20) and, provided  $Q^{(j+1)} \neq 0$ , the existence of  $k_{j+1} > k_j$  is generated.

Let  $v$  be arbitrary in the range  $k_j \leq v < k_{j+1} - 1$  ( $v$  need not be integer [10]!) and consider these unbounded branches of the root-locus such that  $p^{-1} s^v \rightarrow 0$  as  $p \rightarrow \infty$ . Note that  $p^{-1} s^\mu \rightarrow 0$  as  $p \rightarrow \infty$  for all  $\mu \leq v$ .

We now prove that (22) holds on all unbounded branches of the root locus satisfying  $p^{-1} s^v \rightarrow 0$  as  $p \rightarrow \infty$ , by proving the same result for both  $\psi_2^{(j)}$  and  $\psi_3^{(j)}$ . After a little manipulation, we see that

$$\begin{aligned} \lim_{p \rightarrow \infty} s \psi_2^{(j)}(s,p) &= \lim_{p \rightarrow \infty} s \{p P_j(s) + \Omega_j(s,p)\} U_j \left(\frac{s^j}{p} \Lambda_j^{-1} - \right. \\ &\quad \left. (I_{d_j} + \frac{p}{k_j} \Lambda_j)^{-1}\right) V_j \{p P_j(s) + \Omega_j(s,p)\} \\ &= \lim_{p \rightarrow \infty} s \{p P_j(s) + \Omega_j(s,p)\} U_j \frac{s^j}{p} \Lambda_j^{-1} \left(\frac{s^j}{p} I_{d_j} + \Lambda_j\right)^{-1} \\ &\quad \frac{s^j}{p} \{p P_j(s) + \Omega_j(s,p)\} \\ &= 0 \end{aligned} \quad (39)$$

if  $p^{-1} s^v \rightarrow 0$  as it follows that  $p^{-1} s^{k_j} \rightarrow 0$ , and hence that

$$\begin{aligned} \lim_{p \rightarrow \infty} s^\delta \{p P_j(s) + \Omega_j(s,p)\} \frac{s^j}{p} \\ &= \lim_{p \rightarrow \infty} \left\{ s^\delta (s^{k_j} P_j(s)) + \frac{s^j}{p} (s \Omega_j(s,p)) \right\} \\ &= \begin{cases} 0 & , & \delta = 0 \\ \lim_{s \rightarrow \infty} s^{k_j+1} P_j(s) & , & \delta = 1 \end{cases} \end{aligned} \quad (40)$$

(the limit existing from the definition of  $P_j(s)$ ). Also, using (36)

$$\lim_{p \rightarrow \infty} s \psi_3^{(j)}(s,p) = \lim_{p \rightarrow \infty} s \Omega_j(s,p) = 0 \quad (41)$$

where  $p^{-1} s^\nu$  (and hence  $p^{-1} s^{k_j-1}$ )  $\rightarrow 0$  as  $p \rightarrow \infty$ . Clearly (39) and (41)

indicate that (22) holds on all unbounded **branches** of the root-locus where  $p^{-1} s^\nu \rightarrow 0$  as  $p \rightarrow \infty$  and  $\nu$  is any real number in the range  $k_j \leq \nu \leq k_{j+1} - 1$ , and, in particular, for  $\nu = k_{j+1} - 1$ .

Finally, we prove that (5) has no infinite zeros in the range  $k_j < \nu' \leq k_{j+1} - 1$  using the following arguments. Firstly, if  $k_{j+1} = k_j + 1$  we see that the result is trivial as the range is empty. Suppose therefore that  $k_{j+1} - 1 > k_j$  and consider the choice of  $\nu = k_j$ . The above analysis indicates that, if  $p^{-1} s^{k_j} \rightarrow 0$  as  $p \rightarrow +\infty$ , then  $\Omega_{j+1}(s,p) \rightarrow 0$ . Also, if  $p s^{-k_{j+1}} \rightarrow 0$ , then it is easily verified that  $p Q^{(j+1)}(s) \rightarrow 0$  and hence that  $|I + p Q^{(j+1)} + \Omega_{j+1}| \rightarrow 1$  as  $p \rightarrow \infty$ . This clearly contradicts (16) so we must conclude that  $p s^{-k_{j+1}}$  is either finite and non-zero or unbounded on all unbounded branches of the root-locus where  $p^{-1} s^{k_j} \rightarrow 0$ . In other terms, there are no infinite zeros between  $k_j$  and  $k_{j+1} - 1$ . This completes the proof of the result.

We are now in a position to describe a computational method paralleling that due to Kouvaritakis and Edmunds [5] by noting that the results of Proposition 3 ensure that the assumptions of Proposition 2 (with the exception of (b)) are satisfied. They can hence be applied recursively with a starting condition provided by Proposition 1. More precisely:

Step 1: Set  $j=1$ ,  $\Omega_1(s,p) \equiv 0$  and  $Q^{(j)}(s) = Q(s)$

Step 2: If  $Q^{(j)}(s) \neq 0$ , calculate  $k_j$  and, by spectral decomposition of

its leading non-zero Markov Parameter, construct the  $d_j \times d_j$

uniform rank  $k_j$  system  $G_j(s) = V_j Q^{(j)}(s) U_j$ . (Note: condition

(b) of Proposition 2 must be valid for this to be achieved. This

is clearly generically the case!)

Step 3: Use known techniques (e.g. [3] - [6]) to compute the asymptotic directions and pivots of the  $k_j d_j k_j^{\text{th}}$  order infinite zeros of  $G_j$  (Note: Proposition 2 and induction ensures that these asymptotes are also asymptotes of  $Q = Q^{(1)}$ ).

Step 4: If  $\eta_{j+1} = m$ , step. Otherwise construct  $Q^{(j+1)}(s)$ , replace  $j+1$  by  $j$  and return to step 2.

As in [5] the complex transfer function matrix manipulations involved in the construction of  $Q^{(j+1)}$  from  $Q^{(j)}$  can be replaced by elementary manipulations of the systems state-space model  $S(A, B, C)$ .

Proposition 4:  $Q^{(j)}(s)$  has a realization  $S(A_j, B_j, C_j)$  in  $R^n$  of the form

$$S_j \stackrel{\Delta}{=} S(A_j, B_j, C_j) = \begin{cases} S(A, B, C) & \text{if } j = 1 \\ S(T_{j-1} A_{j-1}, B_{j-1} M_{j-1}, N_{j-1} C_{j-1}) & \text{if } j > 1 \end{cases} \quad (42)$$

where  $T_j = I_n - B_j G_j^+ C_j A_j^{j-1}$ ,  $j \geq 1$  (43)

and

$$C_j A_j^{i-1} B_j = 0 \quad (i < k_j) \quad , \quad C_j A_j^{j-1} B_j \neq 0 \quad (44)$$

Proof: If  $j=1$ , the result is trivial as  $Q^{(1)} \stackrel{\Delta}{=} Q$ . Consider therefore the case of  $j > 1$  and suppose, using induction, that  $S_{j-1}$  has the required form and properties. Note also the identity,

$$\begin{aligned} (s I_n - F)^{-1} &= \sum_{i=1}^{\ell} s^{-i} F^{i-1} + F^{\ell} s^{-\ell} (s I_n - F)^{-1} \\ &= \sum_{i=1}^{\ell} s^{-i} F^{i-1} + s^{-j} (s I_n - F)^{-1} F^{\ell} \end{aligned} \quad (45)$$

valid for all  $\ell \geq 0$  and for all  $n \times n$  matrices  $F$ . The transfer function matrix of  $S_j$  is just

$$\begin{aligned}
 & N_{j-1} C_{j-1} (sI_n - T_{j-1} A_{j-1})^{-1} B_{j-1} M_{j-1} \\
 &= \sum_{i=1}^{k_{j-1}} s^{-i} N_{j-1} C_{j-1} (T_{j-1} A_{j-1})^{i-1} B_{j-1} M_{j-1} \\
 &+ s^{-k_{j-1}} N_{j-1} C_{j-1} (T_{j-1} A_{j-1})^{k_{j-1}-1} (sI_n - T_{j-1} A_{j-1})^{-1} B_{j-1} M_{j-1}
 \end{aligned} \tag{46}$$

using (45) with  $F = T_{j-1} A_{j-1}$  and  $\ell = k_{j-1}$ . Using (44) we see that, for  $i \leq k_{j-1} + 1$ ,

$$\begin{aligned}
 & C_{j-1} (T_{j-1} A_{j-1})^{i-1} \\
 &= C_{j-1} (I_n - B_{j-1} G_{j-1}^+ C_{j-1} A_{j-1}^{k_{j-1}-1}) A_{j-1} (T_{j-1} A_{j-1})^{i-2} \\
 &= C_{j-1} A_{j-1} (T_{j-1} A_{j-1})^{i-2} \\
 &\vdots \\
 &= \begin{cases} C_{j-1} A_{j-1}^{i-1} & , i \leq k_{j-1} \\ C_{j-1} A_{j-1}^{i-2} (I_n - B_{j-1} G_{j-1}^+ C_{j-1} A_{j-1}^{k_{j-1}-1}) A_{j-1} & , i = k_{j-1} + 1 \end{cases}
 \end{aligned} \tag{47}$$

and hence, using the properties of the eigenvector matrix  $W_j$ ,

$$\begin{aligned}
 & N_{j-1} C_{j-1} (T_{j-1} A_{j-1})^{i-1} B_{j-1} M_{j-1} = 0 \quad , \quad i \leq k_{j-1} \\
 & N_{j-1} C_{j-1} (T_{j-1} A_{j-1})^{k_{j-1}-1} = N_{j-1} C_{j-1} A_{j-1}^{k_{j-1}-1}
 \end{aligned} \tag{48}$$

The transfer function matrix in (46) now reduces to

$$\begin{aligned}
 & s^{-k_{j-1}} N_{j-1} C_{j-1} A_{j-1}^{k_{j-1}-1} (sI_n - A_{j-1} + B_{j-1} G_{j-1}^+ C_{j-1} A_{j-1}^{k_{j-1}-1})^{-1} B_{j-1} M_{j-1} \\
 &\equiv s^{-k_{j-1}} N_{j-1} C_{j-1} A_{j-1}^{k_{j-1}-1} (sI_n - A_{j-1})^{-1} (I_n + \\
 & B_{j-1} G_{j-1}^+ C_{j-1} A_{j-1}^{k_{j-1}-1} (sI_n - A_{j-1})^{-1})^{-1} B_{j-1} M_{j-1}
 \end{aligned}$$

$$\begin{aligned} &\equiv s^{-k} N_{j-1}^{j-1} C_{j-1}^{j-1} A_{j-1}^{j-1} (sI_n - A_{j-1})^{-1} B_{j-1} (I_{n_{j-1}} + \\ &G_{j-1}^+ C_{j-1}^{j-1} A_{j-1}^{j-1} (sI_n - A_{j-1})^{-1} B_{j-1})^{-1} M_{j-1} \end{aligned} \quad (49)$$

after use of the matrix identity (27). But this is just  $Q^{(j)}(s)$  (as required) as

$$\begin{aligned} C_{j-1}^{j-1} A_{j-1}^{j-1} (sI_n - A_{j-1})^{-1} B_{j-1} &\equiv \sum_{i=1}^{\infty} s^{-i} C_{j-1}^{j-1} A_{j-1}^{j-1+i-1} B_{j-1} \\ &\equiv s^{k_{j-1}} \sum_{i=k_{j-1}+1}^{\infty} s^{-i} C_{j-1}^{j-1} A_{j-1}^{i-1} B_{j-1} \equiv s^{k_{j-1}} P_{j-1}(s) \end{aligned} \quad (50)$$

by (44), and, using (27) again,

$$P_{j-1} (I + G_{j-1}^+ s^{k_{j-1}} P_{j-1})^{-1} \equiv (I + P_{j-1} s^{k_{j-1}} G_{j-1}^+)^{-1} P_{j-1} \quad (51)$$

The result is now completed by noting that if  $S_j$  is a representation of  $Q^{(j)}$ , (44) trivially from the definitions.

(Note: the state space realization of  $Q^{(j)}$  can be used to deduce the state space realization  $S(A_j, B_j U_j, V_j C_j)$  of  $G_j$  and hence its leading Markov parameters that are required to deduce its asymptotic directions and pivots).

In summary the paper has provided a rigorous inductive proof of the results of Kouvaritakis and Edmunds [5] in a form that relates them to the use of uniform rank systems in Owens [3], [4] and [6]. In particular, the nature of the approximation A1 (introduced without justification) in Appendix A of [5] has been identified as being valid for the prediction of the orders, asymptotic directions and pivots of all infinite zeros of the system.

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