
Monograph:

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On the optimal regulator problem

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On the optimal regulator problem

Consider the linear multivariable system

$$\dot{x} = Ax + Bu \qquad (1)$$

$$y = Cx \qquad (2)$$

where x is the n-state vector, u is the r-input vector and y is the m-output vector, and A, B and C are nxn, nxr and mxn constant coefficient matrices respectively, where it is required to choose the input u(t), $0 \leq t \leq \infty$, such that,

$$J = \int_0^\infty \{(y-r)' Q(y-r) + u'Ru\}dt \qquad (3)$$

is a minimum. r is the m-reference vector and Q and R are mxm and rxr weighting matrices respectively.

Initially augment the system to include the reference vector by defining the new (n+m)-state vector z where

$$z = \begin{bmatrix} x \\ r \end{bmatrix} \qquad (4)$$

and, hence,

$$\dot{z} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u \qquad (5)$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} z , \qquad (6)$$

and,

$$J = \int_0^\infty \{\left[(C \;\; 0)z - (0 \;\; I)z\right]' Q\left[(C \;\; 0)z - (0 \;\; I)z\right] + u'Ru\}dt \qquad (7)$$

$$\text{i.e. } J = \int_0^\infty \left\{ z' \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} z + u'Ru \right\} dt \qquad (8)$$

Now substitute the relevant matrices into the matrix Riccati equation to give

$$-\begin{bmatrix} \dot{K}_1 & \dot{K}_2 \\ \dot{K}'_2 & \dot{K}_3 \end{bmatrix} = \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix}$$

$$- \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B' & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \qquad (9)$$

where K_1 and K_3 are nxn and mxm symmetrical matrices respectively and K_2 is an nxm matrix.

Equation (9) then yields the expressions,

$$-\dot{K}_1 = C'QC + K_1A + A'K_1 - K_1BR^{-1}B'K_1 \tag{10}$$

$$-\dot{K}_2 = -C'Q + A'K_2 - K_1BR^{-1}B'K_2 \tag{11}$$

$$-\dot{K}_3 = Q - K'_2BR^{-1}B'K_2 \tag{12}$$

It is known that the optimal control law is given by

$$u = -R^{-1}\begin{bmatrix} B' & 0 \end{bmatrix}\begin{bmatrix} K_{10} & K_{20} \\ K'_{20} & K_{30} \end{bmatrix} z \tag{13}$$

$$\text{i.e. } u = -R^{-1}B'\begin{bmatrix} K_{10} & K_{20} \end{bmatrix} z \tag{14}$$

where K_{10} and K_{20} are the steady-state solutions of eqns (10) and (11) respectively. Equation (12) is always satisfied regardless of the value of K_2 and can be safely excluded from the set of equations.

Thus to solve the optimal regulator problem all that is required is to solve the algebraic nxn matrix Riccati equation

$$C'QC + K_{10}A + A'K_{10} - K_{10}BR^{-1}B'K_{10} = 0 \tag{15}$$

for K_{10} using either the direct method or the eigenvector method or the transition matrix method, and then solve the matrix equation,

$$-C'Q + A'K_{20} - K_{10}BR^{-1}B'K_{20} = 0 \tag{16}$$

$$\text{i.e. } K_{20} = (A' - K_{10}BR^{-1}B')^{-1}C'Q \tag{17}$$

The optimal control law is then given by eqn (14).