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THE CIRCLE THEOREM FOR NONLINEAR PARABOLIC SYSTEMS

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Abstract

The circle theorem is generalized to the case of non-linear parabolic systems using an elementary application of the spectral mapping theorem to the semigroup of the system.

1. Introduction

The circle theorem for non-linear systems gives a simple geometrical criterion for the input-output stability of the system. Since the theorem was first discovered by Sandberg (1964) and Zames (1966), it has been generalized by many authors (e.g. Saeks(1970), Freedman et al, (1969)). In the latter case the theory of abstract harmonic analysis is used to derive a Hilbert space version of the theorem. In this paper we shall give an elementary proof for the case of parabolic systems and relate the critical disc directly to the spectrum of the operator (and the boundary conditions) defining the linear part of the system. This is possible by using the theory of analytic semigroups, which may be found in Yosida (1971).

Our notation is the standard notation for Hilbert spaces and semigroup theory. Thus if an operator A with domain D(A) dense in the Hilbert space H satisfies the conditions of the Hille-Yosida theorem (Yodisa 1971), then A is the infinitesimal generator of a semigroup T_t on H.

For our purpose the proof of Sandberg (1964) will be generalized to the Hilbert space setting, and as such will follow very closely this proof. The main difference will be the use of the spectral mapping theorem to relate the spectrum of A to the critical disc.

2. Preliminary Analytical Results

In this paper we shall be concerned with C_o semigroups T_t defined on a Hilbert Space H. The following results will be needed in the succeeding sections and will be given here for the convenience of the reader. Proofs of these results can be found, for example, in Yosida (1971). We shall assume that the reader is familiar with the notion of C_o-semigroup and Hilbert space valued integration. The first result we need connects the resolvent of the infinitesimal generator $R(\lambda;A)$ with the Laplace transform of the semigroup.

Theorem 2.1 The right half plane of the complex λ- plane is in the resolvent set $\rho(A)$ of A and

$$R(\lambda;A)x = (\lambda I-A)^{-1}x = \int_0^\infty e^{-\lambda t}T_t x dt \quad \text{for } Re(\lambda)>0,$$

and $\forall$ $x\varepsilon H$. □

If the spectrum of A is entirely in the open left half plane, then by the Hille-Yosida theorem, T_t is a stable semigroup and it follows from theorem 2.1 by analytic continuation that we have

Corollary 2.2 The Fourier transform of T_t exists and we have

$$R(i\omega;A)x = \int_0^\infty e^{-i\omega t} T_t x dt \quad , \quad -\infty<\omega<\infty. \quad \square$$

We next describe Dunford's integral calculus for bounded linear operators on H. Let B be a bounded operator on H and let $\mathfrak{F}(B)$ be the class of complex functions which are analytic in some neighbouthood of the spectrum $\sigma(B)$ of B. Then, for $f\varepsilon\mathfrak{F}(B)$, we define

$$f(T) = (2\pi i)^{-1} \int_c f(\lambda)R(\lambda;B)d\lambda,$$

where C is the union of a finite number of rectifiable Jordan curves which contain the spectrum of B. Then we have

Theorem 2.3 (Spectral mapping theorem). If $f\varepsilon\mathfrak{F}(B)$, then

$$f(\sigma(B)) = \sigma\ (f(B)). \quad \square$$

Let $L_2(-\infty,\infty;H)$ denote the set of maps $\alpha:(-\infty,\infty)\rightarrow H$ such that

$$\int_{-\infty}^{\infty} ||\alpha(t)||_H^2\, dt < \infty \quad .$$

If $\{e_n\}_{n\geq 1}$ is a basis for H, then there is an isomorphisom

$$i: L_2(-\infty,\infty;H)\rightarrow \bigoplus_{n=1}^{\infty} L_2(-\infty,\infty)$$

defined by

$$i(\alpha)(t) = \{(\alpha(t),e_n)\}_{n\geq 1} .$$

Composing i with F (the Fourier transform, which is also an isomorphism) we can easily obtain the following generalization of the Riesz-Fischer theorem.

Lemma 2.4 If $\alpha\epsilon L_2(-\infty,\infty;H)$ satisfies

$$\int_{-\infty}^{\infty} <\alpha(t),\alpha(t)>_H \, dt < \infty$$

then there exists $\beta\epsilon L_2(-\infty,\infty;H)$ such that

$$F\beta = \alpha . \quad \square$$

Finally we note that if B is a normal operator on H, then the spectral radius of B , r(B), is given by

$$r(B) = ||B||$$

Thus, for a normal operator B,

$$\sup_{||h||=1} < h,Bh > = r(B) .$$

3. The System Equation

The basic system with which we shall be dealing is a differential equation of the form

(3.1) $\dot{x}(t) = Ax(t) - f(x(t))$

where A is an (in general unbounded) operator defined on H and Q is a non-linear map which satisfies

(3.2) $||Qh - \alpha h ||_{L_2(0,\infty;H)} \leq \eta(\alpha)||h||_{L_2(0,\infty;\, H)}$

for any real α and

$$\eta(\alpha) = \max\left[(\alpha-a),(b-\alpha)\right] .$$

(Here, $(Qh)(t) = f(h(t))$ for almost all $t\geq 0$.)

If A generates a semigroup T_t, then we shall consider (3.1) in the 'mild form'

(3.3) $$x(t) = T_t x_o - \int_o^t T_{t-s} f(x(s))\, ds$$

We shall define the operator $K \in \mathcal{L}(L_2(0,\infty;H))$ by

$$(Kh)(t) = \int_o^t T_{t-s} h(s)\, ds$$

for all $h \in L_2(0,\infty;H)$. Thus, we have reduced equation (3.3) to the form

$$g = x + KQx$$

where $g(t) = T_t x_o \in L_2(0,\infty; H)$ if T_t is a stable semigroup. Clearly K is a causal operator and we shall assume that Q is also causal. In the next section we shall prove the basic circle theorem, by mimicking the proof of Sandberg (1964).

4. The Circle Criterion

The first two results we need are stated below and their proofs are formally the same as in Sandberg (1964), except that the norms are replaced by norms in $L_2(0,\infty; H)$.

Theorem 1. Let $x \in \mathcal{D}(Q) \cap L_2^e(0,\infty;H)$ be such that $Qx \in \mathcal{D}(K) \cap L_2^e(0,\infty;H)$,

$KQx \in L_2^e(0,\infty;H)$ and $g = x+KQx$.

where $g \in L_2(0,\infty;H)$. If x_τ denotes the projection of x on $L_2(0,\tau;H)$ and $x_\tau \in \mathcal{D}(Q)$ for $\tau \in (0,\infty)$ and there exists a complex number α such that

(i) $(I+\alpha K)^{-1} \in \mathcal{L}(L_2(0,\infty;H))$ and is causal

(ii) $\|(I+\alpha K)^{-1}K\| \sup_{x_\tau \neq 0} \frac{\|(Qx_\tau)_\tau - \alpha x_\tau\|}{\|x_\tau\|} < 1.$

Then, $x \in L_2(0,\infty;H)$ and

$$\|x\| \leq (1-r)^{-1} \|(I+\alpha K)^{-1} g\|$$

where

$$r = \|(I+\alpha K)^{-1}K\| \sup_{x_\tau \neq 0} \frac{\|(Qx_\tau)_\tau - \alpha x_\tau\|}{\|x_\tau\|}. \quad \square$$

Lemma 2. If $C \in \mathcal{L}(L_2(0,\infty; H))$ is invertible and for $x \in L_2(0,\infty;H)$,

$$CS_\tau x = S_\tau Cx, \quad \tau \geq 0,$$

then C^{-1} is causal; where

$$(S_\tau x)(t) = 0, \quad t \in [0,\tau)$$
$$= x(t-\tau), \quad t \in [\tau,\infty). \quad \square$$

Lemma 3. Let $u \in L_1(0,\infty;\mathcal{L}(H))$ and let $U \in \mathcal{L}(L_2(0,\infty;H))$ be defined by

$$Ux = \int_0^t u(t-\tau)x(\tau)d\tau, \qquad x \in L_2(0,\infty;H) \ .$$

Then if

$$\overline{U}(s) = \int_0^\infty u(t)e^{-st}dt, \mathrm{Re}\, s \geq 0$$

and

$$0 \notin \sigma(I + \overline{U}(s)) \text{ for } \mathrm{Re}\, s \geq 0$$

we have

(i) $(I + U)$ is invertible on $L_2(0,\infty;H)$

(ii) $||(I + U)^{-1}U|| \leq \sup_\omega \Lambda\{[I+\overline{U}(i\omega)]^{-1}\ \overline{U}(i\omega)\}$

Proof. Consider first the operator $I + V$ defined on $L_2(-\infty,\infty;H)$ by

$$(I+V)x = x + \int_{-\infty}^t u(t-\tau)\ x(\tau)d\tau, \qquad x \in L_2(-\infty,\infty;H)$$

Since $u \in L_1(0,\infty;\mathcal{L}(H))$, we see that $||\overline{U}(i\omega)|| \to 0$ as $|\omega| \to \infty$ ($\omega = \mathrm{Im}\, s$). Also, $\overline{U}(i\omega)$ is a continuous map. Now the spectrum of a bounded operator is a closed and bounded set and if $d(\omega)$ is the distance of the spectrum of $I+\overline{U}(i\omega)$ from the origin, then d is continuous, $d(\omega) \neq 0$ and $d(\omega) \to 1$ as $|\omega| \to \infty$. Hence

$$\sup_\omega \Lambda\{[I + \overline{U}(i\omega)]^{-1}\} < \infty.$$

Now let $\hat{g}$ denote the Fourier transform of an arbitrary element $g \in L_2(-\infty,\infty;H)$. Then,

$$\int_{-\infty}^{\infty} <[I+\overline{U}(i\omega)]^{-1}\hat{g}(\omega),\ [I+\overline{U}(i\omega)]^{-1}\hat{g}(\omega)>_H\ d\omega$$

$$= \int_{-\infty}^{\infty} <\hat{g}(\omega),\ [I+\overline{U}(i\omega)]^{-1^*}.\ [I+\overline{U}(i\omega)]^{-1}\ \hat{g}(\omega)>_H\ d\omega$$

$$\leq \int_{-\infty}^{\infty} <\hat{g}(\omega),\hat{g}(\omega)>_H d\omega.(\sup_\omega \Lambda\ ([I+\overline{U}(i\omega)]^{-1}))^2$$

$$< \infty$$

and so, by lemma 2.4, $\exists\ x \in L_2(-\infty,\infty;H)$ such that

$$\hat{x} = [I+\overline{U}(i\omega)]^{-1}\hat{g}.$$

Hence, I+V is invertible. If we consider the matrix representation of $[I+\overline{U}(s)]^{-1}$ in terms of a basis $\{e_n\}_{n\geq 1}$ of H, then, since $\overline{U}(s) \to 0$ as $|s| \to \infty$

uniformly in the closed right half plane we see that each element of $[I+\bar{U}(s)]^{-1}$ in this representation is analytic and uniformly bounded for >0. Thus, (cf. Titchmarsh,1948), $(I+V)^{-1}$ maps

$$\{x : x \varepsilon L_2(-\infty,\infty:H),\ \ x(t)=0 \text{ for } t<0\}$$

into itself, and so I+U is invertible on $L_2(0,\infty;H)$.

The inequality now follows as in Sandberg(1964), by using Parseval's identity in $L_2(0,\infty;H)$. (That this result is valid in $L_2(0,\infty;H)$ can be seen using the isomorphism i in section 2, the inner product in $\overset{\infty}{\underset{n=1}{\oplus}} L_2(-\infty,\infty)$ being given by

$$<\{h_i^1\},\ \{h_i^2\}> \ = \ \sum_{i=1}^{\infty} < h_i^1, h_i^2 >_{L_2(-\infty,\infty)}$$

for $\{h_i^1\},\ \{h_i^2\} \ \varepsilon \ \overset{\infty}{\underset{n=1}{\oplus}} L_2(-\infty,\infty)$. □

We are now in a position to state the main stability result, which can be obtained from theorem 1 using lemmas 2,3 with α in theorem 1 being replaced by ½(a+b).

Theorem 4. Let T_t be a stable semigroup and let

$$g(t) = x(t) + \int_0^t T_{t-s} f(x(s))ds,$$

where $g \varepsilon L_2(0,\infty;H)$ and f (and therefore Q) satisfies condition (3.2). Let R(s;A) again denote the resolvent operator of the generator A of T_t. If,

(i) $0 \notin \sigma\ (I+\frac{1}{2}(a+b)R(s;A))$ for $\text{Re}S \geq 0$

(ii) $\frac{1}{2}(b-a)\sup_{\omega} \Lambda\{[I + \frac{1}{2}(a+b)R(i\omega;A)]^{-1}R(i\omega;A)\}<1$

then $x \ \varepsilon \ L_2(0,\infty;H)$. □

Now note that

$$\lambda \ \varepsilon \ \sigma(A) \quad \text{iff} \quad \frac{1}{s-\lambda} \ \varepsilon \ \sigma(R(s;A)) .$$

Let $\xi_s : \mathbb{C}\backslash\{o\} \to \mathbb{C}$ be the function defined by

$$\xi_s(\lambda) = \frac{1}{s-\lambda}, \text{ for all } s \in \mathbb{C}^+ .$$

By the spectral mapping theorem, we have

(4.1) $$\sigma(I+\tfrac{1}{2}(a+b)R(s;A)) = 1+\tfrac{1}{2}(a+b)\xi_s(\sigma(A))$$

and

$$\tfrac{1}{2}(b-a)\sigma\{[I+\tfrac{1}{2}(a+b)R(i\omega;A)]^{-1} R(i\omega,A)\}$$

$$= \{\tfrac{1}{2}(b-a)[(1+\tfrac{1}{2}(a+b)\xi_{i\omega}(\lambda))^{-1} \xi_\omega(\lambda)]:\lambda\varepsilon\sigma(A)\}$$

assuming $0 \notin \sigma(I+\tfrac{1}{2}(a+b)R(i\omega;A))$, $-\infty<\omega<\infty$.

Let us examine condition (i) in more detail. According to (4.1), condition (i) is satisfied, if, for each $\lambda\varepsilon\sigma(A)$,

$$0 \neq 1+\tfrac{1}{2}(a+b)\xi_s(\lambda) \quad \text{for} \quad \text{Re } s\geq 0 .$$

However, as is well known, this is true if the polar plot of $\xi_{(i\omega)}(\lambda)$ does not encircle or pass through the point $[-2(a+b)^{-1},0]$. Hence. condition (i) is satisfied if

(4.2) $\left\{\right.$ The region traced out by the set valued map $\omega\to\xi_{(i\omega)}(\sigma(A))$ does not contain a curve which encircles or passes through $[-2(a+b)^{-1},0]$.

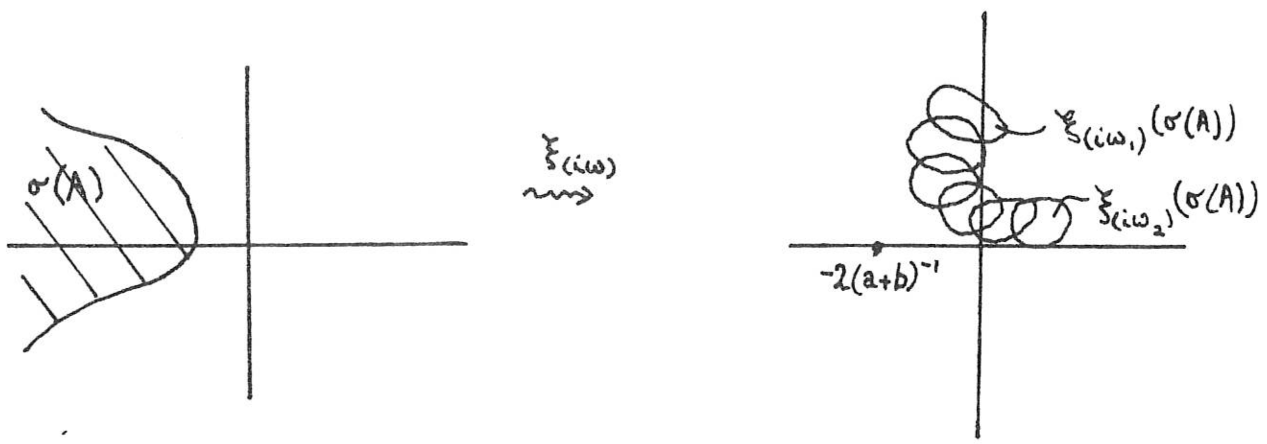

Fig. 1.

Similarly, condition (ii) is satisfied if

(4.3) The region traced out by the set valued map $\omega \to \xi_{(i\omega)}(\sigma(A))$ does not intersect the region $R(a) \subseteq \mathbb{C}$ for $-\infty<\omega<\infty$, where

(a) $R(a)$ = disc of radius $\frac{1}{2}(a^{-1}-b^{-1})$ with centre $\left[-\frac{1}{2}(a^{-1}+b^{-1}),0\right]$ if $a>0$

(b) $R(a)$ = half plane $\operatorname{Re} S \leq -b^{-1}$ if $a=0$

(c) $R(a)$ = exterior of the disc in (a) if $a<0$.

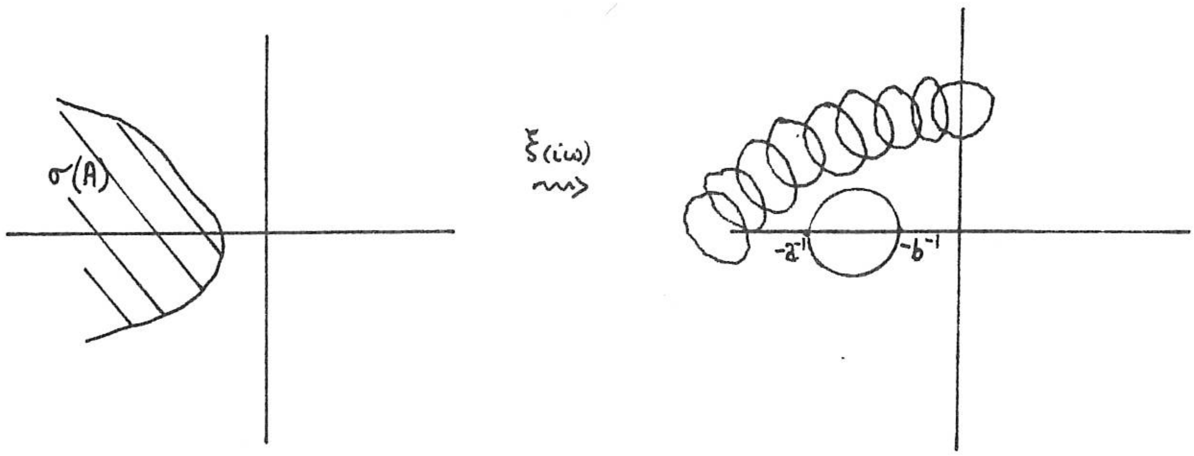

Fig. 2.

We have therefore proved.

Corollary 5. If T_t is a stable semigroup and

$$g(t) = x(t) + \int_0^t T_{t-s} f(x(s))ds$$

where $g \in L_2(0,\infty;H)$, and Q satisfies (3.2). Then if $a>0$ and conditions (4.2), (4.3) or $a \leq 0$ and condition (4.3) is satisfied, then $x \in L_2(0,\infty;H)$. □

5. Examples

We shall now give an example of the theory to the case of a parabolic system

(5.1) $\dot{x}(t) = Ax(t) - f(x(t))$

where A generates an analytic semigroup (cf. Yosida, 1971). In this case the spectrum of A is assumed to lie within a sector

$$S_{d,\phi} = \{\lambda : \pi - \phi < |\arg(\lambda - d)| < \pi \ , \lambda \neq d\}$$

where $d<0$.

We note first that it can be shown by elementary methods that, for each $\omega \varepsilon (-\infty,\infty)$, the function $\xi_{(i\omega)}(\lambda)$ maps $S_{d,\phi}$ into the circle of radius $\frac{1}{2d}$ and centre $(\frac{1}{2d}, 0)$. Also A generates a stable semigroup, and so

$$\int_0^\infty ||T_t x_o||^2 \, dt < \infty$$

for any $x_o \varepsilon H$. Thus, the condition on g in corollary 5 is automatically satisifed. Hence, we have

Theorem 6. If f (or Q) satisfies condition (3.2) and if $a \geq 0$ or $a<0$ and $2d>-a$, then any solution $x(t)$ of (3.2) belongs to $L_2(0,\infty;H)$.

As a specific example of the system (5.1) , consider the partiaal differential equation

$$\frac{\partial z(tx)}{\partial t} = \frac{\partial^2 z(t,x)}{\partial x^2} + pz(t,x) - f\,(z(t,x))$$

where $x \varepsilon [0,1]$, and

$$\frac{\partial z}{\partial x} = 0 \text{ when } x = 0,\ 1.$$

Then the operator

$$Az = \frac{\partial^2 z}{\partial x^2} + p\,z$$

with domain

$$D(A) = \{z \varepsilon L_2(0,1) : \frac{\partial^2 z}{\partial x^2} \varepsilon L_2(0,1), \ \frac{\partial z}{\partial x} = 0 \text{ at } x = 0,1\}$$

generates an analytic semigroup. A has the spectral values

$$\lambda_j = -p - (j-1)^2 \pi^2 \ , \ j \geq 1.$$

Suppose that f satisfies

(5.2) $< f(h(x)) - ah(x),\ f(h(x)) - bh(x) >_{L_2(0,1)} \leq 0,$

for all $h \varepsilon L_2(0,1)$.

Then, we have

$$||Qz-\tfrac{1}{2}(a+b)z\ ||<\tfrac{1}{2}(b-a)||z||$$

for $z \varepsilon L_2(0,\infty;L_2(0,1))$. Hence, if condition (5.2) holds and $a \geq 0$ or $a<0$ and $2p>-a$, then any solution $z(t,x)$ satisfies $z \varepsilon L_2(0,\infty;L_2(0,1))$.

6. Conclusions

In this paper we have obtained a circle theorem for nonlinear parabolic partial differential equations. Our proof is essentially elementary, following Sandberg (1964), and does not require the use of abstract harmonic analysis as in Freedman et al (1969). By using a property of semigroups and the spectral mapping theorem we have related the result directly to the spectrum of the operator A. This is useful in many cases when the boundary conditions make the determination of the spectrum of A very simple.

7. References

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