



Deposited via The University of Sheffield.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/id/eprint/75887/>

Monograph:

Marshall, S.A. (1973) On the Optimal Regulator Problem. Research Report. ACSE Report 12 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

University of Sheffield

Department of Control Engineering

On the optimal regulator problem

S. A. Marshall

Research report No. 12

April 1973

On the optimal regulator problem

Consider the linear multivariable system

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx \quad (2)$$

where x is the n -state vector, u is the r -input vector and y is the m -output vector, and A , B and C are $n \times n$, $n \times r$ and $m \times n$ constant coefficient matrices respectively, where it is required to choose the input $u(t)$, $0 \leq t < \infty$, such that,

$$J = \int_0^{\infty} \{ (y-r)' Q (y-r) + u' R u \} dt \quad (3)$$

is a minimum. r is the m -reference vector and Q and R are $m \times m$ and $r \times r$ weighting matrices respectively.

Initially augment the system to include the reference vector by defining the new $(n+m)$ -state vector z where

$$z = \begin{bmatrix} x \\ r \end{bmatrix} \quad (4)$$

and, hence,

$$\dot{z} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u \quad (5)$$

$$y = [C \ 0] z, \quad (6)$$

and,

$$J = \int_0^{\infty} \{ [(C \ 0)z - (0 \ I)z]' Q [(C \ 0)z - (0 \ I)z] + u' R u \} dt \quad (7)$$

$$\text{i.e. } J = \int_0^{\infty} \left\{ z' \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} z + u' R u \right\} dt \quad (8)$$

Now substitute the relevant matrices into the matrix Riccati equation to give

$$-\begin{bmatrix} \dot{K}_1 & \dot{K}_2 \\ \dot{K}'_2 & \dot{K}_3 \end{bmatrix} = \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} - \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B' & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \quad (9)$$

where K_1 and K_3 are $n \times n$ and $m \times m$ symmetrical matrices respectively and K_2 is an $n \times m$ matrix.

Equation (9) then yields the expressions,

$$-\dot{K}_1 = C'QC + K_1A + A'K_1 - K_1BR^{-1}B'K_1 \quad (10)$$

$$-\dot{K}_2 = -C'Q + A'K_2 - K_1BR^{-1}B'K_2 \quad (11)$$

$$-\dot{K}_3 = Q - K_2'BR^{-1}B'K_2 \quad (12)$$

It is known that the optimal control law is given by

$$u = -R^{-1}[B' \ 0] \begin{bmatrix} K_{10} & K_{20} \\ K_{20}' & K_{30} \end{bmatrix} z \quad (13)$$

$$\text{i.e. } u = -R^{-1}B' [K_{10} \ K_{20}] z \quad (14)$$

where K_{10} and K_{20} are the steady-state solutions of eqns (10) and (11) respectively. Equation (12) is always satisfied regardless of the value of K_2 and can be safely excluded from the set of equations.

Thus to solve the optimal regulator problem all that is required is to solve the algebraic $n \times n$ matrix Riccati equation

$$C'QC + K_{10}A + A'K_{10} - K_{10}BR^{-1}B'K_{10} = 0 \quad (15)$$

for K_{10} , using either the direct method or the eigenvector method or the transition matrix method, and then solve the matrix equation,

$$-C'Q + A'K_{20} - K_{10}BR^{-1}B'K_{20} = 0 \quad (16)$$

$$\text{i.e. } K_{20} = (A' - K_{10}BR^{-1}B')^{-1}C'Q \quad (17)$$

The optimal control law is then given by eqn (14).