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ROBUST CONTROL OF UNKNOWN OR LARGE-SCALE SYSTEMS USING TRANSIENT DATA ONLY

by

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Abstract

The problem of controller design for a (possibly unknown) discrete or continuous multivariable system using only simple graphical and computational steps based on open-loop step response data is considered. Conditions are derived describing when a high-performance controller can be derived for a plant for which rise-time and steady-state data alone is available and the results illustrated by numerical examples.

1. Introduction

There is now a large body of knowledge available concerning the use of frequency response methods in the design of multivariable feedback system (Rosenbrock 1974, Owens 1978, Harris and Owens 1979, Postlethwaite and MacFarlane 1979, MacFarlane 1980a, 1980b) and a number of highly successful techniques for computer-aided-design are now available. Almost all of this work assumes that a model of the process to be controlled is available for use as a basis for design calculations such as simulation, transfer function matrix or frequency response evaluation, calculation of poles and zeros etc. This paper is concerned with the problem of controller design when a plant model is not available in the sense that

(a) the plant model is not known but open-loop plant step responses are available from plant tests, or

(b) the plant model is known but is so complex that design calculations other than simulation are not feasible with available computing facilities.

In either case, the plant model is (from the designers viewpoint) partially unknown, and controller-design must proceed on some other basis. One method is to use identification procedures based on plant responses or simulation of a complex plant model to obtain a reduced-order model that can be analysed with available computing facilities. It is not clear, however, whether identification procedures will produce models that are suitable for controller design in the sense that

(i) if a given controller produces a satisfactory closed-loop performance from the approximate model, it is not necessarily true that the real plant is even stable, and

(ii) the identified model need not necessarily produce a design that is insensitive to the modelling errors!

It is also not always true that the design engineer has access to identification software!

With the above background, this paper addresses its attention to the theoretical identification of a class of multivariable process plant for which neither a detailed process model nor the application of sophisticated identification procedures are required for the design of high-performance feedback controllers. More precisely, we ask

(1) What structural properties must the system have to make this possible?

(2) What form of parametric controller structure will ensure stability and adequate tracking of specified demand signals?

(3) What are the general conditions on available tuning parameters necessary to ensure that (2) is true?

(4) When can the controller structure be deduced from transient open-loop data only?

Some answers to these important practical questions are presented in the following sections based on the use of a conceptual approximate process model as the basis for controller design in a similar manner to Edwards and Owens (1977) and Owens (1978,1979). Throughout the paper attention will be restricted to the practically most important case of proportional plus integral control. Sections 2,3,4 deal with the case of discrete and continuous multivariable systems and illustrative examples respectively.

2. Controllers for Unknown Discrete Plant

Consider an unknown (in the sense defined in section 1) m-input/ m-output dynamic process that can be approximated over its operating range by the linear time-invariant model

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad , \quad x(t) \in R^n$$
$$y(t) = C\,x(t) \quad , \quad y(t) \in R^m \; , \; u(t) \in R^m \qquad (1)$$

If the plant inputs and outputs are synchronously sampled with period h, the plant output sequence $y_o = y(o), y_1 = y(h), y_2 = y(2h), \ldots$ and input sequence $u_o = u(o), u_1 = u(h), \ldots$ can be related by the discrete model

$$x_{k+1} = \Phi\,x_k + \Delta\,u_k$$
$$y_k = C\,x_k \quad , \quad k \geqslant o \qquad (2)$$

where $x_k = x(kh)$, $k \geqslant o$, and

$$\Phi = e^{Ah} \quad , \quad \Delta = e^{Ah} \int_o^h e^{-At}\,B\,dt \qquad (3)$$

This section considers the problem of the design of a robust proportional plus summation controller for this plant using transient response data.

2.1 Properties of the Discrete Model

As might be expected the solution to the problem depends upon the structural properties of the discrete model. The required properties of the model are that

(i) $C\Delta$ is nonsingular, and

(ii) the model $S(\Phi,\Delta,C)$ is minimum phase in the sense that all solutions of the relation

$$\begin{vmatrix} z\,I_n - \Phi & -\Delta \\ C & o \end{vmatrix} = o \qquad (4)$$

(i.e. the system invariant zeros) have modulus strictly less than unity.

The following results provide useful sufficient conditions for the above to hold and generate an important parametric system decomposition.

Proposition 1: If the underlying continuous system (1) is stable and invertible with no zero at the origin of the complex plane, then $C\Delta$ is nonsingular for almost every choice of sample interval h.

Proof: As $C\Delta$ is an entire function of h, it is sufficient to prove that $|C\Delta| \neq 0$ at some point of the complex plane for then every solution of the relation $|C\Delta| = 0$ is isolated! But the stability assumption and the final value theorem lead to the identity

$$\lim_{h\to+\infty} C\Delta = \lim_{h\to+\infty} \int_0^h Ce^{A(h-t)}B\,dt = G_c(0) \qquad (5)$$

where $G_c(s)$ is the mxm transfer function matrix of the continuous underlying system, Both the stability assumption and the assumption that there is no zero at $s = 0$ immediately yields the fact that $G_c(0)$ is nonsingular!

Proposition 2; If CB is nonsingular then $C\Delta$ is nonsingular on some interval $0 < h < h^*$ (i,e, $C\Delta$ is nonsingular at all fast enough sampling rates)

Proof: Follows directly from (3) and the relation

$$\lim_{h\to 0+} h^{-1} C \int_0^h e^{A(h-t)} B\,dt = CB \qquad (6)$$

Proposition 3; If $C\Delta$ is nonsingular then the discrete system (2) can be realised in the form

$$y_{k+1} = (\Phi_{11} - D)y_k + C\Delta(u_k - v_k) \qquad (7)$$

where the signal v_k is generated from the inherent feedback loop

$$z_{k+1} = \Phi_{22} z_k + \Phi_{21} y_k \quad , \; z_k \in R^{n-m}$$

$$v_k = -(C\Delta)^{-1}(\Phi_{12} z_k + D y_k) \tag{8}$$

and the mxm matrix D is arbitrary. Moreover the system is minimum phase if, and only if, all eigenvalues of Φ_{22} lie inside the open unit circle in the complex plane.

Proof: Let $T = [\Delta(C\Delta)^{-1}, M]$ where the columns of M span the kernel of C, then the assumption that $|C\Delta| \neq 0$ trivially implies that T is nonsingular. Consider the change of state variables defined by $\tilde{x}_k = T^{-1} x_k$, $k \geq 0$. Then the transformed system has the form defined by

$$\tilde{\Phi} = T^{-1} \Phi T = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \qquad (\Phi_{11} (mxm))$$

$$\tilde{\Delta} = T^{-1}\Delta = \begin{bmatrix} C\Delta \\ 0 \end{bmatrix} \quad , \; \tilde{C} = CT = [I_m \quad 0] \tag{9}$$

The representation (8) follows easily by noting that the transformed state has the structure $\tilde{x}_k^T = [y_k^T , z_k^T]$. Finally, the last part of the result follows by using similarity transformations and row and column operations to show that

$$\begin{vmatrix} zI_n - \Phi & -\Delta \\ C & 0 \end{vmatrix} \equiv \begin{vmatrix} zI_n - \tilde{\Phi} & -\tilde{\Delta} \\ C & 0 \end{vmatrix}$$

$$\equiv |C\Delta| \cdot |zI_{n-m} - \Phi_{22}| \tag{10}$$

Proposition 4: If the underlying continuous system (1) is minimum phase and CB is nonsingular then the discrete system (2) is minimum phase for h in some interval $0 < h < h^*$ (i.e. the discrete system is minimum phase at fast sampling rates!)

Proof: The result can be deduced from the Appendix in Owens (1979) or Molander (1979) p. 72.

2.2 Construction of First Order Approximate Plant Models

Given the existence of the discrete model (2) of the unknown plant we suppose that controller design is to be undertaken on the basis of an approximating first order (Owens 1979) multivariable model described by the inverse z-transfer function matrix

$$G_A^{-1}(z) = (z-1)\,B_o + B_1 \quad , \quad |B_o| \neq 0 \tag{11}$$

or, equivalently, by the state-variable model

$$y_{k+1} = (I_m - B_o^{-1}B_1)\,y_k + B_o^{-1}\,u_k \quad , \quad |B_o| \neq 0 \tag{12}$$

Assume also that $C\Delta$ is nonsingular (some justification for this assumption can be found from Propositions 1,2) and identify (12) with (7) via the relations

$$B_o^{-1} = C\,\Delta \quad , \quad \Phi_{11} - D = I_m - B_o^{-1}\,B_1 \tag{13}$$

i.e. we regard the approximate model as being generated from (7) by neglecting the inherent feedback loop variable implied by (8). We make no other assumptions about the choice of B_1 but there are clearly two natural choices:

(i) $B_1 = 0$ will simplify the structure of the approximate model but the open-loop input/output behaviours of the real and approximate system will differ greatly in such a case, or

(ii) if we define the system z-transfer function matrix

$$G(z) = C(z\,I_n - \Phi)^{-1}\,\Delta \tag{14}$$

then the choice of $B_1 = G^{-1}(z)|_{z=1}$ will ensure that the real and approximate systems have identical steady state characteristics. We do not reject the possibility that other choices of B_1 may be more useful however!

Finally note that the first order model can be computed quite easily from a model or from simulation/plant transient data as follows:

(a) If a plant model is available then the evaluation of $C\Delta$ is a straightforward task even if the plant model is of very high order. If we choose $B_1 = 0$ then there is no computational problem! If however, we choose B_1 to match the plant and model steady state characteristics, then it can be obtained directly by evaluating $C(I_n - \Phi)^{-1} \Delta$ (involving the numerically feasible inversion of $I_n - \Phi$ even in high order cases) and setting

$$B_1 = (C(I_n - \Phi)^{-1}\Delta)^{-1} = G(1)^{-1} \tag{15}$$

whenever the inverse exists.

(b) Suppose that plant tests or model simulations are undertaken to estimate the output vector sequence $\{y_1^{(i)}, y_2^{(i)}, \ldots\}$ generated by a unit step input in the i^{th} plant input from zero initial conditions and that these experiments are repeated for all inputs, $1 \leq i \leq m$. Then, defining $Y_k = [y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(m)}]$ (mxm), $k \geq 0$, it is clear from the model equation that

$$C\Delta = Y_1 \tag{16}$$

and, provided the plant is stable,

$$G(1) = C(I_n - \Phi)^{-1}\Delta = \lim_{k\to\infty} Y_k \tag{17}$$

2.3 Controller Design for the First Order Approximation

The controller design problem considered is the choice of forward path controller $K(z)$ for the real plant $G(z)$ as illustrated in Fig. 1(a). In the case of an unknown plant G, controller design is taken to proceed on the basis of the approximate model G_A. That is, the controller K is designed to ensure satisfactory stability, steady state, transient and interaction characteristics from the approximate feedback system of Fig. 1 (b). The final design will then be 'hooked up' to the real system as in Fig. 1(a) and final fine-tuning of controller parameters undertaken. This procedure, of course, introduces the question of whether or not the designed controller is capable of producing stable, high-performance sequences from the real plant? An answer to this question is left for the next section. Here we restrict our attention to the design of K for G_A!

Following Owens (1979) the controller $K(z)$ for the approximate model G_A is taken to have the proportional-plus-summation form

$$K(z) = B_o \operatorname{diag}\{1 - k_j c_j + \frac{(1-k_j)(1-c_j)z}{(z-1)}\}_{1\leq j\leq m} - B_1 \tag{18}$$

where k_j, $1 \leq j \leq m$, and c_j, $1 \leq j \leq m$, are available proportional and reset tuning parameters respectively and the matrices B_o and B_1 are obtained from the approximate plant model data. Equivalently, in state-variable form, the controller has the form

$$q_{k+1} = q_k + e_k$$

$$u_k = B_o \operatorname{diag}\{(1-k_j)(1-c_j)\}_{1\leq j\leq m} q_k$$

$$+ (B_o \operatorname{diag}\{2-k_j-c_j\}_{1\leq j\leq m} - B_1)e_k \tag{19}$$

or, in cases where $(1-k_j)(1-c_j) = 0$ for indices $j_1,\ldots,j_\ell$, a minimal realization of (18) obtained from (19) by deleting the $j_1,j_2,\ldots,j_\ell$-th elements of the control state q.

A simple piece of algebra yields the following expression for the closed-loop transfer function matrix

$$H_c(z) = (I_m + G_A(z)K(z))^{-1}G_A(z)K(z) = (G_A^{-1}(z) + K(z))^{-1}K(z)$$

$$= \operatorname{diag}\{\frac{1}{(z-k_j)(z-c_j)}\}_{1\leq j\leq m} (z(\operatorname{diag}\{2-k_j-c_j\}_{1\leq j\leq m}$$

$$- B_o^{-1}B_1) + (B_o^{-1}B_1 - \operatorname{diag}\{1 - k_j c_j\}_{1\leq j\leq m})) \tag{20}$$

and hence that (Owens 1979).

(a) The approximating feedback system (Fig. 1(b)) is asymptotically stable if

$$-1 < k_j < 1 \quad , \quad -1 < c_j \leq +1 \qquad (1 \leq j \leq m) \tag{21}$$

(b) The responses of the approximating feedback system to a step demand in the k^{th} output element have zero steady state errors if (21) holds and $c_k \neq 1$

(c) If the chosen procedure for specifying B_1 is such that

$$\lim_{h \to o+} B_o^{-1} B_1 = O \tag{22}$$

then it is trivially verified that

$$\lim_{h \to o+} H_c(z) = \text{diag}\left\{\frac{z(2-k_j-c_j)-(1-k_jc_j)}{(z-k_j)(z-c_j)}\right\}_{1 \leq j \leq m} \tag{23}$$

uniformly on any closed subset of the complex plane not containing the closed-loop poles $k_1, k_2, \ldots, k_m, c_1, c_2, \ldots, c_m$. That is, the closed-loop system will be almost non-interacting at fast sampling rates with loop dynamics close to those described by the right-hand-side of (23).

Clearly, under fast-sampling conditions, the controller (18) is capable of generating an approximate closed-loop system with excellent steady-state and transient characteristics. Some insight into the required condition (22) is obtained from the following proposition:

Proposition 5: If the chosen procedure for specifying B_1 is such that there exists M such that

$$\limsup_{h \to o+} ||B_1||_m \leq M \tag{24}$$

(where $||\cdot||_m$ is any norm on $L(R^n)$) then relation (22) is satisfied.

Proof: Follows directly from (b) by noting that $B_o^{-1} = C\Delta$.

Specific instances when the conditions of the proposition are valid are obtained by the choice of $B_1 \equiv O$ (when (24) follows trivially) or the choice of $B_1 = G^{-1}(z)|_{z=1}$ (when the real and approximate plants have the same steady state characteristics which are identical to those of the underlying continuous plant and hence independent of sampling rate h^{-1}). It can easily be satisfied in practice therefore.

2.4 Stability and Performance of the Real Feedback System

Consider the problem of predicting the stability and performance characteristics of the feedback system of Fig. 1(a) with the controller (18) in terms of characteristics of the approximating feedback system Fig. 1(b). The main mathematical tool used in the solution is the familiar contraction mapping theorem from functional analysis (see, for example, Dieudonne 1969, Holtzmann 1970 or Martin 1976).

Denote by ℓ^m the real vector space of infinite sequences $f = \{f_o, f_1, f_2, \ldots\}$ with $f_k \in R^m$, $k \geq o$, and the natural definition of vector addition and multiplication by scalars. It is clear that linear systems with m-inputs and m-outputs can be identified in terms of linear operations on ℓ^m. If, also, we denote ℓ^m_∞ the (Banach) subspace of ℓ^m of bounded sequences with norm

$$||f||_\infty = \sup_{k \geq o} ||f_k||_m \tag{25}$$

(where $||x||_m = \max_{1 \leq k \leq m} |x_k|$ is the normal uniform norm on R^n), then stable systems (in the input/output sense) can be identified with operators on ℓ^m that map ℓ^m_∞ into itself. We will use the notation y for the output sequence $\{y_o, y_1, y_2, \ldots\}$ and similarly with other variables. Finally, the linear operator in ℓ^m associated with a linear system with z-transfer function matrix T(z) will be denoted T and the truncation operator P_k is taken to be the linear operator defined by the relation

$$P_k\{f_o, f_1, f_2, \ldots\} = \{f_o, f_1, \ldots, f_k, o, \ldots\} \tag{26}$$

when $k \geq o$ is finite and the identity when $k = +\infty$. An operator T is causal if, and only if, $P_k T = P_k T P_k$ for all $k \geq o$.

Using the identification of equation (13) and assuming zero initial conditions, equation (7) can be written as an equation in ℓ^m of the form

$$y = G_A(u - v) \tag{27}$$

In a similar manner (8) takes the form

$$v = B_o H y \tag{28}$$

with H defined in the obvious manner. The following result forms the foundation of later developments:

<u>Lemma 1</u>: Let L_c be the linear feedback operator in ℓ^m defined by the relation $r \to y$ with (Fig. 1 (b))

$$y = G\,K(r-y) \tag{29}$$

and Let L_{cA} be the linear feedback operator in ℓ^m defined by $r \to y_A$ with (Fig 1(b))

$$y_A = G_A K(r - y_A) \tag{30}$$

Then $L_c|\ell_\infty^m : \ell_\infty^m \to \ell_\infty^m$ and is bounded and causal if

(i) K has an inverse defined in ℓ^m

(ii) $L_{cA}|\ell_\infty^m : \ell_\infty^m \to \ell_\infty^m$ and is bounded and causal

(iii) $L_{cA} K^{-1} B_o H|\ell_\infty^m : \ell_\infty^m \to \ell_\infty^m$ and is bounded and causal

(iv) $\lambda \triangleq ||L_{cA} K^{-1} B_o H||_\infty < 1$ (where $||.||_\infty$ is the operator norm in ℓ_∞^m induced by the norm (25))

Under the conditions, for a given choice of r,

$$||P_k(y - y_A)||_\infty \leqslant \frac{\lambda'}{1-\lambda'} ||P_k\, y_A||_\infty \;\forall\; k \geqslant 0$$

where $\lambda' < 1$ is any upper bound for λ. (31)

<u>Proof</u>: Using the representation (27)-(28) of the plant operator G converts (29) into the form

$$y = G_A(K(r-y)-v) \quad , \quad v = B_o\, H\, y \tag{32}$$

or, more simply,

$$y = G_A K((r - K^{-1}B_o\, Hy) - y) \tag{33}$$

and hence

$$y = L_{cA}(r - K^{-1}B_o\, H\, y) \tag{34}$$

Taking $r \in \ell_\infty^m$ then this equation takes the form $y = W\,y$ where $W: \ell_\infty^m \to \ell_\infty^m$ is a contraction with contraction constant λ'. It follows directly from the contraction mapping theorem that the solution $y \in \ell_\infty^m$ with

$$||y||_\infty \leqslant \frac{1}{1-\lambda'} \;\; ||L_{cA}\, r||_\infty \tag{35}$$

(proving the boundedness of L_c) and

$$||y - y_A||_\infty \leqslant \frac{\lambda'}{1-\lambda'} ||\, y_A||_\infty \tag{36}$$

(proving (31) in the case of $k = +\infty$). Finally, multiplying (34) by P_k and using causality, yields

$$P_k y = (P_k L_{cA}) P_k r - (P_k L_{cA} K^{-1} B_o H) P_k y \tag{37}$$

Equation (31) follows from a similar argument to the above noting that $||P_k L_{cA} K^{-1} B_o H||_\infty \leq ||L_{cA} K^{-1} B_o H|| = \lambda \leq \lambda' < 1$. The proof of the lemma is now complete.

<u>Corollory:</u> With the above assumptions

$$||P_k L_c - P_k L_{cA}||_\infty \leq \frac{\lambda'}{1-\lambda'} ||P_k L_{cA}||_\infty \quad \forall \; k \geq o \tag{38}$$

<u>Proof:</u> The result follows directly from (31) by writing $y = L_c r$ and $y_A = L_{cA} r$.

The above results are of a quite general nature and provide <u>sufficient</u> conditions for the stability of the approximating feedback system of Fig. 1(b) (i.e. the boundedness of L_{cA}) to guarantee the stability of the real feedback system of Fig. 1(a) (i.e. the boundedness of L_c) with the added bonus of providing an upper bound on the error of prediction of <u>transient</u> performance (31). More precisely, for each $k \geq o$,

$$\max_{1\leq i\leq m} |(y_k)_i - (y_{Ak})_i| \leq ||P_k(y-y_A)||_\infty$$

$$\leq \frac{\lambda'}{1-\lambda'} ||P_k y_A||_\infty = \frac{\lambda'}{1-\lambda'} \max_{o\leq j\leq k} \; \max_{1\leq i\leq m} |(y_{Aj})_i| \tag{39}$$

the RHS depending only upon the approximate system responses and the contraction constant λ'.

Clearly the above results can only be applied if the approximation error H is known explicitly or if an upper bound for the contraction constant λ is known. Neither of these situations will apply if the system G is unknown! This does not preclude the possibility that the given controller will prove to be an excellent control system however, but it does imply that the design engineer will have no guarantees of its success! Progress can be made, however, if the system is unknown but is known to have certain structural properties and the available tuning parameters satisfy some simple relations. These ideas are formalized in the following main result of this section, generalizing that of Owens (1979):

Theorem 1: An unknown discrete multivariable plant (2), known to be generated from an unknown continuous multivariable plant (1) that is minimum phase with CB nonsingular, will be stable in the presence of unity negative feedback with forward path proportional plus summation controller of the form of equation (18) if

(i) the tuning parameters $k_1, k_2, \ldots, k_m, c_1, c_2, \ldots, c_m$ satisfy the constraint (21),

(ii) the procedure for choosing B_1 is such that condition (22) holds, and

(iii) the sampling rate h^{-1} is sufficiently fast

Moreover, under these conditions, we have, for each given reference demand sequence r, the relation

$$\lim_{h \to o+} \{y - y_A\} = 0 \tag{40}$$

indicating that the responses of the closed-loop system Fig. 1(a) will be very close to those predicted by the approximating closed-loop system Fig. 1(b) at fast sampling rates.

Proof: We proceed by verifying that the conditions of lemma 1 are satisfied:

(a) We prove initially that K has a causal inverse defined in ℓ^m by writing the realization (19) in the form

$$q_{k+1} = q_k + e_k \quad , \quad u_k = M_1 q_k + M_2 e_k \tag{41}$$

where, using (i), (ii) and (iii), it is true that M_2^{-1} exists at fast sampling and

$$\lim_{h\to o+} M_2^{-1} M_1 = \text{diag}\left\{\frac{(1-k_j)(1-c_j)}{(2-k_j-c_j)}\right\}_{1\leq j\leq m} \tag{42}$$

A little manipulation soon yields the relations

$$q_{k+1} = (I_m - M_2^{-1}M_1)q_k + M_2^{-1}u_k$$

$$e_k = - M_2^{-1} M_1 q_k + M_2^{-1} u_k \tag{43}$$

which proves the invertibility of K on ℓ^m. Also note that (42) implies that

$$\lim_{h\to o+} (I_m - M_2^{-1}M_1) = \text{diag}\left\{\frac{1-k_jc_j}{2-k_j-c_j}\right\}_{1\leq j\leq m} \tag{44}$$

and hence that $K^{-1}|\ell_\infty^m : \ell_\infty^m \to \ell_\infty^m$ as the identity

$$0 < 1 - k_jc_j = 2 - k_j - c_j - (1 - k_j)(1 - c_j)$$
$$< 2 - k_j - c_j \tag{45}$$

indicates that the realization (43) is stable.

(b) The conditions (21) on the available tuning parameters ensure that the approximate closed-loop system is stable and hence that $L_{cA}|\ell_\infty^m: \ell_\infty^m \to \ell_\infty^m$ is bounded and causal.

(c) The result (b), (44) and propositions 3 and 4 imply that each term in the composite operator $L_{cA} K^{-1} B_o H$ represents a stable, causal system and hence that its restriction to ℓ_∞^m is bounded and causal.

(d) We prove that

$$\lim_{h\to o+} ||L_{cA} K^{-1} B_o H||_\infty = 0 \tag{46}$$

and hence that the contraction condition is satisfied at all sampling rates h^{-1} greater than an unknown rate h_*^{-1} and that the contraction constant can be made to be arbitrarily small. The theorem will then be proved as (40) then follows from (31).

To verify (46), note from (22) and (20) that, under fast sampling conditions,

$$H_c(z) \to \text{diag}\left\{\frac{1}{(z-k_j)(z-c_j)}\right\}_{1\leq j\leq m} (z\ \text{diag}\{2-k_j-c_j\}_{1\leq j\leq m}$$
$$- \text{diag}\{1 - k_jc_j\}_{1\leq j\leq m} \tag{47}$$

which is stable. Equivalently, there exists a constant c_1 such that

$$||L_{cA}||_\infty < c_1 \tag{48}$$

in the vicinity of $h = o+$. Next, note that the composite operator $K^{-1}B_o$ can be realized as the map $v \mapsto e$ defined by (see (43))

$$\begin{aligned} q_{k+1} &= (I_m - M_2^{-1}M_1)q_k + M_2^{-1}B_o v_k \\ e_k &= -M_2^{-1}M_1 q_k + M_2^{-1}B_o v_k \end{aligned} \tag{49}$$

which is stable and bounded in the vicinity of $h = o+$ by (44),(45) and the identity,

$$\lim_{h\to o+} M_2^{-1}B_o = \text{diag}\left\{\frac{1}{2-k_j-c_j}\right\}_{1\leq j\leq m} \tag{50}$$

Formally, there exists a constant c_2 such that

$$||K^{-1}B_o||_\infty < c_2 \tag{51}$$

in the vicinity of $h=o+$! Finally, from the results of Boland and Owens(1980), it can be seen that

$$\lim_{h\to o+} ||H||_\infty = 0 \tag{52}$$

and hence that

$$\lim_{h\to o+} ||L_{cA}K^{-1}B_oH||_\infty \leq \lim_{h\to o+} ||L_{cA}||_\infty . ||K^{-1}B_o||_\infty . ||H||_\infty$$

$$\leq c_1c_2 \lim_{h\to o+} ||H||_\infty = 0 \tag{53}$$

which proves (46). The proof of the theorem is now complete.

For the purposes of applications the result guarantees that, with suitable choice of control parameters and sampling rates, the proposed control system is capable of generating excellent responses from the unknown plant. In particular, if the responses in a particular case are not satisfactory, then the theorem states that the use of an increased sampling rate will improve the performance. In general there does not appear to be an easy technique for estimating the required sampling rate so, in general, the choice reduces to trial and error. In the next sub-section, however, a class of unknown processes are identified for which the adequacy of the sampling rate can be checked from a knowledge of the steady-state characteristics of the open-loop real and approximate plants only.

2.5 Unknown Systems with Monotonic Feedback Errors

As noted in the discussion following lemma 1, a knowledge of the contraction constant $\lambda = ||L_{cA}K^{-1}B_oH||_\infty$ or, more probably, an upper bound λ' for λ would enable the stability and transient characteristics of the real feedback system to be directly assessed in terms of the stability and performance of the approximating feedback system. An upper bound λ' is calculated below in the specific case of systems with monotone feedback errors.

Definition 1: A linear m-input/m-output time-invariant system is said to be monotonic (resp. sign-definite) if, and only if, the response from zero initial conditions of the i^{th} output to a unit step in the j^{th} input is either monotically increasing (resp. positive) or monotonically decreasing (resp. negative), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$.
(Note: In effect, this definition generalizes the property used by Astrom (1980) to the multi-input/multi-output case).

Proposition 6: The discrete system

$$x_{k+1} = \Phi x_k + \Delta u_k \; , \; u_k = C\,x_k + D\,u_k \tag{54}$$

is monotonic and sign-definite iff the Markov Parameter matrix sequence

$$H_o = D \quad , \quad H_k = C\,\Phi^{k-1}\Delta \;\; (k \geqslant 1) \tag{55}$$

has the property that the sequence of elements $(H_o)_{ij}, (H_1)_{ij}, (H_2)_{ij}, \ldots$ is either all positive or all negative, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$. If $H_o = D$ is the only violator of this condition, then the system is simply monotonic.

Proof: Write the solution of (54) from zero initial conditions in the form

$$y_k = \sum_{j=o}^{k} H_{k-j} u_j \quad , \quad k \geqslant o \tag{56}$$

and consider the case of unit step inputs in each channel. The result follows then quite easily.

<u>Proposition 7</u>: With the notation of proposition 6, denoting the system z-transfer function matrix by $G(z) = C(zI-\Phi)^{-1}\Delta + D$ and assuming that the system is stable, monotonic and sign-definite, then the operator G in ℓ_∞^m induced by the system has norm equal to the matrix norm $||G(1)||_m$ (where $||.||_m$ is the matrix norm in $L(R^n)$ induced by the vector norm $||.||_m$).

<u>Proof</u>: Consider initially the scalar (m=1) case and note that

$$|y_k| \leqslant \sum_{j=o}^{k} |H_{k-j}| \, |u_j| \leqslant |\sum_{j=o}^{\infty} H_{k-j}| \, \sup_{j\geqslant o} |u_j| \tag{57}$$

with equality holding if $u_j = 1$, $j \geqslant o$ i.e.

$$||G||_\infty = |\sum_{j=o}^{\infty} H_{k-j}| = |G(1)| \tag{58}$$

from the series expansion of G(z) about the point $z^{-1} = 0$. The more general case of m-inputs and m-outputs now follows trivially by noting that, using proposition 6:

$$||y_k||_m = \max_{1\leqslant i\leqslant m} |\sum_{j=o}^{k} \sum_{\ell=1}^{m} (H_{k-j})_{i\ell} (u_j)_\ell|$$

$$\leqslant \max_{1\leqslant i\leqslant m} \sum_{\ell=1}^{m} \sum_{j=o}^{k} |(H_{k-j})_{i\ell}| \, ||u||_\infty$$

$$= \max_{1\leqslant i\leqslant m} \sum_{\ell=1}^{m} |\sum_{j=o}^{k} (H_{k-j})_{i\ell}| \, ||u||_\infty$$

$$\leqslant \max_{1\leqslant i\leqslant m} \sum_{\ell=1}^{m} |\sum_{j=o}^{\infty} (H_{k-j})_{i\ell}| \cdot ||u||_\infty$$

$$= \max_{1\leqslant i\leqslant m} \sum_{\ell=1}^{m} |G_{i\ell}(1)| \cdot ||u||_\infty$$

$$= ||G(1)||_m \, ||u||_\infty \quad , \quad k \geqslant o \tag{59}$$

equality holding when $\{u_k\}_{k\geqslant o}$ is a constant sequence $\{\alpha\}_{k\geqslant o}$ with all elements of α of unit magnitude and carefully selected sign. This completes the proof of the proposition.

The following is the main result of this section and identifies a class of multivariable plant for which transient data above is sufficient to design the controller (18) and guarantee the stability and performance of the real closed-loop system.

Theorem 2: Given an unknown discrete multivariable plant (2) where $\mathbb{C}\Delta$ is nonsingular and where the feedback operator H is stable, monotonic and sign-definite, it follows that the unity negative feedback system with forward path controller (18) will be stable if

(i) the tuning parameters k_j, c_j, $1 \leq j \leq m$ satisfy (21)

(ii) the controller K(z) is invertible and minimum phase, and

(iii) the inequality

$$\lambda' \triangleq \| B_o^{-1}(G^{-1}(1) - B_1) \|_m \max_{1 \leq j \leq m} \gamma(k_j, c_j) < 1 \tag{60}$$

is satisfied, where the function $\gamma : R^2 \to R$ is any upper bound for the norm in ℓ_∞^m of a minimal realization of $(z-1)/(z-k)(z-c)$. For example, if $k_j > c_j$, $1 \leq j \leq m$, we can choose

$$\gamma(k_j, c_j) = \begin{cases} \dfrac{(1-k_j)}{(c_j-k_j)(1-|k_j|)} + \dfrac{(1-c_j)}{(c_j-k_j)(1-|c_j|)}, & c_j \neq 1 \\[2ex] \dfrac{1}{1-|k_j|}, & c_j = 1 \end{cases} \tag{61}$$

Moreover, under these conditions, equation (39) provides an upper bound on the approximation error due to the design of K based on the approximate model.

Proof: The proof follows the general lines of theorem 1 but computes an explicit expression for an upper bound $\lambda' \geq \| L_{CA} K^{-1} B_o \|_\infty \cdot \| H \|_\infty \geq \| L_{CA} K^{-1} B_o H \|_\infty = \lambda$. More precisely, the theorem is proved if we verify that

$$\| L_{CA} K^{-1} B_o \| \leq \max_{1 \leq j \leq m} \gamma(k_j, c_j) \tag{62}$$

and

$$||H||_\infty = ||B_o^{-1} (G^{-1}(1) - B_1)||_m \tag{63}$$

Equation (63) follows quite simply by applying Proposition 7 as, taking z-transforms of (27) and (28) leads to the identity

$$G(z) = (I_m + G_A(z)B_oH(z))^{-1}G_A(z) \tag{64}$$

and hence

$$H(z) = B_o^{-1} (G^{-1}(z) - G_A^{-1}(z)) \tag{65}$$

Equation (62) follows in a similar manner by noting that the operator $L_{cA}K^{-1}B_o$ can be represented by the diagonal z-transfer function matrix

$$(I_m + G(z)K(z))^{-1}G(z)B_o = (G^{-1}(z) + K(z))^{-1}B_o$$

$$\equiv \text{diag} \left\{ \frac{z-1}{(z-k_j)(z-c_j)} \right\}_{1\leq j\leq m} \tag{66}$$

and that a minimal realization of $(z-1)/(z-k_j)(z-c_j)=((1-k_j)/(c_j-k_j))(1/(z-k_j)) + ((1-c_j)/(k_j-c_j))(1/(z-c_j))$ has norm bounded by $\gamma(k_j,c_j)$. This completes the proof.

This result clearly provides a means of checking the stability and performance of the closed-loop system for a given sampling rate at the expense of needing to know or compute the required properties of H and K! If however, H has the desired properties, the required properties of K are easily checked and the constant λ' evaluated from the steady state characteristics G(1) of the real plant deduced either from a model using (15) or, if the system is open-loop stable, from plant tests or plant simulations using (17). The remaining problem is to find a systematic technique for checking the stability, monotonicity and sign-definiteness of H from transient data only. For this purpose, write H as the map $v' \rightarrow v = Hv'$ defined by

$$v_k = \sum_{j=o}^{k} H_{k-j} v'_j \tag{67}$$

and note the easily proven proposition:

Proposition 8: H is stable iff $\lim_{k\to\infty} H_k = o$

A check on monotonicity can be obtained if plant tests or model simulations are undertaken to estimate the vector output sequences $\{y_1^{(i)}, y_2^{(i)}, \ldots\}$ and $\{y_{A1}^{(i)}, y_{A2}^{(i)}, \ldots\}$ from the real and approximate plants generated by a unit step input in the i^{th} plant input from zero initial conditions, and if these experiments are repeated for all inputs $1 \leq i \leq m$. Defining

$$Y_k = [y_k^{(1)}, \ldots, y_k^{(m)}] \quad , \; Y_{Ak} = [y_{Ak}^{(1)}, \ldots, y_{Ak}^{(m)}] \tag{68}$$

then it is easily verified that the plant equations (27, (28) take the form

$$Y_A - Y = G_A B_o H Y \tag{69}$$

or, in difference form, noting that $Y_o = Y_{Ao} = 0$,

$$F_k \stackrel{\Delta}{=} (E_{k+1} - E_k) + B_o^{-1} B_1 E_k = \sum_{j=1}^{k} H_{k-j} Y_j$$

$$E_k = Y_{Ak} - Y_k \qquad , \qquad k \geq 0 \tag{70}$$

Suppose now that data is available for $1 \leq k \leq M$. Equation (70) can be expressed as an equation of the form

$$[H_o, H_1, \ldots, H_{M-1}] \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_M \\ 0 & Y_1 & \cdots & Y_{M-1} \\ \vdots & & & \\ 0 & & 0 & Y_1 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_M \end{pmatrix}^T \tag{71}$$

which can be solved recursively for $H_o, H_1, \ldots, H_{M-1}$ as (equation (16)) $Y_1 = C\Delta$ is, by assumption, nonsingular. As assessment of whether or not H is monotonic and sign-definite can then be undertaken by applying the test of proposition 6 to the finite number of Markov Parameter matrices available and an assessment of its stability undertaken by considering the validity of proposition 8 in the light of the finite data set available.

Note that the choice of model data B_1 could have some impact on the success of the above procedure in the sense that, from (8), it is clear that the Markov Parameters H_k, $k \geq 1$, are independent of the choice of D (and hence, by (13), B_1) whilst $H_o = -D = I_m - \Phi_{11} - B_o^{-1}B_1$ depends explicitly on our choice of B_1. This observation could be of some significance if the computed Markov parameters $H_1, H_2, \ldots, H_{M-1}$ satisfy the requirements of proposition 6 whilst H_o does not (i.e. H is monotonic but not sign-definite). It is easy to see that we can introduce an addition δH_o to H_o to remedy this problem (e.g. $\delta H_o = - H_o$) and that this beneficial change can be incorporated in the model by changing B_1 to $B_1 - B_o \, \delta H_o$ in the approximate plant model and designed controller. Finally, note that the existence of a choice of B_1 ensuring the monotonicity and sign-definiteness of H can be guaranteed in a certain special case:

Proposition 9: If the discrete model (2) is minimum-phase with $|C\Delta| \neq 0$ and $n = m + 1$, then there exists a choice of B_1 such that H is monotonic and sign-definite.

Proof: The assumptions guarantee that H is stable with transfer function matrix of the form $H(z) = (1/(z+\gamma))H_1 + H_o$. Clearly H is monotonic and, using the argument preceding the proposition, we can choose B_1 to ensure sign-definiteness! This proves the proposition.

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3. Controllers for Unknown Continuous Plant

Consider an unknown m-input/m-output dynamic process given in equation (1) and the problem of designing a robust proportional plus integral controller for this plant using transient data only. The development follows a similar line to that of section 2 and, for this reason, proofs will, at times, be simply outlined. Throughout, however, the use of variation of sample rate to investigate system behaviour is replaced by variations in controller gains.

3.1 An Important Property of the Continuous Model

Most of the properties described in section 2.1 are related to the discretization process and hence plan no role in the analysis of continuous plant. The following is the exception and represents the continuous equivalent to Proposition 3:

Proposition 10: If CB is nonsingular then the continuous system (1) can be realized in the form

$$\dot{y}(t) = (A_{11} - D)y(t) + CB(u(t) - v(t)) \qquad (72)$$

where the signal v(t) is generated from the inherent feedback loop

$$\dot{z}(t) = A_{22}z(t) + A_{21}y(t) \quad , \quad z(t) \in R^{n-m}$$

$$v(t) = -(CB)^{-1}(A_{12}z(t) + D\,y(t)) \qquad (73)$$

and the mxm matrix D is arbitrary. Moreover the system is minimum phase if, and only if, all eigenvalues of A_{22} lie in the open left-half complex plane.

Proof: Identical to that of Proposition 3 with Φ,Δ replaced by A,B respectively.

3.2 Construction of First Order Approximate Plant Models

Suppose that controller design for the unknown system (1) is to be undertaken on the basis of an approximating first order multivariable model (Edwards and Owens 1977, Owens 1978) described by the inverse transfer function matrix

$$G_A^{-1}(s) = s\,A_o + A_1 \qquad (74)$$

or, equivalentely, by the state-variable model

$$\dot{y}(t) = - A_o^{-1} A_1\ y(t) + A_o^{-1}\ u(t) \tag{75}$$

Assume also that CB is nonsingular and identify (75) with (72) via the relations

$$A_o^{-1} = CB \quad , \quad - A_o^{-1} A_1 = A_{11} - D \tag{76}$$

i.e. we regard the approximate model as being generated from (72) by neglecting the feedback loop (73). As in section 2.2 we make no other assumptions about the choice of A_1 except to note the two natural choices:

(i) $A_1 = o$ will simplify the structure of the approximate model, or

(ii) if we define the system transfer function matrix

$$G(s) = C(sI_n - A)^{-1} B \tag{77}$$

then the choice of $A_1 = G^{-1}(s)|_{s=o}$ will ensure that the real and approximate systems have identical steady state characteristics.

Other choices of A_1 are not dismissed however!

Finally note that a suitable first order model can be computed from a model or from simulation/plant transient data as follows:

(a) CB (and hence A_o) can be calculated easily if a plant model is available. The choice of $A_1 = o$ causes no computational problems whilst the choice of A_1 to match the plant steady state characteristics is obtained from the relation

$$A_1 = - (CA^{-1}B)^{-1} = G(o)^{-1} \tag{78}$$

whenever the inverse exists.

(b) Plant tests or model simulations can yield the response vector $y^{(i)}(t)$ generated from zero initial conditions by a unit step in the i^{th} plant input. Defining

$Y(t) = [y^{(1)}(t), y^{(2)}(t), \ldots, y^{(m)}(t)]$, it is clear that

$$CB = \left.\frac{dY(t)}{dt}\right|_{t=o} \tag{79}$$

or, equivalently, $1 \leqslant i,j \leqslant m$,

$$(CB)_{ij} = \frac{d}{dt} \left. y_i^{(j)}(t) \right|_{t=o} \tag{80}$$

Also, if the plant is stable,

$$G(o) = -CA^{-1}B = \lim_{t\to+\infty} Y(t) \tag{81}$$

Clearly, both CB and G(o) can be estimated by graphical constructions on the system response curves.

3.3 Controller Design for the First Order Approximation

As in section 2.3 the design problem is considered as the choice of forward path controller K(s) for the real plant G(s) (see Fig. 1(a)) by designing K on the basis of the approximate model $G_A(s)$ to ensure that the approximate feedback system (Fig. 1(b)) has the required stability, steady state, transient and interaction characteristics. The final design is then implemented on the real system and final tuning of control parameters undertaken. In this section the design of K for G_A is considered.

The controller K(s) for $G_A(s)$ is a generalization of the proportional-plus-integral controller originally proposed by Owens (1978), namely the parametric form

$$K(s) = A_o \operatorname{diag}\{k_j + c_j + \frac{k_j c_j}{s}\}_{1\leq j\leq m} - A_1 \tag{82}$$

where k_j, $1\leq j\leq m$, and c_j, $1\leq j\leq m$, are proportional and integral tuning parameters respectively. Equivalently, in state-variable form, the controller has a realization

$$\dot{q}(t) = e(t)$$

$$u(t) = A_o \operatorname{diag}\{k_j c_j\}_{1\leq j\leq m} q(t) + (A_o \operatorname{diag}\{k_j + c_j\}_{1\leq j\leq m} - A_1)\, e(t) \tag{83}$$

or, in cases when $c_j = o$ for indices $j_1, j_2, \ldots, j_\ell$, a minimal realization of (83) obtained by deleting the corresponding states of q(t).

In a similar manner to (20), the closed-loop transfer function matrix

$$H_c(s) = \mathrm{diag}\{\frac{1}{(s+k_j)(s+c_j)}\}_{1\leqslant j\leqslant m} (s(\mathrm{diag}\,\{k_j + c_j\}_{1\leqslant j\leqslant m}$$

$$- A_o^{-1}A_1) + \mathrm{diag}\,\{k_j c_j\}_{1\leqslant j\leqslant m}) \tag{84}$$

and hence, using a simple pole-residue analysis, we conclude that

(a) The approximating feedback system is stable if

$$k_j > o \quad , \quad c_j \geqslant o \quad (1 \leqslant j \leqslant m) \tag{85}$$

(b) The responses of the approximating feedback system to a unit step demand in the k^{th} output have zero steady state errors if (85) holds and $c_k > o$.

(c) As $k \triangleq \min\limits_{1\leqslant j\leqslant m} k_j$ increases the responses of the closed-loop system approach those of the system

$$\tilde{H}_c(s) \triangleq \mathrm{diag} \frac{\{s(k_j+c_j) + k_j c_j\}}{(s+k_j)(s+c_j)}\ 1\leqslant j\leqslant m \tag{86}$$

i.e. response spreads increase and interaction effects decrease as the 'gain' k increases, the system exhibiting rise times and reset times of the order of k_j^{-1} and c_j^{-1} respectively in loop j.

Clearly, under high enough gain conditions, the controller (82) is capable of generating an approximate closed-loop system with excellent steady-state and transient characteristics and the overall form and structure of the response can be shaped by judicious choice of tuning parameters k_j and c_j, $1\leqslant j\leqslant m$.

3.4 Stability and Performance of the Real Feedback System

The problem of predicting the stability and performance of the real feedback system with the controller (82) in terms of the response characteristics of the approximating feedback system can be treated in a similar manner to the discrete case considered in section 2.4. More precisely, replacing ℓ^m by the real vector space $C^m(o,\infty)$ of continuous mappings of $[o,+\infty)$ into R^m and ℓ_∞^m by the (Banach) subspace of ℓ^m of bounded mappings with the uniform norm

$$||f||_\infty = \sup_{t \geq o} ||f(t)||_m \tag{87}$$

Linear systems are then identified as linear operators on $C^m(o,\infty)$ and stable linear systems as linear operators whose restriction to $C_m(o,\infty)$ is an injection. We will denote the linear operator on $C^m(o,\infty)$ associated with a linear system with transfer function matrix T(s) by T and the normal truncation operator P is defined by

$$(P_\tau f(t) \triangleq \begin{cases} f(t) & , \quad o \leq t \leq \tau \\ 0 & , \quad t > \tau \end{cases} \tag{88}$$

for $\tau \geq 0$ and the identity if $\tau = +\infty$. An operator T is causal if $P_\tau T = P_\tau T P_\tau$ for all $\tau \geq 0$.

Using the identification of (76) and assuming zero initial conditions, equations (72) and (73) can be written in $C^m(o,\infty)$ in the form of (27) and (28) respectively with B_o replaced by A_o and a natural identification of H. In particular it is easily verified that Lemma 1 (and its corollary) remain valid with ℓ^m (resp. ℓ^m_∞) replaced by $C^m(o,\infty)$ (resp. $C^m_\infty(o,\infty)$), B_o replaced by A_o and the integer variable k replaced by the continuous variable τ. In particular (39) is replaced by the bound

$$\max_{1\leq i\leq m} |y_i(t) - y_{Ai}(t)| \leq ||P_t(y-y_A)||_\infty$$

$$\leq \frac{\lambda'}{1-\lambda'} ||P_t y_A||_\infty = \frac{\lambda'}{1-\lambda'} \max_{o\leq s\leq t} \max_{1\leq i\leq m} |y_{Ai}(s)| \tag{89}$$

providing an explicit, computable estimate of the error involved in using the approximate model for control design puposes in terms of the 'contraction constant' λ' and the approximate response y_A. As in the discrete case the lemma can only be used explicitly if either H or an upper bound for λ is known. Neither of these situations will normally exist if G is not known! We can however deduce the following main result of this section (c.f. Theorem 1).

<u>Theorem 3</u>: An unknown continuous multivariable plant (1) that is known to be minimum phase with CB nonsingular will be stable in the presence of unity negative feedback with forward path proportional plus integral controller (82) if

(i) the tuning parameters k_j and c_j, $1 \leq j \leq m$, satisfy (85), and

(ii) the 'overall gain' $k = \min_j k_j$ is sufficiently high.

Moreover, under the conditions, we have, for each given bounded reference demand,

$$\lim_{k \to +\infty} (y(t) - y_A(t)) = 0 \tag{90}$$

uniformly on $[0,+\infty)$, indicating that the responses of the real and approximating closed-loop systems will be extremely close at high gains. (Note: condition (ii) can be replaced by the conditions $k > k^*$ where k^* is an unknown 'minimum overall gain'. In general terms, it is the value of k^* that dictates the success of the technique for a given application. More precisely, if k^* is very high the technique will fail if high gains cannot be generated. It is anticipated, however, that, in many applications, k^* will be small and hence that the technique will succeed using only modest control gains and also that the approach to the limit in (90) will be rapid ensuing that $y_A(t)$ is a good working approximation to $y(t)$).

<u>Proof of Theorem 3</u>: The proof follows the same lines as the proof of theorem 1 by verifying that all conditions of lemma 1 are satisfied at high gains.

(i) K has an inverse on $C^m(0,\infty)$ as $A_0 \operatorname{diag}\{k_j + c_j\}_{1\leq j\leq m} - A_1$ has a causal inverse at high gains and hence (83) can be inverted in the form

$$\begin{aligned}\dot{q}(t) &= (A_0 \operatorname{diag}\{k_j+c_j\}_{1\leq j\leq m} - A_1)^{-1}(u(t) - A_0 \operatorname{diag}\{k_j c_j\}_{1\leq j\leq m} q(t)) \\ &= e(t)\end{aligned} \tag{91}$$

Note also that $K^{-1}|C_\infty^m(0,\infty)$ is an injection at high gains as it can be approximated by

$$\begin{aligned}\dot{q}(t) &= -\operatorname{diag}\{\frac{k_j c_j}{k_j+c_j}\}_{1\leq j\leq m} q(t) + \operatorname{diag}\{\frac{1}{k_j+c_j}\}_{1\leq j\leq m} A_0^{-1} u(t) \\ &= e(t)\end{aligned} \tag{92}$$

Moreover, it is trivially verified that

$$\lim_{k \to +\infty} ||K^{-1}||_{\infty} = 0 \qquad (93)$$

(ii) The conditions (85) ensure that the approximate closed-loop system is stable and hence that $L_{cA}|C_{\infty}^{m}(o,\infty)$ is bounded and causal.

(iii) Using proposition 9 and (i) and (ii) it is clear that each term in $L_{cA}K^{-1}A_oH$ represents a stable, causal system and hence that its restriction to $C_{\infty}^{m}(o,\infty)$ is injective, causal and bounded.

(iv) Finally the theorem will follow if we prove that $||L_{cA}K^{-1}A_oH||_{\infty} \to o$ as $k \to +\infty$. But the approximation (86) to (84) at high gains indicates that there exists $k^* \geq o$ such that (48) holds for $k > k^*$ and hence that $||L_{cA}K^{-1}A_oH||_{\infty} \leq c_1 ||K^{-1}||_{\infty}. ||A_oH||_{\infty} \to 0(k \to +\infty)$ by (93).

In applications situations the result guarantees that the proposed controller is capable of generating excellent responses from the unknown plant if enough controller gain is available. The result cannot provide any information on the gains required - this is to be discovered during the tuning phase and is an inevitable consequence of our assumed ignorance of plant dynamics! It does indicate however that increasing the gain can only improve system responses and, in many (but not necessarily all) practical applications, it is anticipated that the required controller gains will be quite modest in the sense that k^* will be small. In general there is no easy technique for estimating k^* unless a detailed model of the plant is known. In the next, section, however, a class of unknown processes are identified for which k^* can be computed from the steady state characteristics of the open-loop real and approximate plants only. The analysis parallels that of section 2.5.

3.5 Unknown Systems with Monotonic Feedback Errors

As in section 2.5 we can compute an upper bound λ' for the contraction constant $\lambda = ||L_{cA}K^{-1} A_oH||_{\infty}$ if H has a monotocity property. More precisely, note that Definition 1 makes sense for continuous-time systems and Proposition 6 becomes:

Proposition 11: The continuous system

$$\dot{x}(t) = Ax(t) + B\,u(t)\ ,\ y(t) = C\,x(t) + D\,u(t) \tag{94}$$

is monotonic and sign-definite iff the impulse response matrix

$$I(t) = C\,e^{At}\,B + D\,\delta(t) \tag{95}$$

has the property that each element is either positive or negative for all $t \geqslant o$..

Proof: Follows as in proposition 6 by writing

$$y(t) = \int_o^t I\,(t-s)\,u(s)\,ds \quad , \quad t \geqslant o \tag{96}$$

The important application of the result is in the proof of the following equivalent to proposition 7:

Proposition 12: With the notation of proposition 11, denoting $G(s) = C(sI-A)^{-1}B + D$ and assuming that the system is stable, monotonic and sign-definite, then the operator $\mathcal{G}$ in $C_\infty^m(o,\infty)$ induced by the system has norm equal to $||G(o)||_m$.

Proof: Follows as in proposition7'replacing summations by integrals' and noting that

$$G(o) = \int_o^\infty I(t)\,dt \tag{97}$$

the integral existing by stability.

The following theorem is the analogue of theorem 2 and identifies a class of multivariable plant for which transient data alone is sufficient to design the controller and guarantee the stability and performance of the real closed-loop system.

Theorem 4: An unknown continuous multivariable system (1) which is minimum-phase with CB nonsingular and such that the feedback operator H is monotonic and sign-definite will be stable in the presence of unity negative feedback with forward path controller (82) if

(i) the tuning parameters $k_j, c_j, 1 \leqslant j \leqslant m$, satisfy (85) with $k_j > c_j$, $1 \leqslant j \leqslant m$,

(ii) the controller K(s) is invertible and minimum phase, and

(iii) the following inequality is satisfied

$$\lambda' \triangleq \| A_o^{-1}(G^{-1}(0) - A_1) \|_m \max_{1\leqslant j\leqslant m} \gamma(k_j, c_j) < 1 \tag{98}$$

where $\gamma(k,c)$ is an upper bound for the norm of a minimal realization of $s/(s+k)(s+c)$ in $C_\infty^m(o,\infty)$. For example $\gamma(k,c) = 2/(k-c)$.

Moreover, under these conditions, equation (89) provides an upper bound on the approximation error due to the design of K based on the approximate model.

(Note: In terms of the notation of theorem 3, equation (98) yields on estimate of k^*, namely the upper bound

$$k^* \leqslant \max_{1\leqslant j\leqslant m} c_j + 2 \| A_o^{-1}(G^{-1}(o) - A_1) \|_m \tag{99}$$

that is easily computed from transient data and anticipated integral action)

Proof: The proof is similar to that of theorem 3 but uses an explicit expression for an upper bound λ' for λ i.e. $\lambda' \geqslant \| L_{cA} K^{-1} A_o H \|_\infty . \| H \|_\infty \geqslant \| L_{cA} K^{-1} A_o H \|_\infty = \lambda$. Firstly note that $L_{cA} K^{-1} A_o$ can be represented by the transfer function matrix

$$(I_m + G(s)K(s))^{-1} G(s) A_o = (G^{-1}(s) + K(s))^{-1} A_o$$

$$\equiv \text{diag} \left\{ \frac{s}{(s+k_j)(s+c_j)} \right\}_{1\leqslant j\leqslant m} \tag{100}$$

and hence by the diagonal impulse response matrix

$$I_G(t) = \text{diag}\left\{\frac{k_j}{k_j-c_j} e^{-k_jt} - \frac{c_j}{k_j-c_j} e^{-c_jt}\right\}_{1\leqslant j\leqslant m} \tag{101}$$

It is clear therefore that

$$||L_{CA}K^{-1}A_o||_\infty = \max_{1\leqslant j\leqslant m} \int_o^\infty |(I_G(t))_{jj}|dt$$

$$\leqslant \max_{1\leqslant j\leqslant m} \int_o^\infty \left\{\frac{k_j}{k_j-c_j} e^{-k_jt} + \frac{c_j}{k_j-c_j} e^{-c_jt}\right\}dt$$

$$\leqslant \max_{1\leqslant j\leqslant m} \frac{2}{k_j-c_j} \tag{102}$$

Also note that $||H||_\infty = ||A_o^{-1}(G^{-1}(o) - A_1)||_m$ as Laplace transformation of (27) and (28) leads to the identity

$$G(s) = (I_m + G_A(s)\ A_oH(s))^{-1}\ G_A(s) \tag{103}$$

and hence (c.f. (65))

$$H(s) = A_o^{-1}(G^{-1}(s) - G_A(s)) \tag{104}$$

yielding the required value for $||H||_\infty$ by Proposition 12 noting that the minimum phase assumption guarantees the stability of H by Proposition 10. This completes the proof of the theorem.

In practice the application of the result is straightforward if it is known that H is monotonic and sign-definite. If there are no a priori grounds for supposing this to be the case then the engineer must either proceed with the assumption of monotonicity and using (99) (or some other estimation of k^*) as a guide to the required gains or he must use a systematic technique for checking for the existence of the property. This does not appear to be quite as straightforward as in the discrete case (see section 2.5) as the basic equation (c.f.(69))

$$y - y_A = G_A\ A_o\ H\ y \tag{105}$$

relating open-loop approximate and real plant requires contains the 'low-pass filter' G_A and hence, in principle, requires differentiation of data to deduce H. This could be a problem if the transient data is noise-contaminated. It would appear to be necessary therefore to attempt identification of H by other approximate, numerical means. This problem will be the topic of future studies.

Finally, note that the results of Proposition 9 carry through to the continous case with the condition $|C\Delta| \neq 0$ replaced by $|CB| \neq 0$. It is of interest to note that the monotonicity and sign-definiteness property of H can be guaranteed in this case by suitable choice of A_1.

4. Illustrative Examples

4.1 A single-input/single-output Discrete System

In this section we illustrate the application of the ideas above by detailed analysis of the system model.

Consider the unstable, minimum-phase continuous system

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \qquad (106)$$

subjected to synchronized piecewise constant input and output sampling of frequency $h^{-1} = 20$. The induced discrete model takes the form

$$x_{k+1} = \begin{pmatrix} 1.05 & 0 \\ 0 & 0.86 \end{pmatrix} x_k + \begin{pmatrix} 0.05 \\ 0.047 \end{pmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k \quad , \quad k \geq 0 \qquad (107)$$

with the natural identification of Φ, Δ and C and has poles at the points 1.05, 0.86 and one zero at the point 0.95 inside the unit circle. The approximating first order lag that matches the high and low frequency plant behaviour is obtained from equations (11),(13) to be defined by the data B_o = 10.31 and the inverse transfer function

$$G_A^{-1}(z) = (z-1)B_o + B_1$$

$$= (z-1)10.31 + B_1 \tag{108}$$

where we will leave our choice of B_1 free for the moment as, using Proposition 9, it is clear that we can choose B_1 to ensure that H is both monotonic and sign-definite. Using (65), the transfer function of H is

$$H(z) = \frac{(z-1.05)(z-0.86)}{(z-0.95)} - (z-1) - 0.097\ B_1$$

$$= 0.04 - 0.097\ B_1 - \frac{0.005}{z-0.905} \tag{109}$$

which is clearly monotonic and sign-definite if $0.047 - 0.097B_1 \leqslant 0$ i.e. $B_1 \geqslant 0.4$. The required controller is derived from (18) to be

$$K(z) = 10.31\left(1 - kc + \frac{(1-k)(1-c)z}{(z-1)}\right) - B_1 \tag{110}$$

Consider now the data required to satisfy the conditions of theorem 2. Clearly (60) is satisfied if

$$\lambda' = \left|0.097\ (B_1 + 1.5)\right|\gamma(k,c) < 1 \tag{111}$$

and we stand the best chance of satisfying this inequality if B_1 = 0.41 when (111) reduces to

$$\lambda' = 0.185\ \gamma(k,c) < 1 \tag{112}$$

Considering, for simplicity the case of proportional control (i.e. c=1) it follows from (61) we can take $\gamma(k,c) = 1/(1-|k|)$ when (112) reduces to the constraint

$$|k| < 0.815 \tag{113}$$

Remembering that k is simply the pole of the approximate first order model with the designed controller, we will choose k = o to attempt to obtain a deadbeat response. With this choice λ' = 0.185 and it is trivially verified that the (proportional) controller is minimum phase. The unit step response y_A of the approximate closed-loop system is deadbeat with steady state value 0.96. All conditions of theorem two are clearly satisfied. The real feedback system is hence stable and applying the estimate (39) of the error involved in the use of the approximation, it is seen that the error involved in using the approximate model to predict the closed-loop response is less than 24% of the approximate response. This value is disappointingly large but is not pessimistic as can be seen from the exact responses shown in Fig. 2.

4.2 A Multi-input/multi-output Continuous System

In this section we apply the results of section 3 to design a proportional plus integral level controller for the three tank system illustrated in Fig. 3. Although we will use a process model to illustrate results it will be clear that the designed controller could have been obtained (in the manner outlined in Section 3) from graphical analysis of plant step response data.

Using the data shown on the figure (where a_i denotes the cross-sectional area of tank i and the β's are orifice resistances (head/flowrate)) the system is assumed to be described by the 2x2 linear state-variable model

$$\dot{x}(t) = \begin{pmatrix} -0.5 & 0.17 & 0.0 \\ 0.25 & -1.75 & 1.0 \\ 0.0 & 2.0 & -3.0 \end{pmatrix} x(t) + \begin{pmatrix} 0.33 & 0.0 \\ 0.0 & 0.5 \\ 0.0 & 1.0 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t) \qquad (114)$$

which clearly has only stable real poles and the single zero at $s = -2.75$. The system is hence minimum phase and $CB = \mathrm{diag}\{0.33, 1\}$ is nonsingular!

Following the ideas of section 3.2 we construct a first order approximate model of open-loop system dynamics of the form of (74) with (see (76))

$$A_o = (CB)^{-1} = \begin{pmatrix} 3.0 & 0 \\ 0 & 1.0 \end{pmatrix} \tag{115}$$

and, choosing A_1 to ensure that real and approximate plant have identical steady state characteristics, with

$$A_1 = G^{-1}(o) = -(CA^{-1}B)^{-1} = \begin{pmatrix} 1.47 & -0.47 \\ -0.18 & 1.18 \end{pmatrix} \tag{116}$$

The real and approximate system open-loop step responses are compared in Fig. 4.

The next step is the design of a control system for G_A. Initially, suppose that a proportional control system is to be designed of the form of (82) with $c_1 = c_2 = o$. From (84) the resultant transfer function matrix of the approximate closed-loop system is

$$(I + G_A K)^{-1} G_A K = \begin{pmatrix} \frac{k_1}{s+k_1} & 0 \\ 0 & \frac{k_2}{s+k_2} \end{pmatrix} \begin{pmatrix} 1 - k_1^{-1}0.49 & k_1^{-1}0.16 \\ k_2^{-1}0.18 & 1 - k_2^{-1}1.18 \end{pmatrix} \tag{117}$$

and the predicted response in loop i is first order with time constant k_i^{-1}. Suppose, for simplicity that we wish to have time constants of $k_1^{-1} = 0.6$ in loop one and $k_2^{-1} = 0.3$ in loop 2 corresponding to an increase in system response speed of the order of five times. Equation (117) then predicts that interaction effects and steady state errors in response to unit step demand in loop one (resp. two) are 5% and 30% (resp. 10% and 35%) respectively. We will remove these steady state errors by introducing integral action into the controller (82) by choosing $c_1 = 0.4$ and $c_2 = 0.8$ to produce loop reset times of approximately four times the loop time constant. The step responses of the resulting approximate feedback systems are shown in Fig. 5 and take the form expected from the analysis.

Returning now to the design of the two-term controller for the real plant, theorem 3 can be invoked to indicate that, as our chosen tuning parameters satisfy (85), the use of the controller designed above will guarantee the stability of the closed-loop system provided that our chosen gains are sufficiently high. Moreover, if this is the case, (90) then indicates that the responses of the closed-loop system will be very close to those predicted by the approximate model. At this stage of the design process a model of the plant (if known) could be used to assess stability by Nyquist methods (Owens 1978, MacFarlane 1980) or simple calculation of the closed-loop system poles. We are primarily interested in this paper, however, in the case when the plant model is unknown. In this situation, theorem 3 simply implies the existence of 'gains' k_1 and k_2 assuming stability and states that, if a given choice of gains does not produce stability and good response characteristics, the situation can always be improved (on-line if necessary!) by increasing these gains. For the particular choice of parameters given above, stability and adequate transient characteristics are obtained as illustrated in Fig. 6. Note that $y_A(t)$ was an excellent prediction of the overall form of $y(t)$!

5. Conclusions

The paper has identified a large class of multivariable discrete and continuous process plant for which a simple multivariable first order lag model of plant dynamics (Owens 1978, 1979), obtained from a known plant model or, when a plant model is not known, simply from graphical analyses of open-loop step response data, is sufficient information to enable the design of stable, high-performance, low-interaction feedback control schemes based on widely-used proportional or proportional plus integral control elements. One of the key points of the theory is that the controller structure is easily obtained leaving the required performance to be obtained by (possibly on-line!) tuning of a few loop parameters. A guide to the required parameters is, however, easily obtained by analysis of the first order approximate model and the main theorems of the paper guarantee conditions under which suitable tuning parameters exist. There are clear connections between the structure of these results and those of Davison (1976), Astrom (1980) and Penttinen and Koivo (1980) in that conditions for the stability and tracking properties of the closed-loop system are obtained. In our paper however, we use different controller-structures and obtain the added bonus that explicit information on the form of the closed-loop transient performance is obtained in terms of the performance of the approximating first order feedback system.

Finally, the theoretical basis of the techniques takes the form of theorems 1 and 3 guaranteeing the existence of suitable tuning parameters and cannot provide computable estimates of these parameters in general. It has been demonstrated in theorems 2 and 4, however, that a generalization of the notion of monotonicity used by Astrom (1980) is an important property of the modelling error that enables these results to be refined in this direction.

6. Acknowledgements

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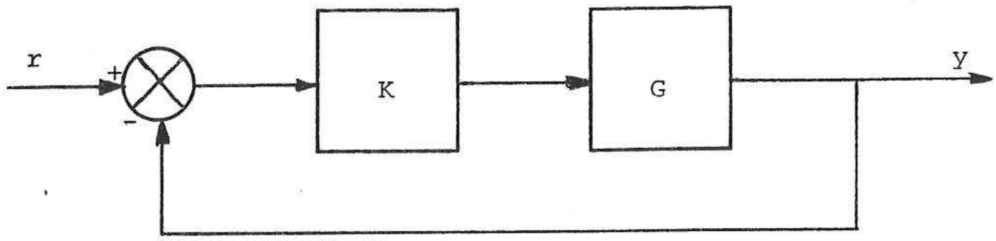
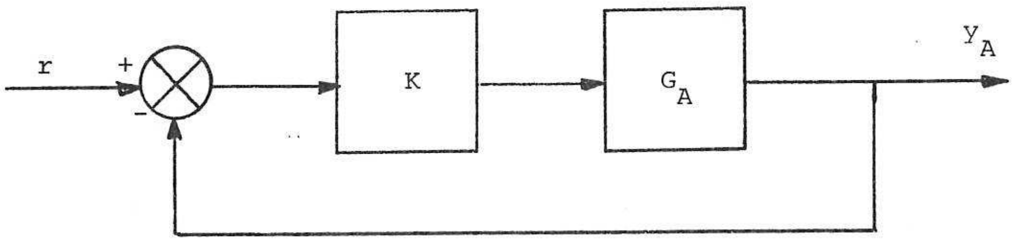


(a)



(b)

Fig. 1. <u>Real (a) and Approximating (b) Feedback Systems</u>

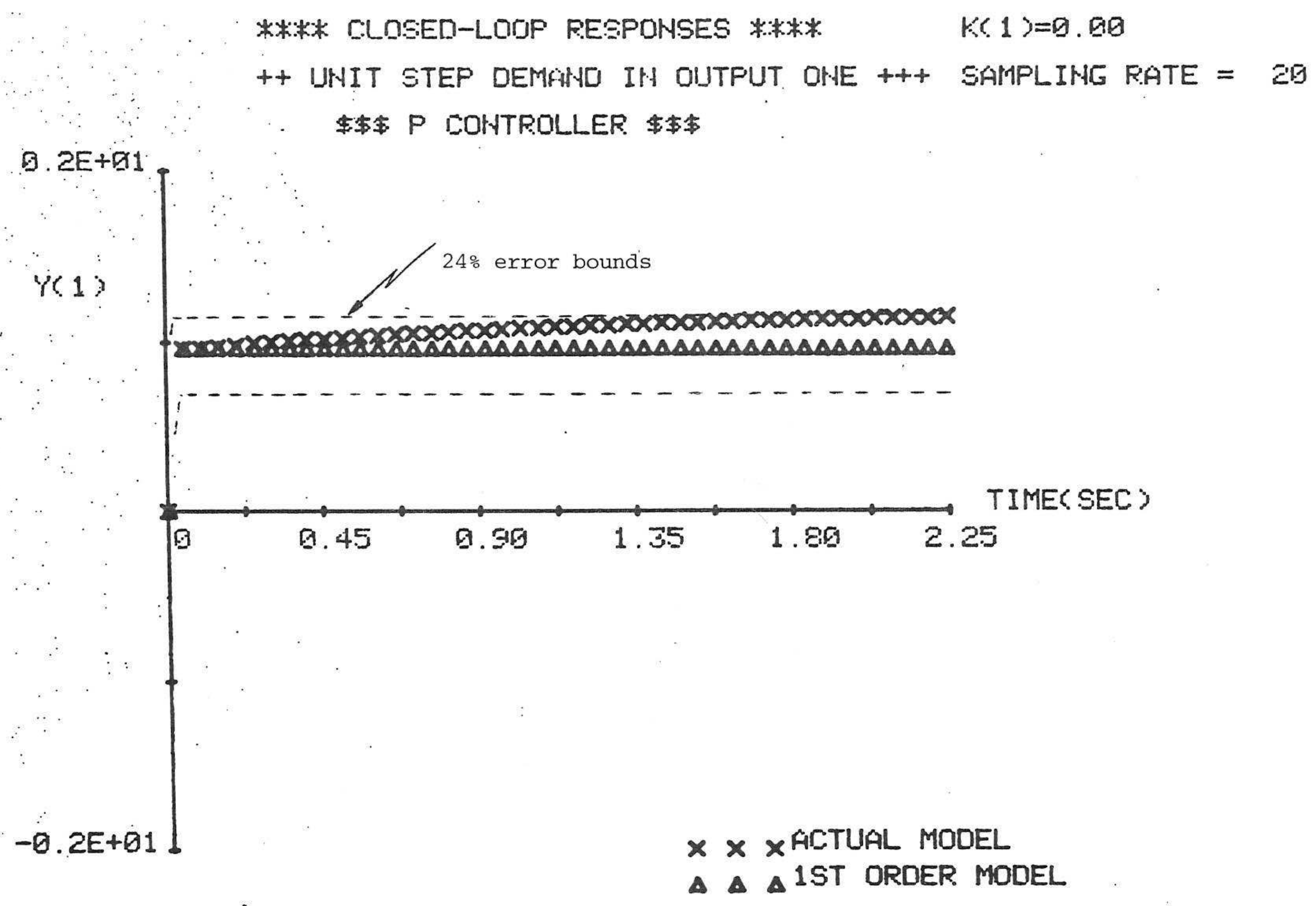


Fig. 2. Closed-loop Responses to Unit Step Demand

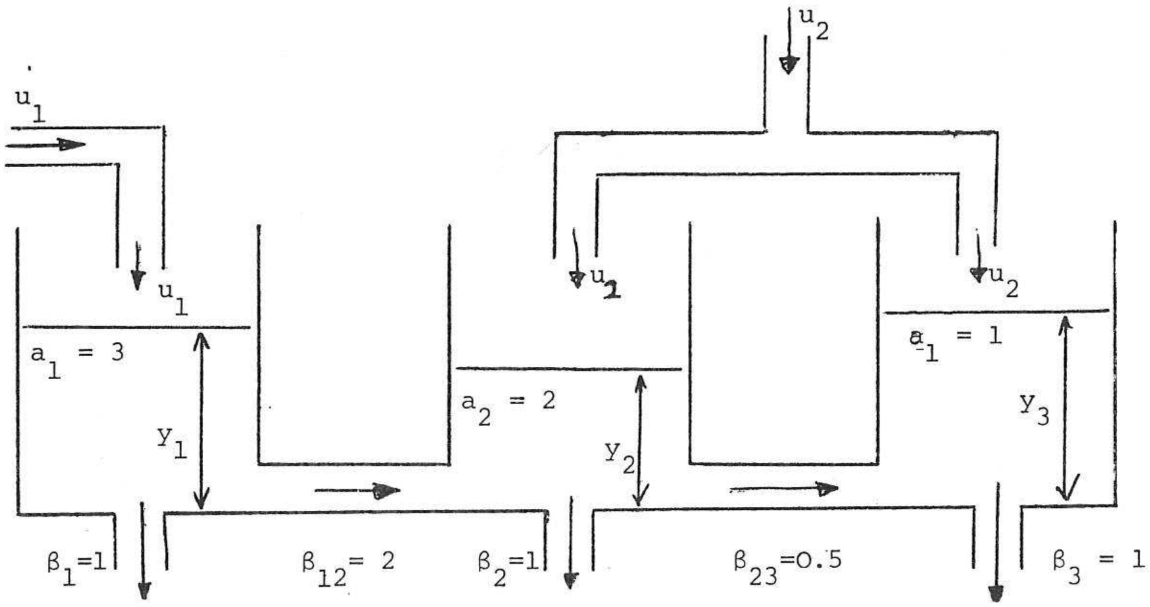


Fig. 3. Liquid Level Systems

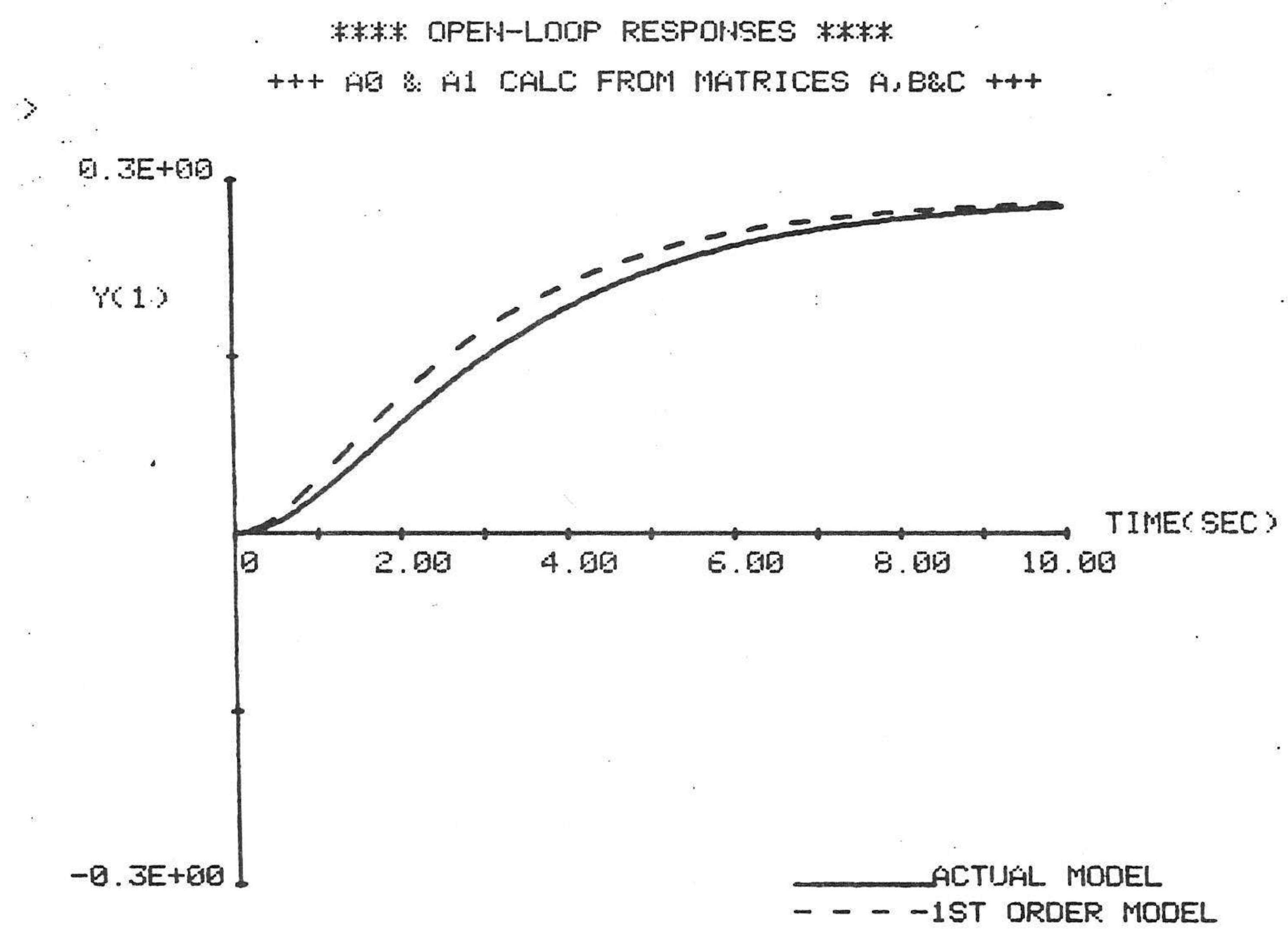


Fig. 4(a) Open loop response of $y_1(t)$ to unit step input in $u_2(t)$

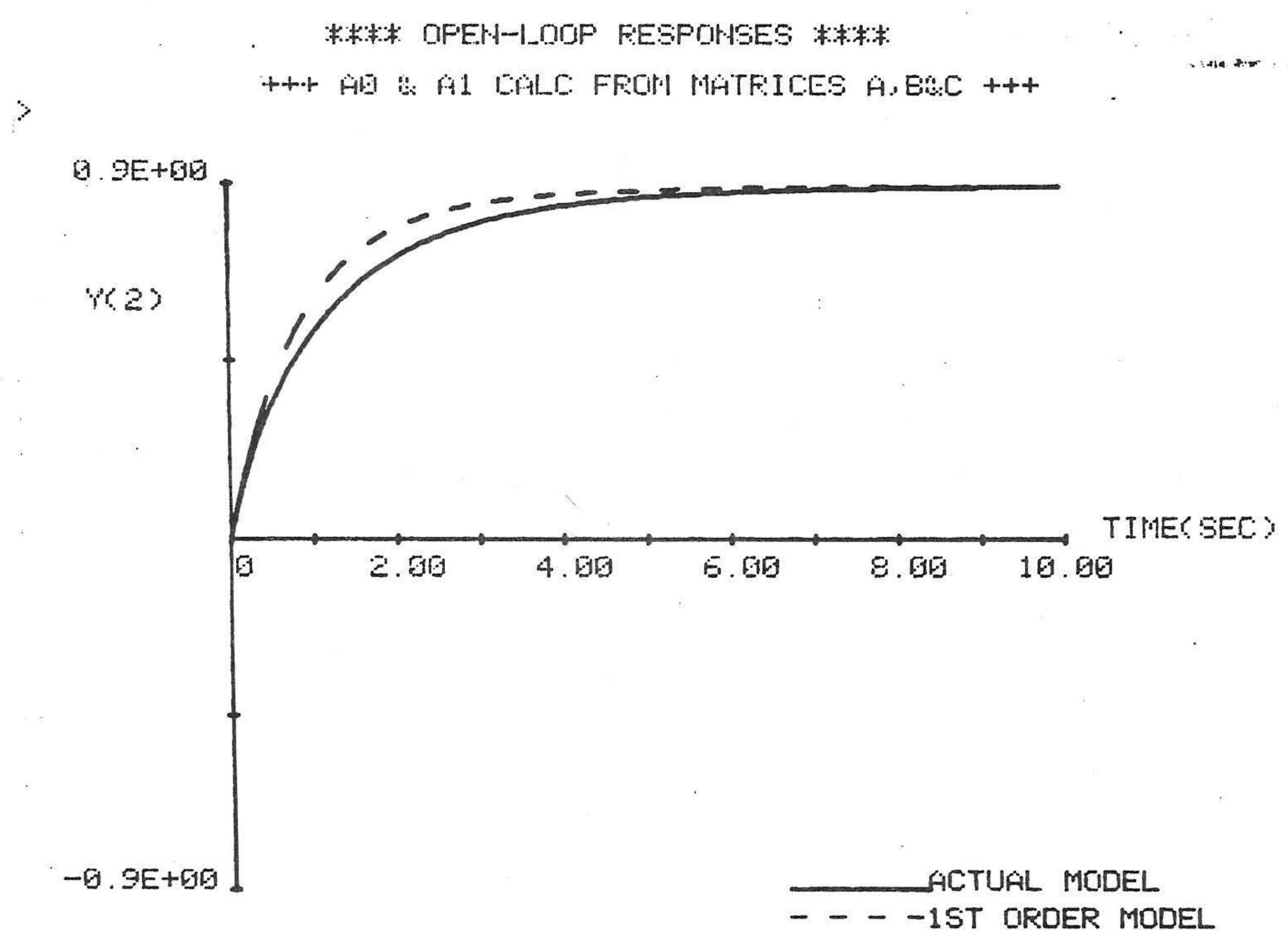


Fig. 4(b) Open loop response of $y_2(t)$ to unit step input in $u_2(t)$

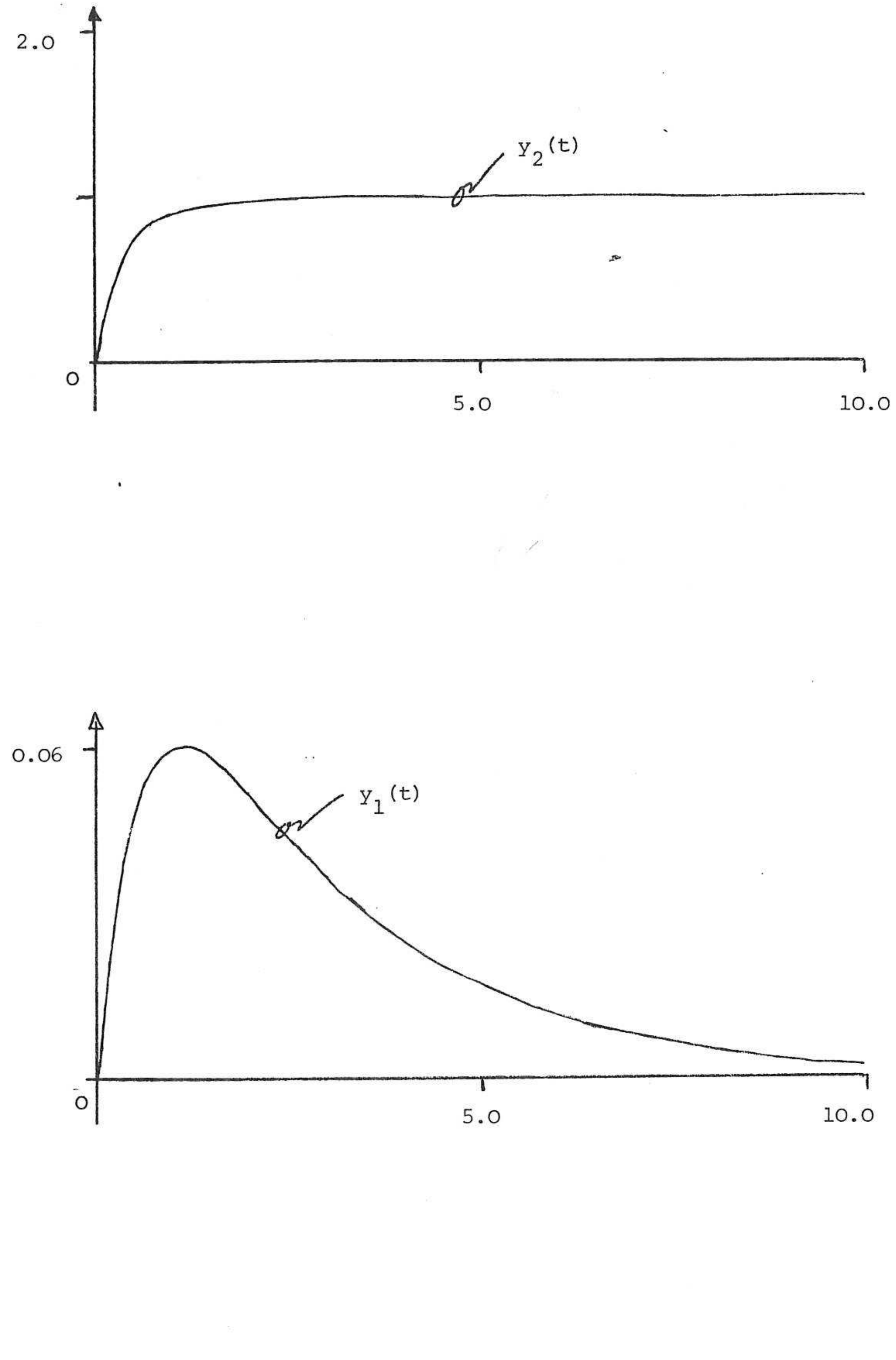


Fig. 5 Response of Approximating Closed loop Systems to a Unit Step Demand

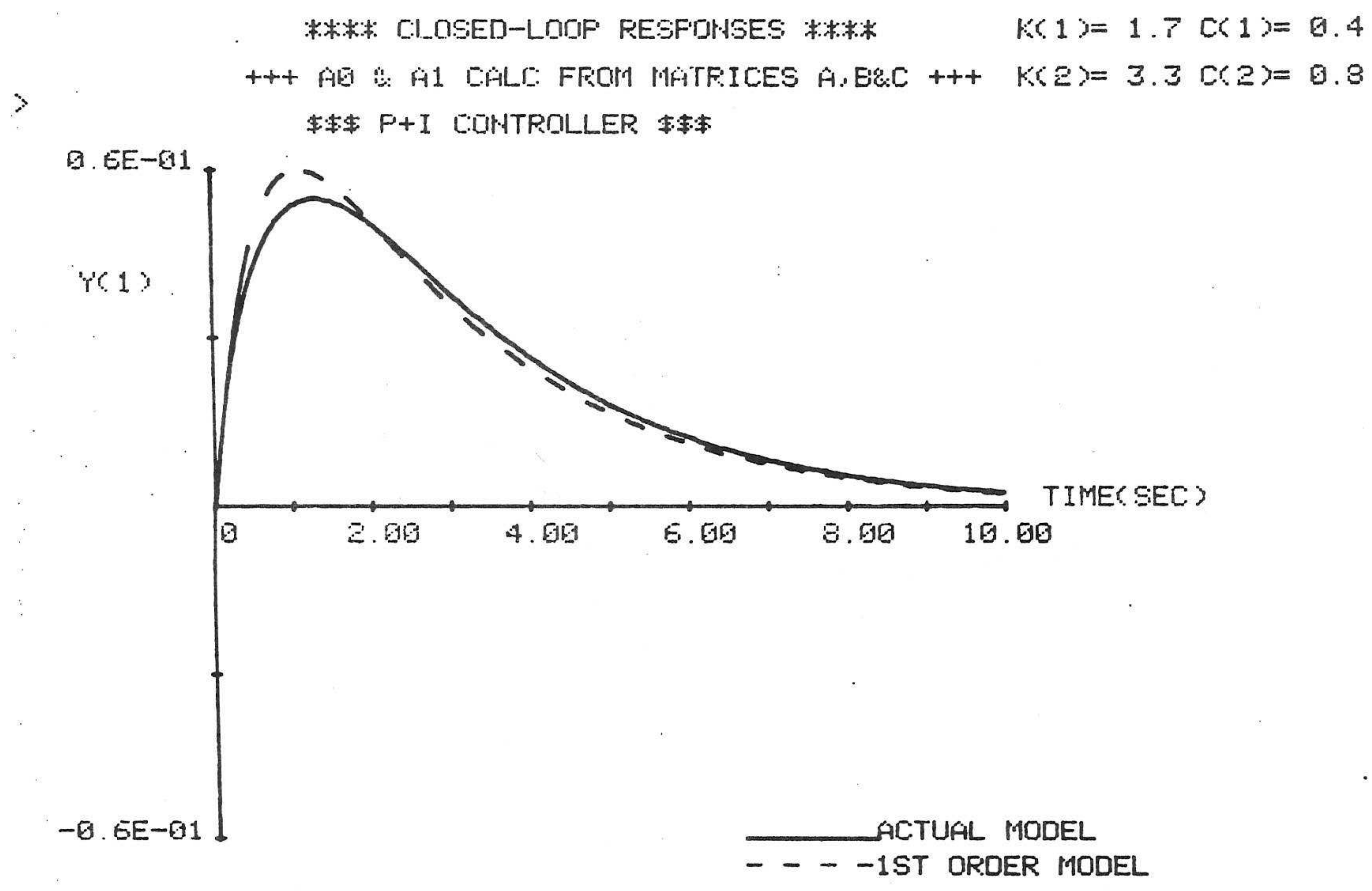


Fig. 6(a) Closed loop Responses to a Unit Step Demand in $y_2(t)$