PROJECTED MARKOV PARAMETERS AND

MULTIVARIABLE ROOT-LOCI

BY

D.H. Owens B.Sc., A.R.C.S., Ph.D., A.F.I.M.A., C.Eng., M.I.E.E.

Department of Control Engineering,
University of Sheffield,
Mappin Street, Sheffield S1 3JD.

Research Report No. 146

February 1981

ABSTRACT

An inductive proof of the relationship between the asymptotic behaviour of multivariable root-loci and the projected Markov parameters introduced by Kouvaritakis and Edmunds is provided.

Work on the generalization of the root-loci methodology to multivariable systems is now well advanced with contributions in the areas of algebraic function theory [1]-[2], numerical methods [3]-[5] and compensation theory [5]-[7]. A brief review of the state of the art to 1980 can be found in [8] and to 1978 in [9].

The purpose of this note is to provide a concise inductive proof of the basic results provided by Kouvaritakis and Edmunds [5] in a generalized form that relates to the techniques of dynamic transformation published by Owens [3].

Consider the m-input/m-output, linear, time-invariant system $S(A,B,C)$ in R^n subjected to unity negative feedback with scalar gain p and, in particular, consider the unbounded solutions of the resulting return-difference relation

$$|I_m + p\,Q(s)| = 0 \tag{1}$$

where $Q(s) = C(sI_n - A)^{-1}B$ is the system transfer function matrix. Note that

$$Q(s) = s^{-1}Q_1 + s^{-2}Q_2 + \dots \tag{2}$$

for 'large enough' values of s where $\{Q_j\}$ are the system Markov parameter matrices and

$$Q_j = C\,A^{j-1}\,B \quad , \quad j \geq 1 \tag{3}$$

Note also [3], [4], [6] that $S(A,B,C)$ is said to have uniform rank k if

$$Q_j = 0 \quad (j < k) \quad , \quad |Q_k| \neq 0 \tag{4}$$

or, equivalently, if $\lim_{s\to\infty} s^k Q(s)$ is finite and non-zero

---

<u>Proposition 1</u>: If $Q^{(1)}(s) \triangleq Q(s) \not\equiv 0$, there exists an integer $k_1 \geqslant 1$ such that the limit $\lim_{|s|\to\infty} s^{k_1} Q^{(1)}(s)$ exists and is non-zero (i.e. k_1 is the index of the first non-zero Markov parameter of $Q^{(1)}$). Moreover, $Q^{(1)}$ has no infinite zeros of order $< k_1$.

<u>Proof</u>: If $ps^{-k_1} \to 0$ as $p \to \infty$ then $pQ^{(1)}(s) \to 0$ and equation (1) cannot be satisfied.

The next step in the analysis is to embed the unbounded solutions of (1) in the unbounded solutions of the sequence of extended relations

$$|I_{\eta_j} + p\, Q^{(j)}(s) + \Omega_j(s,p)| = 0 \qquad (5)$$

where $\eta_j = m - \sum_{i=1}^{j-1} d_i > 0$, $Q^{(j)}$ is strictly proper and $\Omega_j(s,p)$ is a mxm matrix function of s and p. We clearly retrieve (1) by setting j=1 and $\Omega_1(s,p) \equiv 0$

The basic propositions are as follows:

<u>Proposition 2</u>: If $Q^{(j)}(s) \not\equiv 0$, there exists an integer $k_j \geqslant 1$ such that the limit

$$\lim_{|s|\to+\infty} s^{k_j} Q^{(j)}(s) \triangleq Q^{(j)}_{k_j} \qquad (6)$$

exists and is non-zero. Assume that

(a) $$\lim_{p\to\infty} s\, \Omega_j(s,p) = 0 \qquad (7)$$

on all unbounded branches of the root-locus satisfying $\lim_{p\to\infty} p^{-1} s^{k_j - 1} = 0$, and

(b) $Q^{(j)}_{k_j}$ has 'simple *null* structure' in the sense [5] that we can construct a nonsingular eigenvector matrix W_j of $Q^{(j)}_{k_j}$ such that

$$W_j^{-1} Q^{(j)}_{k_j} W_j = \begin{pmatrix} \Lambda_j & 0 \\ 0 & 0 \end{pmatrix} \qquad (8)$$

where the matrix Λ_j is nonsingular of dimension $d_j x d_j$ and d_j = Rank $Q^{(j)}_{k_j}$.

Then equation (5) has $k_j\ d_j\ k_j^{th}$ order infinite zeros whose asymptotic directions and pivots are identical to those of the uniform rank k_j system

$$G_j(s) \triangleq V_j\, Q^{(j)}(s)\, U_j \tag{9}$$

where V_j, U_j are $d_j x \eta_j$ and $\eta_j x d_j$ matrices respectively obtained by partitioning of W_j^{-1} and W_j as follows

$$W = [U_j \quad M_j] \quad , \quad W_j^{-1} = \begin{pmatrix} V_j \\ - - - \\ N_j \end{pmatrix} \tag{10}$$

(Note: (i) The assumptions on Ω_j are trivially verified if $\Omega_j = 0$. Under the stated conditions, however, note that the number, asymptotic directions and pivots of the k_j^{th} order infinite zeros are independent of Ω_j.

(ii) The matrices M_j and N_j are clearly related to the annihilators described in [5]. In fact they are identical if j=1 and k_j=1).

<u>Proof of Proposition 2</u>: The proof is a straightforward application of previous techniques and is outlined below.

We consider only unbounded branches of the root-locus satisfying $\lim_{p\to\infty} p^{-1}s^{k_j-1} = 0$ and note that $p^{-1}s^{k_j}$ is bounded on all other branches. To prove this, write (5) in the form

$$\left| I_{\eta_j} + \frac{p}{s^{k_j}} \{s^{k_j} Q^{(j)}(s)\} + \Omega_j(s,p) \right| = 0 \tag{11}$$

and note from (6) and (7) that the left-hand-side approaches 1 if $p^{-1}s^{k_j}$ is unbounded. This is clearly impossible! Next write (5) in the form

$$0 = \left| I_{\eta_j} + W_j^{-1}\{pQ^{(j)}(s) + \Omega_j(s,p)\}\, W_j \right|$$

$$\equiv \begin{vmatrix} I_{d_j} + p\, G_j(s) + V_j\, \Omega_j(s,p) U_j, & p\, V_j Q^{(j)}(s) M_j + V_j \Omega_j(s,p) M_j \\ p\, N_j\, Q^{(j)}(s)\, U_j + N_j \Omega_j(s,p) U_j, & I_{\eta_j - d_j} + p\, N_j Q^{(j)}(s) M_j + N_j \Omega_j(s,p) M_j \end{vmatrix}$$

$$\equiv \left| I_{\eta_j - d_j} + p\, N_j\, Q^{(j)}(s)\, M_j + N_j\, \Omega_j(s,p)\, M_j \right| \times$$

$$\left| I_{d_j} + p\, G_j(s) + p\, \psi_1^{(j)}(s,p) \right| \tag{12}$$

where, from Schurs' formula,

$$\psi_1^{(j)}(s,p) \triangleq -p^{-1}(pV_j Q^{(j)}(s)M_j + V_j \Omega_j(s,p)M_j)(I_{\eta_j-d_j}$$
$$+ p N_j Q^{(j)}(s)M_j + N_j \Omega_j(s,p)M_j)^{-1}(p N_j Q^{(j)}(s) U_j$$
$$+ N_j \Omega_j(s,p)U_j) + p^{-1}V_j \Omega_j(s,p)U_j \qquad (13)$$

Note that $G_j(s)$ is clearly of uniform rank k_j as $\lim_{s\to\infty} s^{k_j}G_j(s) = V_j Q_k^{(j)} U_j = \Lambda_j$ which is nonsingular by assumption. Note also that

$$\lim_{s\to\infty} s^{k_j} V_j Q^{(j)}(s) M_j = \lim_{s\to\infty} s^{k_j} N_j Q^{(j)}(s) M_j$$
$$= \lim_{s\to\infty} s^{k_j} N_j Q^{(j)}(s) U_j = 0 \qquad (14)$$

We now focus attention on those branches of the root-locus where $p^{-1}s^{k_j-1} \to 0$ and $p^{-1} s^{k_j}$ has only <u>non-zero</u> cluster points as $p\to+\infty$ (i.e. only k_j^{th} order infinite zeros). On such branches it follows from (7) and (14) that

$$\lim_{p\to\infty} s^{k_j+1} \psi_1^{(j)}(s,p) = 0 \qquad (15)$$

and hence that (12) is dominated by $G_j(s)$ and, in particular, the k_j^{th} and $(k_j+1)^{th}$ Markov parameter matrices of $G_j(s)$. The result now follows using identical reasoning used in the case of uniform rank systems in references [3] and [6].

<u>Proposition 3</u>: With the notation of Proposition 2, suppose that $d_j < \eta_j$ and consider only those branches of the root-locus where $\lim_{p\to\infty} p^{-1}s^{k_j} = 0$. On such branches, equation (5) can be replaced by

$$|I_{\eta_{j+1}} + p\, Q^{(j+1)}(s) + \Omega_{j+1}(s,p)| = 0 \qquad (16)$$

where

$$Q^{(j+1)}(s) = N_j(I_{\eta_j} + P_j(s)s^{k_j}G_j^{+})^{-1}P_j(s)\, M_j \qquad (17)$$

$$\lim_{|s|\to+\infty} s^{k_j} Q^{(j+1)}(s) = 0 \qquad (18)$$

$$G_j^+ = U_j \Lambda_j^{-1} V_j \tag{19}$$

and

$$P_j(s) = Q^{(j)}(s) - s^{-k_j} Q_{k_j}^{(j)} = \sum_{i=k_j+1}^{\infty} s^{-i} Q_i^{(j)} \tag{20}$$

If $Q^{(j+1)}(s) \not\equiv 0$, there exists a integer $k_{j+1} > k_j$ such that the limit

$$\lim_{|s|\to\infty} s^{k_{j+1}} Q^{(j+1)}(s) \triangleq Q_{k_{j+1}}^{(j+1)} \tag{21}$$

exists and is non-zero (i.e. k_{j+1} is the index of the first non-zero Markov parameter of $Q^{(j+1)}$). Moreover, equation (5) has no infinite zero of order ν in the range $k_j < \nu \leqslant k_{j+1} - 1$ and

$$\lim_{p\to\infty} s\, \Omega_{j+1}(s,p) = 0 \tag{22}$$

on all unbounded branches of the root-locus satisfying $\lim_{p\to\infty} p^{-1} s^{k_{j+1}-1} = 0$.

<u>Proof of Proposition 3</u>: Using Schur's formula, definition (20) and the identities (14), write (12) in the form

$$0 = \left| I_{n_j} + W_j^{-1}\{p\, Q^{(j)}(s) + \Omega_j(s,p)\}\, W_j \right|$$

$$= \left| I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j \right| \cdot \left| I_{n_j} + \begin{pmatrix} (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} & 0 \\ 0 & I_{n_{j+1}} \end{pmatrix} W_j^{-1}\{ \right.$$

$$\left. p\, P_j(s) + \Omega_j(s,p)\} W_j \right|$$

$$= \left| I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j \right| \cdot \left| I_{d_j} + (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} \{p\, V_j P_j(s) U_j \right.$$

$$\left. + V_j \Omega_j(s,p) U_j\} \right| \cdot \left| I_{n_{j+1}} + N_j \Gamma_j(s,p) M_j \right| \tag{23}$$

where

$$\Gamma_j(s,p) = \{p\, P_j(s) + \Omega_j(s,p)\} - \{p\, P_j(s) + \Omega_j(s,p)\} U_j\, (I_{d_j} +$$

$$(I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j \{p\, P_j(s) + \Omega_j(s,p)\}\, U_j)^{-1} (I_{d_j} +$$

$$\frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j \quad \{p\, P_j(s) + \Omega_j(s,p)\} \tag{24}$$

The nonsingularity of Λ_j ensures that the first determinant in (23) is never zero on unbounded branches of the root-locus such that $p^{-1}s^{k_j} \to 0$ as $p \to \infty$. A similar result also holds for the second determinant as

$$(I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} \{p V_j P_j(s) U_j + V_j \Omega_j(s,p) U_j\}$$

$$\equiv (\Lambda_j + \frac{s^{k_j}}{p})^{-1} \{V_j s^{k_j} P_j(s) U_j + V_j \frac{s^{k_j}}{p} \Omega_j(s,p) U_j\}$$

$$\to 0 \quad (\text{as } p \to \infty) \tag{25}$$

due to the definition of P_j, property (7) of Ω_j and the fact that we are focussing attention only on those unbounded parts of the root-locus where $p^{-1}s^{k_j}$ (and hence $p^{-1}s^{k_j-1}$) tend to zero as $p \to \infty$. Equation (23) therefore reduces to

$$0 = |I_{n_{j+1}} + N_j \Gamma_j(s,p) M_j| \tag{26}$$

Our proof of (16) proceeds by analysis of $\Gamma_j(s,p)$. First, we use the matrix identity

$$(I + XY)^{-1}X = X(I + YX)^{-1} \tag{27}$$

to write Γ_j in the form

$$\Gamma_j(s,p) = \{p P_j + \Omega_j\} - \{p P_j + \Omega_j\} U_j(I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1}$$

$$V_j(I_{n_j} + \{p P_j + \Omega_j\} U_j(I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j)^{-1}$$

$$\{p P_j + \Omega_j\} \tag{28}$$

Next note that

$$X - XY(I + XY)^{-1} X = (I + XY)^{-1} X \tag{29}$$

and hence that we can write

$$\Gamma_j(s,p) = (I_{n_j} + \{p P_j(s) + \Omega_j(s,p)\} U_j (I_{d_j} +$$

$$\frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j)^{-1} \{p P_j(s) + \Omega_j(s,p)\} \tag{30}$$

The identity

$$(I + XY)^{-1} - (I + XZ)^{-1}$$
$$= (I + XY)^{-1} X(Z - Y) (I + XZ)^{-1} \quad (31)$$

then leads to

$$\Gamma_j(s,p) = (I_{\eta_j} + \{p P_j(s) + \Omega_j(s,p)\} \frac{s^{k_j}}{p} U_j \Lambda_j^{-1} V_j)^{-1}$$

$$\{p P_j(s) + \Omega_j(s,p)\} + \psi_2^{(j)}(s,p) \quad (32)$$

where

$$\psi_2^{(j)}(s,p) = (I_{\eta_j} + \{p P_j(s) + \Omega_j(s,p)\} U_j (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} \{p P_j(s) +$$

$$\Omega_j(s,p)\} (\frac{s^{k_j}}{p} U_j \Lambda_j^{-1} V_j -$$

$$U_j (I_{d_j} + \frac{p}{s^{k_j}} \Lambda_j)^{-1} V_j)(I_{\eta_j} + \{p P_j(s) +$$

$$\Omega_j(s,p)\} \frac{s^{k_j}}{p} U_j \Lambda_j^{-1} V_j)^{-1} \{p P_j(s) + \Omega_j(s,p)\} \quad (33)$$

Finally, we use the matrix identity

$$(I + XY)^{-1} X - (I + ZY)^{-1} Z$$
$$= (I + XY)^{-1} X - Z(I + YZ)^{-1} \qquad \text{(using (27))}$$
$$= (I + XY)^{-1} (X-Z)(I + YZ)^{-1} \quad (34)$$

to write

$$\Gamma_j(s,p) = p(I_{\eta_j} + P_j(s)\, s^{k_j} U_j \Lambda_j^{-1} V_j)^{-1} P_j(s)$$
$$+ \psi_2^{(j)}(s,p) + \psi_3^{(j)}(s,p) \quad (35)$$

where

$$\psi_3^{(j)}(s,p) = (I_{\eta_j} + \{p P_j(s) + \Omega_j(s,p)\} \frac{s^{k_j}}{p} U_j \Lambda_j^{-1} V_j)^{-1}$$
$$x\Omega_j(s,p) (I_{\eta_j} + \frac{s^{k_j}}{p} U_j \Lambda_j^{-1} V_j\, p P_j(s))^{-1} \quad (36)$$

Using these relations and (17) and (19) we see that (26) becomes

$$0 = |I_{\eta_{j+1}} + p\, Q^{(j+1)}(s) + N_j \psi_2^{(j)}(s,p) M_j + N_j \psi_3^{(j)}(s,p) M_j| \quad (37)$$

which is just (16) if we set

$$\Omega_{j+1}(s,p) = N_j(\psi_2^{(j)}(s,p) + \psi_3^{(j)}(s,p))M_j \tag{38}$$

Equation (18) follows trivially from (20) and, provided $Q^{(j+1)} \not\equiv 0$, the existence of $k_{j+1} > k_j$ is generated.

Let ν be arbitrary in the range $k_j \leq \nu < k_{j+1} - 1$ (ν need not be integer [10]!) and consider those unbounded branches of the root-locus such that $p^{-1} s^{\nu} \to 0$ as $p \to \infty$. Note that $p^{-1}s^{\mu} \to 0$ as $p \to \infty$ for all $\mu \leq \nu$.

We now prove that (22) holds on all unbounded branches of the root locus satisfying $p^{-1} s^{\nu} \to 0$ as $p \to \infty$, by proving the same result for both $\psi_2^{(j)}$ and $\psi_3^{(j)}$. After a little manipulation, we see that

$$\lim_{p\to\infty} s\,\psi_2^{(j)}(s,p) = \lim_{p\to\infty} s\,\{p\,P_j(s) + \Omega_j(s,p)\}\; U_j\Big(\frac{s^{k_j}}{p}\Lambda_j^{-1} - (I_{d_j} + \frac{p}{s^{k_j}}\Lambda_j)^{-1}\Big)V_j\,\{p\,P_j(s) + \Omega_j(s,p)\}$$

$$= \lim_{p\to\infty} s\,\{p\,P_j(s) + \Omega_j(s,p)\}\; U_j\,\frac{s^{k_j}}{p}\;\Lambda_j^{-1}\Big(\frac{s^{k_j}}{p}I_{d_j} + \Lambda_j\Big)^{-1}\frac{s^{k_j}}{p}\;\{p\,P_j(s) + \Omega_j(s,p)\}$$

$$= 0 \tag{39}$$

if $p^{-1}s^{\nu} \to 0$ as it follows that $p^{-1}s^{k_j} \to 0$, and hence that

$$\lim_{p\to\infty} s^{\delta}\{p\,P_j(s) + \Omega_j(s,p)\}\;\frac{s^{k_j}}{p}$$

$$= \lim_{p\to\infty}\{\,s^{\delta}\,(s^{k_j}P_j(s)) + \frac{s^{k_j}}{p}\,(s\,\Omega_j(s,p))\}$$

$$= \begin{cases} 0 , & \delta = 0 \\ \lim_{s\to\infty} s^{k_j+1}\;P_j(s) , & \delta = 1 \end{cases} \tag{40}$$

(the limit existing from the definition of $P_j(s)$). Also, using (36)

$$\lim_{p\to\infty} s\,\psi_3^{(j)}(s,p) = \lim_{p\to\infty} s\,\Omega_j(s,p) = 0 \tag{41}$$

where $p^{-1}s^{\nu}$ (and hence $p^{-1}s^{k_j-1}$) $\to 0$ as $p \to \infty$. Clearly (39) and (41) indicate that (22) holds on all unbounded branches of the root-locus where $p^{-1}s^{\nu} \to 0$ as $p \to \infty$ and ν is <u>any</u> real number in the range $k_j \leq \nu \leq k_{j+1}-1$, and, in particular, for $\nu = k_{j+1} - 1$.

Finally, we prove that (5) has no infinite zeros in the range $k_j < \nu' \leq k_{j+1} - 1$ using the following arguments. Firstly, if $k_{j+1} = k_j + 1$ we see that the result is trivial as the range is empty. Suppose therefore that $k_{j+1} - 1 > k_j$ and consider the choice of $\nu = k_j$. The above analysis indicates that, if $p^{-1}s^{k_j} \to 0$ as $p \to +\infty$, then $\Omega_{j+1}(s,p) \to 0$. Also, if $p\,s^{-k_{j+1}} \to 0$, then it is easily verified that $p\,Q^{(j+1)}(s) \to 0$ and hence that $|I + p\,Q^{(j+1)} + \Omega_{j+1}| \to 1$ as $p \to \infty$. This clearly contradicts (16) so we must conclude that $p\,s^{-k_{j+1}}$ is either finite and non-zero or unbounded on all unbounded branches of the root-locus where $p^{-1}s^{k_j} \to 0$. In other terms, there are no infinite zeros between k_j and $k_{j+1} - 1$. This completes the proof of the result.

We are now in a position to describe a computational method paralleling that due to Kouvaritakis and Edmunds [5] by noting that the results of Proposition 3 ensure that the assumptions of Proposition 2 (with the exception of (b)) are satisfied. They can hence be applied recursively with a starting condition provided by Proposition 1. More precisely:

Step 1: Set $j=1$, $\Omega_1(s,p) \equiv 0$ and $Q^{(j)}(s) = Q(s)$

Step 2: If $Q^{(j)}(s) \not\equiv 0$, calculate k_j and, by spectral decomposition of its leading non-zero Markov Parameter, construct the $d_j \times d_j$ uniform rank k_j system $G_j(s) = V_j\,Q^{(j)}(s)\,U_j$. (Note: condition (b) of Proposition 2 must be valid for this to be achieved. This is clearly generically the case!)

Step 3: Use known techniques (e.g. [3] - [6]) to compute the asymptotic directions and pivots of the $k_j d_j$ k_j^{th} order infinite zeros of G_j (Note: Proposition 2 and induction ensures that these asymptotes are also asymptotes of $Q = Q^{(1)}$).

Step 4: If $\eta_{j+1} = m$, step. Otherwise construct $Q^{(j+1)}(s)$, replace j+1 by j and return to step 2.

As in [5] the complex transfer function matrix manipulations involved in the construction of $Q^{(j+1)}$ from $Q^{(j)}$ can be replaced by elementary manipulations of the systems state-space model S(A,B,C).

Proposition 4: $Q^{(j)}(s)$ has a realization $S(A_j, B_j, C_j)$ in R^n of the form

$$S_j \triangleq S(A_j, B_j, C_j) = \begin{cases} S(A,B,C) & \text{if } j = 1 \\ \\ S(T_{j-1}A_{j-1}, B_{j-1}M_{j-1}, N_{j-1}\ C_{j-1}) & \text{if } j>1 \end{cases} \tag{42}$$

where
$$T_j = I_n - B_j G_j^+ C_j A_j^{k_j-1} \quad , \quad j \geq 1 \tag{43}$$

and
$$C_j A_j^{i-1} B_j = 0 \quad (i < k_j) \quad , \quad C_j A_j^{k_j-1} B_j \neq 0 \tag{44}$$

Proof: If j=1, the result is trivial as $Q^{(1)} \triangleq Q$. Consider therefore the case of j > 1 and suppose, using induction, that S_{j-1} has the required form and properties. Note also the identity,

$$(s\,I_n - F)^{-1} = \sum_{i=1}^{\ell} s^{-i} F^{i-1} + F^{\ell} s^{-\ell} (sI_n - F)^{-1}$$
$$= \sum_{i=1}^{\ell} s^{-i}\, F^{i-1} + s^{-\ell} (sI_n - F)^{-1} F^{\ell} \tag{45}$$

valid for all $\ell \geq 0$ and for all nxn matrices F. The transfer function matrix of S_j is just

$$N_{j-1}\,C_{j-1}(sI_n - T_{j-1}A_{j-1})^{-1}\,B_{j-1}\,M_{j-1}$$

$$= \sum_{i=1}^{k_{j-1}} s^{-i}\,N_{j-1}\,C_{j-1}\,(T_{j-1}A_{j-1})^{i-1}B_{j-1}M_{j-1}$$

$$+ s^{-k_{j-1}}\,N_{j-1}C_{j-1}(T_{j-1}A_{j-1})^{k_{j-1}}(sI_n - T_{j-1}A_{j-1})^{-1}B_{j-1}M_{j-1} \tag{46}$$

using (45) with $F = T_{j-1}A_{j-1}$ and $\ell = k_{j-1}$. Using (44) we see that, for $i \leq k_{j-1}+1$,

$$C_{j-1}(T_{j-1}A_{j-1})^{i-1}$$

$$= C_{j-1}(I_n - B_{j-1}G^+_{j-1}\,C_{j-1}A^{k_{j-1}-1})A_{j-1}(T_{j-1}A_{j-1})^{i-2}$$

$$= C_{j-1}A_{j-1}(T_{j-1}A_{j-1})^{i-2}$$

$$\vdots \qquad \vdots$$

$$= \begin{cases} C_{j-1}A^{i-1}_{j-1} & , \; i \leq k_{j-1} \\ C_{j-1}A^{i-2}_{j-1}\,(I_n - B_{j-1}G^+_{j-1}C_{j-1}A^{k_{j-1}-1}_{j-1})A_{j-1}, & i=k_{j-1}+1 \end{cases} \tag{47}$$

and hence, using the properties of the eigenvector matrix W_j,

$$N_{j-1}C_{j-1}(T_{j-1}A_{j-1})^{i-1}B_{j-1}M_{j-1} = O \quad , \quad i \leq k_{j-1}$$

$$N_{j-1}C_{j-1}(T_{j-1}A_{j-1})^{k_{j-1}} = N_{j-1}C_{j-1}A^{k_{j-1}}_{j-1} \tag{48}$$

The transfer function matrix in (46) now reduces to

$$s^{-k_{j-1}}N_{j-1}C_{j-1}A^{k_{j-1}}_{j-1}(sI_n - A_{j-1} + B_{j-1}G^+_{j-1}C_{j-1}A^{k_{j-1}})^{-1}B_{j-1}M_{j-1}$$

$$\equiv s^{-k_{j-1}}N_{j-1}C_{j-1}A^{k_{j-1}}_{j-1}(sI_n - A_{j-1})^{-1}(I_n +$$

$$B_{j-1}G^+_{j-1}C_{j-1}A^{k_{j-1}}(sI_n - A_{j-1})^{-1})^{-1}B_{j-1}M_{j-1}$$

$$\equiv s^{-k_{j-1}} N_{j-1} C_{j-1} A_{j-1}^{k_{j-1}} (sI_n - A_{j-1})^{-1} B_{j-1} (I_{n_{j-1}} +$$

$$G_{j-1}^{+}\ C_{j-1} A^{k_{j-1}} (sI_n - A_{j-1})^{-1} B_{j-1})^{-1} M_{j-1} \qquad (49)$$

after use of the matrix identity (27). But this is just $Q^{(j)}(s)$ (as required) as

$$C_{j-1} A^{k_{j-1}} (sI_n - A_{j-1})^{-1} B_{j-1} \equiv \sum_{i=1}^{\infty} s^{-i}\ C_{j-1} A_{j-1}^{k_{j-1}+i-1}\ B_{j-1}$$

$$\equiv s^{k_{j-1}} \sum_{i=k_{j-1}+1}^{\infty} s^{-i}\ C_{j-1} A_{j-1}^{i-1}\ B_{j-1} \equiv s^{k_{j-1}}\ P_{j-1}(s) \qquad (50)$$

by (44), and, using (27) again,

$$P_{j-1}(I + G_{j-1}^{+} s^{k_{j-1}} P_{j-1})^{-1} \equiv (I + P_{j-1}\ s^{k_{j-1}} G_{j-1}^{+})^{-1} P_{j-1} \qquad (51)$$

The result is now completed by noting that if S_j is a representation of $Q^{(j)}$, (44) trivially from the definitions.

(Note: the state space realization of $Q^{(j)}$ can be used to deduce the state space realization $S(A_j, B_j U_j, V_j C_j)$ of G_j and hence its leading Markov parameters that are required to deduce its asymptotic directions and pivots).

In summary the paper has provided a rigorous inductive proof of the results of Kouvaritakis and Edmunds [5] in a form that relates them to the use of uniform rank systems in Owens [3],[4] and [6]. In particular, the nature of the approximation A1 (introduced without justification) in Appendix A of [5] has been identified as being valid for the prediction of the orders, asymptotic directions and pivots of all infinite zeros of the system.

References

[1] I. Postlethwaite: 'Algebraic-function theory in the analysis of multivariable feedback systems', Proc. IEE, 126, 1979, 555-562.

[2] A.G.J. MacFarlane, I. Postlethwaite: 'A complex variable approach to the analysis and design of linear multivariable feedback systems', Springer-Verlag, 1979.

[3] D.H. Owens: 'Dynamic transformations and the calculation of multivariable root-loci', Int. J. Control, 28, 333-343.

[4] D.H. Owens: 'Multivariable root-loci and the inverse transfer function matrix', Int. J. Control, 28, 345-351.

[5] B. Kouvaritakis, J.M. Edmunds: 'Multivariable root-loci: a unified approach to finite and infinite zeros', Int. J. Control, 29, 393-428.

[6] D.H. Owens: 'Feedback and multivariable systems', Peter Peregrinus,1978.

[7] D.H. Owens: 'Compensation theory for multivariable root-loci', Proc. IEE, 126, 1979, 538-541.

[8] D.H. Owens: 'Multivariable root-loci: an emerging design tool?' Proceedings of the IEE International Conference, 'Control and Its Applications', Univ. of Warwick, March 23-25th, 1981.

[9] A.G.J. MacFarlane: 'Frequency response methods in control systems', IEEE Press, 1979.

[10] D.H. Owens: 'A note on the orders of the infinite zeros of linear multivariable systems', Int. J. Control, 1980, 31, 409-412.