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ON OPTIMAL CONTROL OF PLASMA CONFINEMENT

by

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ABSTRACT

The optimal control of plasmas is considered by minimizing a quadratic cost functional subject to Vlasov's equation. The infinite dimensional Riccati equation is obtained and a finite dimensional basis representation is considered.

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1. Introduction

Many attempts have been made, both in theory and in experiment, to stabilize plasma instabilities by feedback control techniques. In the theoretical approach Arsenin and Chynov (1968) considered the stabilization of the flute mode, with sensing and suppressing carried out from surfaces outside the plasma, while Furth and Rutherford (1969) presented a technique to suppress drift modes by modulated electron surfaces. The theory of feedback stabilization of various plasma instabilities has also been investigated by many other authors (Chu and Hendel, 1970), including Lowder and Thomassen (1973) who described models of kink modes and their stabilization conditions, and Kammash and Uckan (1975) who considered the use of (ϕ-ϕ) feedback control where the control variable is the local potential perturbation; they examined this method for some instabilities of mirror machines. We may also mention the work of Sen and Sunderman (1976) who investigated the feedback control of dissipative trapped-particle instabilities by neutral beam injection.

The major problem in the above work is the fact that the feedback control components are frequency independent (constant phase shifts) and so can stabilize one mode only and may destabilize some other modes. Melcher (1965, 1966), and Lindgren and Birsdall (1970) have suggested the use of more than one suppressor pairs for multimode stabilization. This technique is obviously not desirable. Using frequency-dependent feedback control and one sensor/suppressor pair, Taylor and Lashmore-Davies (1970) have given a general theory for stabilizing electrostatic instabilities leading to the conditions for stabilizing reactive and dissipative instabilities. Lashmore-Davies (1971) has considered the effects of the above method on a flute-type instability occurring in a low-density plasma confined in a simple magnetic mirror. In fact he only examined the effects of fixed phase shifts, time derivative and time integral controllers. Sen (1974) developed a method of using a suitable frequency-dependent feedback and considered its application to multimode weak reactive and dissipative instability. Using one sensor/suppressor pair, Sen (1975a) has also shown that most plasma instabilities of a discrete spectrum are observable and controllable. He used multivariable control theory considering the discrete normal modes as the states of the system. Frequency-dependent stabilizers have also been studied by Uckan and Kammash (1975) who gave a general theory of stabilization of multimode fluctuations in collisionless magnetized plasma. As an example, they considered (ϕ-ϕ) feedback control of flute-like drift instabilities. However Kitao and Higuchi (1976) have shown that (ϕ-ϕ) feedback with constant phase shift can not stabilize these instabilities. Liberman and Wong (1977) considered axial stabilization of flute mode in a simple mirror reactor. By using proportional and derivative feedback controllers they showed that stability could be obtained over a wide range of phase shifts.

In all the above methods the mathematical tool for stabilization is the dispersion relation of the plasma. The dispersion relation is in fact equivalent to the characteristic equation of a linear lumped system. The eigenvalues of the characteristic equation of the system when it is not externally excited, are the natural frequencies (eigen frequencies) from which the stability of the system can be analysed. To design a controller the knowledge of eigenfrequencies of the system is not sufficient and one also needs to know its transfer function. The transfer function of a system is, by definition, the input-output ratio in the Laplace domain ; its poles are the system's eigenfrequencies and its zeros are the residues. Sen (1975b) was the first to consider the use of transfer functions in plasma systems; he presented a general theory of the determination of plasma transfer functions with boundary and internal excitation. In another paper some general procedures, based on classical control theory and the knowledge of the transfer function, are presented by Sen (1978) to stabilize any plasma instabilities. The synthesis technique for constructing the suitable compensators are also given in this paper. Sen (1979a) studied the feedback control of some important plasma instabilities such as drift waves, flute and trapped particle modes by applying the transfer function method. He discusses their observability and controlability in the sense defined by Sen (1975a) and also the use of feedback control as a diagonastic tool for the measurement of the instability growth rate. State feedback for control of multimode instabilities in plasma was also presented by Sen (1979b). He applies the theory of the state reconstructor to recover all dynamic states of plasma instabilities by a single sensor. The sensor and suppressor are placed in accordance with observability and controlability, respectively.

Feedback stabilization of plasma instabilities has also been the subject of experimental work. Both constant and frequency-dependent controllers have been used and results in successful agreement with theories are obtained. Valuab reports and reviews are given by Chu and Hendel (1970) and by Thomassen (1971). There are also some other reports on this work; such as Brown et al (1971), Keen and Fletcher (1971), Richard et al (1975), Wong and Liberman (1978), and Richard and Emmert (1980). An experimental measurement and theoretical calculati of the plasma transfer function have been given in Richard and Emmert (1977) with results in reasonable agreement with the theory presented by Sen (1975b).

In all the techniques of stabilization mentioned above, the plasmas are considered as lumped systems for certain specific modes. Even when the multi-variable theories are applied to express the plasma dynamics, discrete normal modes are taken as the base states, again using the plasma dispersion relation. In fact plasmas are distributed parameter systems and a more rigorous approach is to write their dynamics in standard state space form using the Vlasov equation. As far as we are aware only limited contributions have been made to this approach. Wang (1969), Wang and Janas (1970), and Wang (1974), have presented different optimal control problems of confining collisionless plasmas by means of external electromagnetic fields. The control in these problems are the external electric and/or magnetic fields, and the states are the distribution functions.

In this paper we shall consider the plasmas as distributed parameter systems and use as our state equation the lowest order approximation to Lionville's equation

$$\frac{\partial F}{\partial t} + \{H, F\} = 0,$$

(which describes the probability $F(x)\,dx$ of finding a plasma particle in the phase-space region $dx = (dx_1, dx_2. dx_3, dv_1, dv_2, dv_3))$ given by the linearized Vlasov equation for a collisionless plasma

$$\frac{\partial f_{1j}}{\partial t} + \underline{v} \cdot \frac{\partial f_{1j}}{\partial \underline{r}} + \frac{q_j}{m_j c} \underline{v} \times \underline{B} \cdot \frac{\partial f_{1j}}{\partial \underline{v}} + \frac{q_j}{m_j c} \underline{E} \cdot \frac{\partial f_{oj}}{\partial v} = 0, \tag{1.1}$$

where $j = e$ for electron and i for ions, f_{1e} and f_{1j} are the respective perturbed distribution functions for these particles species and f_{oj} is the appropriate equilibrium distribution. For a derivation of these equations, see G. Ecker (1972).

Introducing a control term Bu in (1.1), we shall write the controlled plasma dynamics in the standard state-space form

$$\dot{x} = Ax + Bu$$

and construct a quadratic cost functional on an appropriate Hilbert space. The optimal control will then be obtained in a standard way via the Riccati equation and we shall indicate how to obtain reasonable finite-dimensional approximations to the solution.

2. Notations and Terminology

Let X denote a Banach space, and consider the Cauchy problem

$$\frac{dx}{dt} = Ax, \quad x(0) = x_o \in D(A) \tag{2.1}$$

The operator A has domain (denoted henceforth by $D(A)$) which is dense in X and is a closed operator (i.e. its graph $\{(x, Ax) : x \in D(A)\}$ is closed in the product space $X \times X$). Then, under certain conditions it is well known (Yosida, 1974) that there exists an operator-valued map $t \to T(t)$ from R^+ to $\mathcal{L}(X)$ (the space of bounded operators on X) which satisfies

$$T(0) = I \qquad \text{(identity operator)}$$

$$T(t_1 + t_2) = T(t_1)T(t_2) \quad (t_1, t_2 \geq 0),$$

$$\lim_{t \to 0^+} T(t)x = x \qquad \forall\, x \varepsilon X,$$

and

$$Ax = \lim_{t \to 0^+} \frac{1}{t}(T(t) - I)x, \quad x \varepsilon D(A) .$$

T(t) is called the semigroup of operators generated by A and the solution of equation (2.1) is given by

$$x = T(t)\, x_o, \quad \text{for } x_o \varepsilon D(A)$$

The semigroup T(t) essentially replaces the exponential exp (At) for finite-dimensional systems. If B is another closed, densely defined operator such that A + B generates a semigroup S(t) then we say that S(t) is the perturbation of T(t) by B.

In the case of nonautonomous systems, i.e. when the operator A in equation (2.1) depends on t, then one must consider evolution operators, which generalize the transition matrices for finite-dimensional systems. A (mild) evolution operator (Curtain and Pritchard, 1978) is a map

$$U(t,s) : \{(t,s) : 0 \leq s \leq t \leq t_1\} \to L(x)$$

such that

$$U(t,t) = I, \; t \varepsilon [0, t_1]$$

$$U(t,r)\, U(r,s) = U(t,s), \quad 0 \leq s \leq t \leq t_1$$

$U(.,s)$ is strongly continuous on $[s, t_1]$

$U(t,.)$ is strongly continuous on $[0,t]$.

Thus, if A generates a semigroup T(t) and $B(t) : [0, t_1] \to X$ then A + B(t) generates a mild evolution operator.

In this paper, $\Omega \subseteq R^3$ will denote the region of space occupied by the plasma container and $V_c = \{ v \varepsilon R^3 : \|v\| \leq c \}$, i.e. the ball of radius c (the speed of light) in velocity space. We shall consider the subset of

$\Omega \times V_c \subseteq R^6$ of phase-space and $C^o(\Omega \times V_c)$ will denote the space of continuous real-valued functions defined on $\Omega \times V_c$. (It is, of course, a Banach space). $L^2(\Omega \times V_c)$ will denote the (Hilbert) space of real-valued functions under the inner product

$$< f, g >_{L^2(\Omega \times V_c)} = \int_\Omega \int_{V_c} f\, g\, d\underline{z}, \qquad \underline{z} \in R^6$$

We shall also use the direct sum $H_1 \oplus H_2$ of two Hilbert spaces under the inner product

$$< (f_1, f_2), (g_1, g_2) >_{H_1 \oplus H_2} = < f_1, g_1 >_{H_1} + < f_2, g_2 >_{H_2},$$

for $f_i, g_i \in H_i$. Note that, if $\{ e_n \}$, $1 \leq n < \infty$, is an orthonormal basis of both H_1 and H_2, then an element $(f, g) \in H_1 \oplus H_2$ may be represented by

$$\sum_{n=1}^{\infty} < f, e_n > \begin{pmatrix} e_n \\ o \end{pmatrix} + \sum_{n=1}^{\infty} < g, e_n > \begin{pmatrix} o \\ e_n \end{pmatrix}.$$

A basis of $H_1 \oplus H_2$ is then $\begin{pmatrix} e_1 \\ o \end{pmatrix}, \begin{pmatrix} o \\ e_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ o \end{pmatrix}, \begin{pmatrix} o \\ e_2 \end{pmatrix}, \ldots$

This basis will be denoted by

$$\{ \bar{e}_n \}_{n \geq 1}, \text{ i.e. } \bar{e}_n = \begin{cases} \begin{pmatrix} e_{(n+1)/2} \\ o \end{pmatrix}, & n \text{ odd}, \\ \begin{pmatrix} o \\ e_{n/2} \end{pmatrix}, & n \text{ even}. \end{cases}$$

Finally, note that if $T(t)$ is a semigroup on H generated by A, then we shall denote by the same symbol $T(t)$ the semigroup generated by $A \oplus A$ on $H \oplus H$. (i.e. $T(t)$ will also denote $T(t) \oplus T(t)$ when the context is clear). A useful reference for this section is Balakrishnan (1976).

3. Solution of the Linearized Equation

The linearized Vlasov equation governing the uncontrolled collisionless magnetized plasma dynamics has been given above. This equation including the control term is given by

$$\frac{\partial f_{1j}}{\partial t} + \underline{v} \cdot \frac{\partial f_{1j}}{\partial \underline{r}} + \frac{q_j}{m_j c} \underline{v} \times \underline{B} \cdot \frac{\partial f_{1j}}{\partial \underline{v}} + \frac{q_j}{m_j} \underline{E}_p \cdot \frac{\partial f_{oj}}{\partial \underline{v}} = F_j, \tag{3.1}$$

where $\underline{B}$ is the external magnetic field (it is assumed that the magnetic field induced by the plasma is negligible), $\underline{E}_p$ is the electric field induced by the plasma and satisfies the Poisson's equation

$$\nabla \cdot \underline{E}_p = 4 \pi \sum_j q_j \int f_{1j} \, d\underline{v},$$

and $j = e$ or i. F_j is the control term.

In order to solve (3.1) consider first the operator

$$A_j = -\left(\underline{v} \cdot \frac{\partial}{\partial \underline{r}} + \alpha_j \, \underline{v} \times \underline{B} \cdot \frac{\partial}{\partial \underline{v}} \right) , \tag{3.2}$$

where $\alpha_j = q_j/m_j c$, defined on $C^1 (R^6)$.

Written out more fully, (3.2) becomes

$$A_j = - \Big[v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} + \alpha_j \Big\{ (v_2 B_3 - v_3 B_2) \frac{\partial}{\partial v_1} + (v_3 B_1 - v_1 B_3) \frac{\partial}{\partial v_2} + (v_1 B_2 - v_2 B_1) \frac{\partial}{\partial v_3} \Big\} \Big] \tag{3.3}$$

It is well known (see Goursat, 1959, for example) that the local group action defined by the operator (3.3) is given by solving the system of equations

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} = \frac{dv_1}{\alpha_j (v_2 B_3 - v_3 B_2)} = \frac{dv_2}{\alpha_j (v_3 B_1 - v_1 B_3)} = \frac{dv_3}{\alpha_j (v_1 B_2 - v_2 B_1)} = dt .$$

Writing these equations in the form

$$\dot{\underline{z}} = \underline{A}_j \, \underline{z} ;$$

where $\underline{z} = (x_1, x_2, x_3, v_1, v_2, v_3)^T \in R^6$, and

$\underline{A}_j \in \mathcal{L}(R^6, R^6)$ is a matrix defined by

$$\underline{A}_j = \begin{pmatrix} \underline{0}_3 & \underline{I}_3 \\ \underline{0}_3 & \alpha_j \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix} \end{pmatrix}$$

(provided that $\underline{B}$ is uniform), where $\underline{0}_3$ and $\underline{I}_3$ are the 3×3 zero and identity matrices respectively, It is easy to see that the operator A_j given by (3.2) generates a semigroup $T_j(t)$ on $C^o(R^6)$ or $L^2(R^6)$ defined by

$$(T_j(t)f)(\underline{z}) = f(\exp(-\underline{A}_j t)\underline{z}), \quad t \geq 0 \tag{3.4}$$

for all $f \in C^o(R^6)$, or $f \in L^2(R^6)$.

Consider now the region $\Omega \times V_c \subseteq R^6$. Extending functions by zero outside $\Omega \times V_c$ it is clear that we can embed $C^o(\Omega \times V_c)$ in $C^o(R^6)$ and consider the semigroup $T_j(t)$ defined on $C^o(\Omega \times V_c)$. Note however that $T_j(t)\{C^o(\Omega \times V_c)\}$ is not necessarily contained in $C^o(\Omega \times V_c)$ for all $t \geq 0$. However, the plasma is initially contained in a region Ω_1, say such that $\Omega_1 \subseteq \Omega$ and * $\partial\Omega_1 \cap \partial\Omega = \phi$ (i.e. the plasma should not touch the containing vessel). For any function $f \in C^o(R^6)$ we denote by Supp f the <u>support</u> of f defined by Supp $f = \overline{f^{-1}(R - o)}$. It follows from the above remarks that the semigroup (3.4) may be regarded as being defined in $C^o(\Omega \times V_c)$ for all $t \geq 0$ such that

$$\text{Supp } f(\exp(-\underline{A}_j t\, \underline{z})) \subseteq \Omega \tag{3.5}$$

for $f \in C^o(\Omega \times V_c)$, i.e. provided the plasma does not touch the vessel wall. If it is possible to stabilize the plasma with feedback control, condition (3.5) can be guaranteed for all $t \geq 0$, and so we will have a well defined solution on $C^o(\Omega \times V_c)$, (or $L^2(\Omega \times V_c)$).

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* $\partial\Omega \triangleq$ boundary of Ω.

Note that

$$\| T_j(t) \|_{\mathcal{L}(C^o(R^6))} \leq 1 \quad , \quad t \geq 0$$

and if (3.5) holds for all $t \geq 0$, then

$$\| T_j(t) \|_{\mathcal{L}(C^o(\Omega \times V_c))} \leq 1 \quad , \quad t \geq 0.$$

Hence, $T_j(t)$ is a contraction semigroup. A more explicit characterization of $T_j(t)$ may be obtained by noting that

$$\underline{A}_j = \begin{pmatrix} \underline{O}_3 & \underline{I}_3 \\ & \\ \underline{O}_3 & \underline{C}_j \end{pmatrix}$$

where,

$$\underline{C}_j = \alpha_j \begin{pmatrix} o & B_3 & -B_2 \\ -B_3 & o & B_1 \\ B_2 & -B_1 & o \end{pmatrix}$$

Thus,

$$\exp(-\underline{A}_j t) = \begin{pmatrix} \underline{I}_3 & \int_0^t \exp(-\underline{C}_j t)\, dt \\ & \\ \underline{O}_3 & \exp(-\underline{C}_j t) \end{pmatrix}$$

and so

$$(T_j(t)f)\ (\underline{z}) = f\ (\underline{x} + \int_0^t \exp(-\underline{C}_j t)\ \underline{v}\ dt,\ \exp(-\underline{C}_j t)\ \underline{v}) \qquad (3.6)$$

where

$$\underline{z} = (x_1, x_2, x_3, v_1, v_2, v_3)^T \in R^6$$

In the case where the magnetic field is undirectional, say $B_1 = B_2 = o$, then an even more explicit representation of the semigroup may be obtained.

First note that the operator (3.3) reduces to the form

$$A_j = - \left[v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} + \alpha_j B_3 \left(v_2 \frac{\partial}{\partial v_1} - v_1 \frac{\partial}{\partial v_2} \right) \right] ,$$

and it is easy to see that (3.4) becomes

$$(T_j(t)f)\ (\underline{z}) = f\left(\left[\underline{D}\ (x_1, x_2, v_1, v_2)^T ,\ x_3 - v_3 t,\ v_3 \right]_p \right) ,$$

where $\underline{z}_p$ is the vector obtained from $\underline{z}$ by applying the permutation

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 3 & 4 & 6 \end{pmatrix}$$

to its elements and

$$D = \begin{pmatrix} \underline{I}_2 & -\left[\underline{C}_2\ (\cos(\alpha_j B_3 t) - 1) + \underline{I}_2\ \sin(\alpha_j B_3 t) \right] / \alpha_j B_3 \\ \\ \underline{0}_2 & -\underline{I}_2 \cos(\alpha_j B_3 t) - \underline{C}_2 \sin(\alpha_j B_3 t) \end{pmatrix},$$

where $\underline{I}_2$ is the 2×2 identity matrix and

$$\underline{C}_2 = \begin{pmatrix} 0 & 1 \\ \\ -1 & 0 \end{pmatrix} .$$

Hence,

$$(T_j(t)f)\ (\underline{z}) = f \left(\begin{bmatrix} x_1 - \dfrac{S(t)v_1}{\alpha_j B_3} - \dfrac{(C(t)-1)v_2}{\alpha_j B_3} \\[2ex] x_2 + \dfrac{(C(t)-1)v_1}{\alpha_j B_3} - \dfrac{S(t)v_2}{\alpha_j B_3} \\[2ex] x_3 - v_3 t \\[1ex] -C(t)v_1 - S(t)v_2 \\[1ex] S(t)v_1 - C(t)v_2 \\[1ex] v_3 \end{bmatrix}^T \right) , \qquad (3.7)$$

where $C(t) = \cos\ (\alpha_j B_3 t)$, $S(t) = \sin(\alpha_j B_3 t)$.

Consider finally that the term $\frac{q_j}{m_j} \underline{E}_p \cdot \frac{\partial f_{oj}}{\partial \underline{v}}$ in the equation (3.1); we have,

$$\nabla \cdot \underline{E}_p = 4\pi \sum_j q_j \int_{\underline{V}_c} f_{1j}\, d\underline{v} \quad ,$$

and so, introducing the electric potential ϕ_p,

$$\nabla^2 \phi_p = 4\pi \sum_j q_j \int_{V_c} f_{1j}\, d\underline{v} \quad .$$

Using Poisson's solution we have

$$\phi_p(\underline{r}) = 4\pi \sum_j q_j \int_{\Omega} \frac{1}{\| \underline{r} - \underline{r}' \|} \int_{V_c} f_{1j}\, d\underline{v}\, d\underline{r}' \quad .$$

Hence

$$\underline{E}_p = \nabla\phi_p = 4\pi \sum_j q_j \int_{\Omega} \underline{\nabla}_r \left(\frac{1}{\| \underline{r} - \underline{r}' \|} \right) \int_{V_c} f_{1j}\, d\underline{v}\, d\underline{r}' , \qquad (3.8)$$

and so

$$\left\| \frac{q_j}{m_j} \underline{E}_p \cdot \frac{\partial f_{oj}}{\partial \underline{v}} \right\|_{C^o(\Omega \times V_c)} \leq \frac{|q_j|}{m_j} \| \underline{E}_p \|_{C^o(\Omega \times V_c)} \left\| \frac{\partial f_{oj}}{\partial \underline{v}} \right\|_{C^o(\Omega \times V_c)}$$

$$\leq C_e \| f_{1e} \|_{C^o(\Omega \times V_c)} + C \| f_{1i} \|_{C^o(\Omega \times V_c)} \qquad (3.9)$$

for some constants, C_e and C_i, since, as it is well-known from elementary potential theory (see, for example, Kellogg, 1965), the integral

$$\int_{\Omega} \underline{\nabla}_r \left(\frac{1}{\| r - r' \|} \right) d\underline{r}'$$

exists. Hence writing $f_1 = (f_{1e}, f_{1i})^T \in C^o(\Omega \times V_c) \oplus C^o(\Omega \times V_c)$ we see that the operator $\overline{Q}$ defined by

$$\overline{Q} f_1 = \left(\frac{q_e}{m_e} \underline{E}_p \cdot \frac{\partial f_{oe}}{\partial \underline{v}} \;,\; \frac{q_i}{m_i} \underline{E}_p \cdot \frac{\partial f_{oi}}{\partial \underline{v}} \right)^T \quad ,$$

where $\underline{E}_p$ is given by equation (3.8), is a bounded operator defined on $C^o(\Omega \times V_c) \oplus C^o(\Omega \times V_c)$.

Note that since (3.8) represents $\underline{E}_p$ as a convolution of $(f_{1e}, f_{1i})^T$ with an L^1 function it is easy to see that $\underline{E}_p$ is also a bounded operator on $L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c)$ with the obvious inner product, i.e.

$$\| \overline{Q} f_1 \|_{L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c)} \leq C_L \quad ,$$

for some constant C_L.

We now note that in the expression (3.6) for the semigroup action, we have that $\underline{C} = - \underline{C}^T$ and so $e^{-\underline{C}t}$ is an isometry; i.e. $e^{-\underline{C}t} e^{-\underline{C}^T t} = \underline{I}$. Hence, T(t) is, in fact, a semigroup on the space $C^o(R^3 \times V_c)$. It is also clear from above that $\underline{E}_p$ defined by (3.8) is also a bounded operator on $C^o(R^3 \times V_c) \oplus C^o(R^3 \times V_c)$ with the same as in (3.9). A standard result in the theory of the perturbation of semigroups (Kato, 1976) now shows that the operator

$$\mathcal{A} = \begin{pmatrix} A_e & 0 \\ 0 & A_i \end{pmatrix} + \overline{Q} \, ,$$

defined on (a dense subspace of) $C^o(R^3 \times V_c) \oplus C^o(R^3 \times V_c)$ or $L^2(R^3 \times V_c) \oplus L^2(R^3 \times V_c)$, generates a semigroup U(t) such that

$$\| U(t) \|_{C^o(R^3 \times V_c) \oplus C^o(R^3 \times V_c)} \leq \exp \left(\| \overline{Q} \| \, t \right).$$

Also,

$$\| U(t) \|_{L^2(R^3 \times V_c) \oplus L^2(R^3 \times V_c)} \leq \exp \left(\omega + C_L t \right),$$

where

$$\omega = \exp \left(\Sigma \, \sigma \right),$$

and $\Sigma\sigma$ represents the sum of the eigenvalues of $\underline{A}_j$ counted with multiplicity. This latter formula follows from the inequality

$$\| T(t) \|_{L^2(R^3 \times V_c)} \leq \exp \left(\omega \, t \right)$$

which is easily derived from (3.4).

Note that the equation (3.1) may be written in the form

$$\frac{df_1}{dt} = \mathcal{A} f_1 + F \ , \tag{3.10}$$

where $f_1 = (f_{1e}, f_{1i})^T \in C^o(R^3 \times V_c) \oplus C^o(R^3 \times V_c)$ and $F = (F_e, F_i)^T$ is the control term. It follows that the solution of equation (3.10) is given by

$$f_1 = U(t)\ f_1(0) + \int_o^t U(t-s)\ F(s)\ ds \ , \tag{3.11}$$

where $f_1(0)$ is the initial distribution. Note that, as stated earlier, if the plasma is to be confined to the region of space Ω, then (3.11) is only valid for t such that

$$\{\text{Supp } f_1(t)\} \cup \{\text{Supp } f_1(0)\} \subseteq \Omega.$$

If this condition is satisfied at time $t = o$, then there will certainly exist a time $T > 0$ such that it is satisfied for $t \in [0, T]$. Hence it is meaningful to consider the control in equation (3.10) in the interval $[0, T]$ for $f_1 \in C^o(R^3 \times V_c) \oplus C^o(R^3 \times V_c)$. All that remains now is to specify the control F and to define the cost functional. These problems will be dealt with in the next section.

4. The Optimal Control Problem

In the present problem control is achieved by placing a probe in the plasma and varying the voltage u applied to the probe. Then the field $\underline{E}_c$ produced by this control is proportional to u and so the control term F in equation (3.10) may be written in the form

$$\left(\frac{q_e}{m_e} \nabla \psi \cdot \frac{\partial f_{oe}}{\partial \underline{v}} \ , \ \frac{q_i}{m_i} \nabla \psi \cdot \frac{\partial f_{oi}}{\partial \underline{v}} \right) u \tag{4.1}$$

for some specified function ψ such that Supp $\psi \subseteq \Omega$). (ψ is the potential in space which results from the voltage u applied to the probe neglecting the plasma electric field. This assumes that superposition of electric fields is valid here). Since u is the (scalar) control, it follows that the control term in equation (3.10) may be written in the form Bu where $B \in \mathcal{L}(C^0(\Omega \times V_c) \oplus C^0(\Omega \times V_c))$ is a bounded operator defined in an obvious way from (4.1). The plasma equation (3.10) now takes the form

$$\dot{f}_1 = \mathcal{A} f_1 + Bu \tag{4.2}$$

which is the standard state space representation of a system for control theoritic purposes.

In order to specify an appropriate cost functional, note that it is desired to keep f_1 "near" to the equilibrium distribution $f_o = (f_{oe}, f_{oi})^T$ and so a reasonable cost functional to choose is

$$J[u] = \langle f_1(T) - f_o, G(f_1(T) - f_o)\rangle + \int_o^T [\langle f_1(t) - f_o, M(f_1(t) - f_o)\rangle + R\, u^2(t)]\, dt, \tag{4.3}$$

where the inner products $\langle.,.\rangle$ are with respect to $L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c)$ and $G, M \in \mathcal{L}(L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c))$. R is a positive real number, and represents a penalty on using too much control. We therefore have a tracking problem defined in the standard form on a Hilbert space $H =$ $L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c)$, the solution of which is given in Curtain and Pritchard (1978). The results will be summarized here for the convenience of the reader. Firstly, we define a sequence of controls of the form

$$u_k(t) = - F_k(t)\, f_1(t) - R^{-1}B^*\, S_{k-1}(t) \tag{4.4}$$ *

where

$$F_k(t) = - R^{-1}B^*\, Q_{k-1}(t) \quad ; \quad F_o(t) = 0$$

$$M_k(t) = M + F_k^*(t) R F_k(t)$$

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* B* is the adjoint or dual operator of B

$$Q_k(t)h = U_k^*(T,t)\ G\ U_k\ (T,t)h$$

$$+ \int_t^T U_k^*(s,t)\ M_k(s)\ U_k(s,t)\ h\ ds$$

for any $h \in L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c)$,

$$S_k(t) = - U_k^*(T,t)\ G\ r\ (T)$$

$$- \int_t^T U_k^*(p,t)\ \left[Mr(p) - (Q_k(p) - Q_{k-1}(p))B\ R^{-1}\ B^*\ S_{k-1}(p) \right] dp,$$

$$S_o(t) = 0,$$

where $U_k(t,s)$ is the perturbation of $U(t)$ by $B\ F_k(t)$, i.e. $U_k\ (t,s)$ satisfies the equation

$$\underline{\dot{\xi}} = (\mathcal{A} + B\ F_k(t))\ \xi \quad ; \quad \xi(0) = \xi_o \in D(\mathcal{A}).$$

As one would expect, the control u_k is of the form of a feedback term and an open loop term. We have the following result:

<u>Theorem 4.1</u> (Curtain and Pritchard, 1978)

$Q_k(t)$ converges strongly (i.e. $Q_k(t)h$ converges for each h) as $k \to \infty$ to a self-adjoint operator $Q(t) \in \mathcal{L}(L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c))$ which is the unique solution of the inner product Riccati equation:

$$\frac{d}{dt} \langle Q(t)h, k\rangle + \langle Q(t)h, \mathcal{A}k\rangle + \langle \mathcal{A}h, Q(t)\ k\rangle + \langle Mh, k\rangle$$

$$= \langle Q(t)BR^{-1}B^*\ Q(t)h, k\rangle \quad \text{on} \quad [0, T] \tag{4.5}$$

$$Q(T) = G,$$

for $h, k \in D(\mathcal{A})$. Furthermore, $S_k(t)$ converges strongly as $k \to \infty$ to the operator

$$S_\infty(t) = - U^*\ (T,t)\ Gf_o - \int_t^T U^*\ (p,t)Mf_o\ dp, \tag{4.6}$$

where $U(t,s)$ is the perturbation of $U(t)$ by $-BR^{-1}B^*Q(t)$.

Finally, the optimal control is given by

$$u_\infty(t) = - R^{-1}B^*Q(t)f_1(t) - R^{-1}B^*S_\infty(t) \tag{4.7}$$

The cost for this control is given by

$$\begin{aligned} J_o\left[u_\infty;\, t_o,\, f(0)\right] &= \langle f(0),\, Q(0)\, f(0)\rangle + \langle f_o,\, Gf_o\rangle \\ &+ \int_0^T \langle f_o,\, Mf_o\rangle\, dp - 2\,\langle f(0),\, S_\infty(0)\rangle \\ &- \int_0^T \langle S_\infty(p),\, BR^{-1}B^*S_\infty(p)\rangle\, dp. \quad \square \end{aligned} \tag{4.8}$$

Now introduce an orthonormal basis $\{\bar{e}_i\}$ of $H = L^2(\Omega \times V_c) \oplus L^2(\Omega \times V_c)$ consisting of elements which belong to $D(\mathcal{A})$. (This is possible since $D(\mathcal{A})$ is dense in H). Thus since $B : R \to H$ and $B^* : H^*\ (=H) \to R$, we can write

$$B(1) = \sum_{i=1}^{\infty} b_i \bar{e}_i \qquad , \qquad B^*(\bar{e}_i) = b_i$$

Hence, from (4.5)

$$\langle \dot{Q}(t)\, \bar{e}_i,\, \bar{e}_j\rangle + \langle Q(t)\, \bar{e}_i, \mathcal{A}\bar{e}_j\rangle + \langle \mathcal{A}\bar{e}_i,\, Q(t)\, \bar{e}_j\rangle + \langle M\bar{e}_i,\, \bar{e}_j\rangle$$

$$= \langle Q(t)BR^{-1}B^*Q(t)\, \bar{e}_i,\, \bar{e}_j\rangle,$$

or

$$\dot{q}_{ij} + \sum_{k=1}^{\infty} q_{ik}\, a_{jk} + \sum_{k=1}^{\infty} a_{ik}\, q_{jk} = R^{-1} \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} q_{ik}\, b_k\, b_\ell\, q_{\ell j} - m_{ij} \tag{4.9}$$

where

$$Q(t)\, \bar{e}_i = \sum_{j=1}^{\infty} q_{ij}\, (t)\, \bar{e}_j \quad ,$$

$$\mathcal{A}\, \bar{e}_i = \sum_{j=1}^{\infty} a_{ij}\, \bar{e}_j \quad ,$$

$$M\, \bar{e}_i = \sum_{j=1}^{\infty} m_{ij}\, \bar{e}_j \quad .$$

The final condition for equation (4.9) is $Q(T) = G$, or

$$q_{ij}(T) = g_{ij} \tag{4.10}$$

In order to solve (4.9) and (4.10) numerically it is necessary to realize, à priori, that one can control only a finite number of modes of the system in practice. Hence we assume that

$$m_{ij} = g_{ij} = 0 \text{ for } i > N \text{ or } j > N, \text{ say.}$$

(In other words, M and G operator in the subspace of H generated by the functions $\{ \overline{e}_i \}, 1 \leqslant i \leqslant N$). It follows from (4.9) and (4.10) that if $i > N$ or $j > N$ then $q_{ij}(t) = 0$ is a solution of these equations. But the equations have unique solution and so (4.9) and (4.10) may be replaced by the system

$$\dot{q}_{ij} + \sum_{k=1}^{N} q_{ik}\, a_{jk} + \sum_{k=1}^{N} a_{ik}\, q_{jk} = R^{-1} \sum_{\ell=1}^{N} \sum_{k=1}^{N} q_{ik}\, b_k\, b_\ell\, q_{\ell j} - m_{ij},$$

for $1 \leqslant i, j \leqslant N$

$$q_{ij}(T) = g_{ij}$$

$$q_{ij}(t) = 0 \quad , \quad \text{for } i \text{ or } j > N \text{ and all } t \in [0, T] .$$

This is a finite-dimensional matrix equation and can be solved off-line for a fixed T by standard methods.

The remaining problem is the calculation of the operator $U(t,s)$ in equation (4.6) for the open-loop term. Now $U(t,s)$ is the perturbation of $U(t)$ by $-BR^{-1}B^* Q(t)$ and $U(t)$ is the perturbation of $T(t)$ by $\overline{Q}$. It follows that $U(t,s)$ is the solution of the equation

$$U(t,s)h = T(t-s)h + \int_s^t T(t-\alpha) \left[\overline{Q} - BR^{-1}B^* Q(\alpha) \right] U(\alpha,s)h \, d\alpha \tag{4.11}$$

for any $h \in H$. The solution of this equation can be written in the form

$$U(t,s) = \sum_{n=o}^{\infty} U_n(t,s), \tag{4.12}$$

where

$$U_o(t,s) = T(t-s)$$

$$U_n(t,s) = \int_s^t T(t-\alpha) \left[\overline{Q} - BR^{-1}B^* Q(\alpha) \right] U_{n-1}(\alpha,s) \, d\alpha, \tag{4.13}$$

$n=1, 2, 3, \ldots$

and convergence of the series (4.12) is in the uniform operator topology. Again, a good approximation to the solution of (4.11) could be obtained by evaluating (off-line) the integral terms (4.13) in the sum (4.12) at a finite number of basis vectors $\{\bar{e}_i\}$, $i \leq 1 \leq N$. Consider, therefore, the operation of $\bar{Q}-BR^{-1}B^* Q(\alpha)$ on $\bar{e}_k$; we have

$$\zeta_k \triangleq (\bar{Q}-BR^{-1}B^* Q(\alpha))\bar{e}_k = \bar{Q}\,\bar{e}_k - BR^{-1}B^* \sum_{j=1}^{N} q_{kj}(\alpha)\,\bar{e}_j$$

$$= \bar{Q}\,\bar{e}_k - R^{-1}\sum_{j=1}^{N} q_{kj}(\alpha)\, B \Big[O(j) < e_{(j+1)/2}\,,\, \eta_e >_{L^2(\Omega \times V_c)} + E(j) < e_{j/2}, \eta_i >_{L^2(\Omega \times V_c)} \Big]$$

$$= \left(\frac{q_e}{m_e} \underline{E}_p (\bar{e}_k) \cdot \frac{\partial f_{oe}}{\partial \underline{v}} \;,\; \frac{q_i}{m_i} \underline{E}_p (\bar{e}_k) \cdot \frac{\partial f_{oi}}{\partial \underline{v}} \right)^T$$

$$- R^{-1} \sum_{j=1}^{N} q_{kj}(\alpha)\, (\eta_e, \eta_i)^T \Big[O(j) < e_{(j+1)/2}\,,\, \eta_e >_{L^2} + E(j) < e_{j/2}\,,\, \eta_i >_{L^2} \Big]$$

where,

$$E(j)\ (O(j)) = \begin{cases} 1 & \text{if } j \text{ is even (odd)} \\ o & \text{if } j \text{ is odd (even)} \end{cases} ,$$

$$\underline{E}_p(\bar{e}_k) = \begin{cases} 4\pi q_e \displaystyle\int_\Omega \underline{\nabla}_r \left(\frac{1}{\| \underline{r} - \underline{r}' \|} \right) \int_{V_c} e_{(k+1)/2}\, d\underline{v}\, d\underline{r}' & , k \text{ odd} \\ 4\pi q_i \displaystyle\int_\Omega \underline{\nabla}_r \left(\frac{1}{\| \underline{r} - \underline{r}' \|} \right) \int_{V_c} e_{k/2}\, d\underline{v}\, d\underline{r}' & , k \text{ even} \end{cases}$$

and

$$\eta_j = \frac{q_j}{m_j} \nabla \psi \cdot \frac{\partial f_{oj}}{\partial \underline{v}} \quad , \quad \text{for } j = e \text{ or } i.$$

Hence, by (4.13),

$$U_o(t,s)\,\bar{e}_k\,(\underline{z}) = \bar{e}_k\,(\exp\,(-\underline{A}t)\,\underline{z})$$

$$U_n(t,s)\,\bar{e}_k\,(\underline{z}) = \sum_{\ell=1}^{\infty} \int_s^t u_{n-1,k\ell}(\alpha,s)\,\zeta_\ell\,(\exp\,(-\underline{A}(t-\alpha))\,\underline{z})\,d\alpha,$$

where,

$$u_{n-1,k\ell}(\alpha,s) = < \bar{e}_\ell, U_{n-1}(\alpha,s)\,\bar{e}_k > .$$

Alternatively, one could substitute $\bar{e}_n$ for h in equation (4.11) and, making the above identifications for the operators $\bar{Q}$, Q and B, solve this integral equation numerically for $U(t,s)\,\bar{e}_n$. For a general $h \in H$, we then have

$$U(t,s)h \simeq \sum_{n=1}^{N} U(t,s)\,\bar{e}_n < h,\bar{e}_n >$$

provided

$$\| h - \sum_{n=1}^{N} \bar{e}_n < h,\bar{e}_n > \|$$

is small.

Once the above calculations are made the control law may be implemented in the way illustrated in Fig. (1).

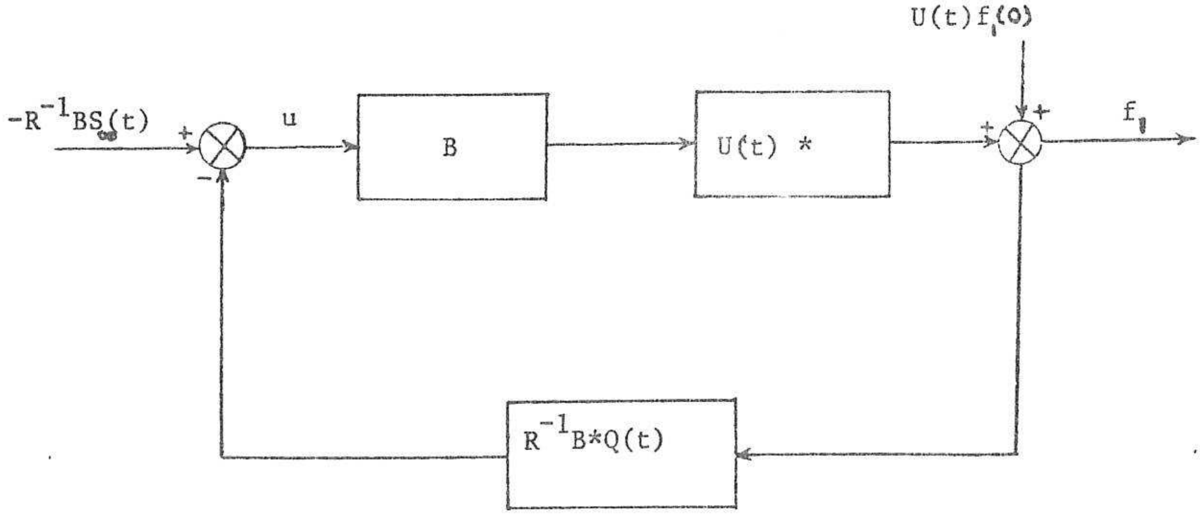

Fig. (1)

5. Conclusions

In this paper we have presented a theoretical study of the optimal control of plasma confinement, and have indicated the way in which the control would be evaluated. The feedback law (4.7) requires, of course, a knowledge of the states (distribution in this case) of the system throughout the phase-space $\Omega \times V_c$, even though we are applying control in a confined region (namely by applying a voltage on a probe). The problem of optimal estimation of the state of the system will be considered in a further paper, since this represents an interesting problem on its own right. The separation theorem can then be used to implement the control in the form of Fig. (2), rather than that of Fig. (1), where C is an observation operator and $\hat{f}$ is the optimal estimate of f and ν is a white noise process.

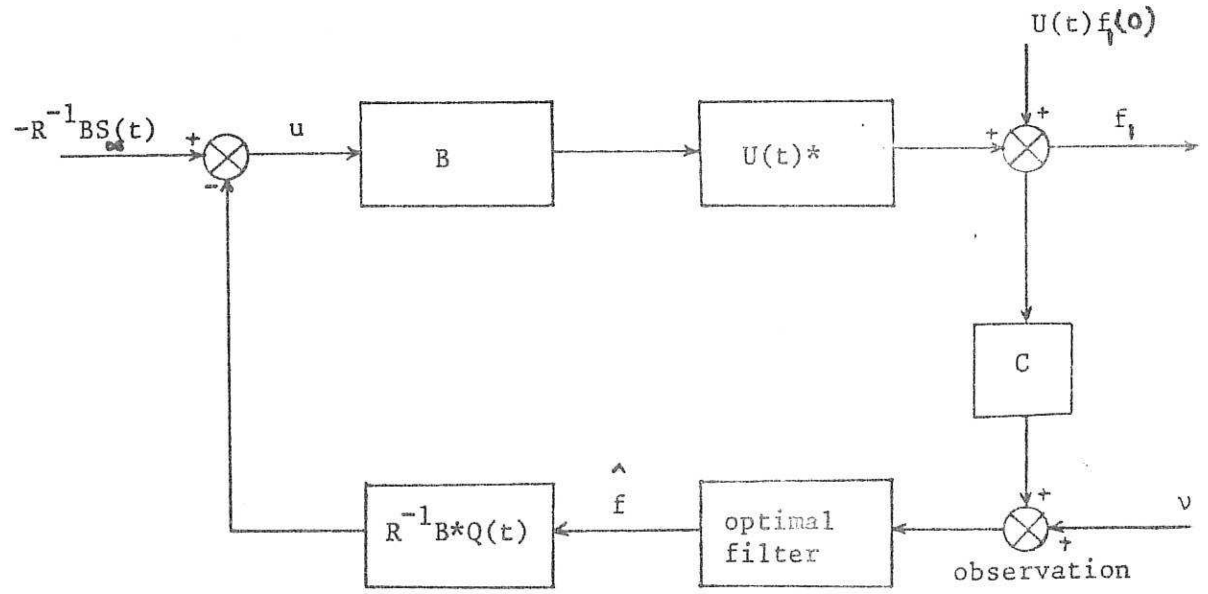

Fig. 2.

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