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Dyadic Expansion, Characteristic Loci and Multivariable-Control-Systems-Design

by

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Abstract

Recent results in the dyadic representation of system interactions are used to derive a systematic approach to the manipulation and compensation of the characteristic loci of a system described by the NxN transfer function matrix G(s), using rational transfer function approximations to the characteristic loci. The approach is thought to strengthen the link between single-input and multivariable control system design techniques by releasing well-known classical compensation techniques for application to multivariable systems.

List of Figures

List of Symbols

s	=	Laplace-transform variable
$G(s)$	=	NxN matrix of plant transfer functions
$K(s)$	=	NxN matrix of controller transfer functions
$K(s,\omega_1)$	=	NxN controller in the vicinity of $s = i\omega_1$
A^+	=	adjoint or conjugate transpose of matrix A
A^T	=	transpose of matrix A
$\|A\|$	=	determinant of matrix A
$\delta_{j\ell}$	=	Kronecker delta

$\mathrm{diag}\{\lambda_1,\lambda_2,..,\lambda_N\} = \mathrm{diag}\{\lambda_j\}_{1\leq j\leq N}$ = diagonal matrix with $\lambda_1,\lambda_2,..$ along the diagonal

1. Introduction

In recent papers[1,2] the dyadic description of the interaction structure of a plant described by an NxN transfer function matrix G(s) has been applied to the analysis and synthesis of feedback control systems for a class of nuclear reactor spatial instabilities. In such cases the approach has several important advantages_

(a) If G(s) is a dyadic transfer function matrix, a dyadic description of system interaction structure makes possible the determination of a simple controller structure which decouples the output modes of the system. The N modal loops can then be designed independently using single-loop design concepts.

(b) The interpretation of the magnitude, form, and frequency dependence of the plant dyadic structure is an important link between the physical insight available into process dynamics and the theoretical frequency domain analysis. Such information can provide estimates of the effect of modelling errors on the final closed-loop design[3] and a technique for synthesizing fail-safe feedback control systems[1,2].

It has been demonstrated[4] that similar considerations can alleviate difficulties arising in the application of SRD[5] and INA[6] design techniques.

Despite their relevance to reactor control system design, dyadic and approximately dyadic structures represent only a subclass of possible physical systems. This paper extends the previous results using a combination with the general principles of the characteristic locus design method[8] and recent results[7] on the general dyadic representation of system interactions to propose a straightforward approach to the modification and compensation of general system characteristic loci. The basic theoretical ideas are described in section 2 where it is shown that an interplay between the concepts of modal decoupling[2,7] and characteristic loci[8] and the use of Gershgorn's theorem[9] enables the exact allocation of the characteristic loci at a particular frequency and the approximation of the loci in the vicinity of that point by rational polynomial transfer functions. An illustrative example is described in section 3.

Finally, in section 6, an analytic solution is obtained for a simple class of multivariable feedback control problems. This solution is used as a conceptual aid in section 3.

2. Dyadic Expansion, Modal Decoupling and Characteristic Loci

Consider a unity negative feedback configuration for the control of a system described by the NxN transfer function matrix G(s). Let K(s) be the NxN forward path controller and D the usual Nyquist contour in the complex plane, traversed in the clockwise direction. It is well known[8] that the ratio of the system closed-loop characteristic polynomial to the system open-loop characteristic polynomial is given by the determinant of the return-difference matrix T(s) = I+G(s)K(s). If $\{t_k(s)\}_{1\leqslant k\leqslant N}$ are the eigenvalues of G(s)K(s), then

$$|T(s)| = \prod_{k=1}^{N} (1 + t_k(s)) \qquad \ldots(1)$$

Defining the image of D under $t_k(s)$ to be the kth characteristic locus[8] then, if p_o is the number of right-half-plane zeros of the open-loop characteristic polynomial, the closed-loop system is stable if, and only if,[8]

$$\sum_{k=1}^{N} n_k = -p_o \qquad \ldots(2)$$

where n_k is the number of clockwise encirclements of the kth characteristic locus about the (-1,0) point of the complex plane.

The above result has been used to form the basis of the characteristic locus method for multivariable feedback control systems design[8]. The approach regards the design objective as the gain and phase compensation of the characteristic loci of the original plant G(s) by the choice of a suitable controller K(s). Unfortunately, as there is no general formula giving the eigenvalues of the product of two matrices as a function of the eigenvalues of the individual matrices, the systematic choice and modification of $K(s)$ to produce the required loci compensation is a major practical problem. The aim of this section is to provide a framework for a systematic approach to the solution of this problem using recent results[7] on the dyadic representation of system interactions. The problem is regarded in three stages,

(i) The choice of a controller $K(s,\omega_1)$ such that the eigenvalues of $G(s)K(s,\omega_1)$ take specified values at $s = i\omega_1$. For practical purposes, both $G(s)$ and $K(s,\omega_1)$ are assumed to be analytic at $s = i\omega_1$ and non-singular.

(ii) The systematic investigation of the effects of a particular choice of controller $K(s,\omega_1)$ on the characteristic loci in the vicinity of $s = i\omega_1$, and the use of such information in the choice of suitable compensation elements in $K(s,\omega_1)$.

(iii) By repeating (i) and (ii) at selected frequencies $\omega_1, \omega_2, \ldots, \omega_\ell$, the controller matrix

$$K(s) = \sum_{j=1}^{\ell} K(s,\omega_j) \qquad \ldots(3)$$

will produce the desired characteristic loci provided that each term $K(s,\omega_j)$ dominates the summation in the vicinity of $s = i\omega_j$.

An important practical aspect of the approach is the facility of obtaining rational scalar transfer function approximations to the system characteristic loci, valid in the vicinity of the frequency point of interest. It is considered that this feature strengthens the link between single and multivariable design concepts by releasing the well-known single-variable compensation techniques for application to multivariable systems.

2.1 Manipulation of Characteristic Loci at a Given Frequency

Consider the problem of the choice of a controller $K(s,\omega_1)$ such that the eigenvalues of $G(s)K(s,\omega_1)$ take specified values at $s = i\omega_1$. A solution is obtained below using the concepts of dyadic expansion and modal decoupling[7].

Consider the case when $|G(i\omega_1)| \neq 0$ and the matrix $G(-i\omega_1)G^{-1}(i\omega_1)$ has a complete set of eigenvectors $\{\alpha_k(\omega_1)\}_{1\leq k\leq N}$. Previous results[7] imply that,

$$G(i\omega_1) = \sum_{k=1}^{N} h_k(i\omega_1)\alpha_k(\omega_1)\beta_k^{+}(\omega_1) \qquad \ldots(4)$$

where $\{h_k(i\omega_1)\}_{1\leq k\leq N}$ are non-zero complex numbers and $\{\alpha_k(\omega_1)\beta_k^{+}(\omega_1)\}_{1\leq k\leq N}$ is a set of linearly independent dyads which are real or exist in complex conjugate pairs. In general[7] $\{\alpha_k(\omega_1)\}_{1\leq k\leq N}$ are not the eigenvectors of $G(i\omega_1)$ and $\{h_k(i\omega_1)\}_{1\leq k\leq N}$ are not the eigenvalues. However, defining

$$K_D(\omega_1) = \{\sum_{k=1}^{N} \alpha_k(\omega_1)\beta_k^{+}(\omega_1)\}^{-1} \qquad \ldots(5)$$

then,[7] $K_D(\omega_1)$ is real and non-singular and $(K_D(\omega_1))^{-1}K_D(\omega_1) = I$ implies that, $1\leq j,\ k\leq N$,

$$\beta_j^{+}(\omega_1)K_D(\omega_1)\alpha_k(\omega_1) = \delta_{jk} \qquad \ldots(6)$$

and hence, $1\leq \ell\leq N$,

$$G(i\omega_1)K_D(\omega_1)\alpha_\ell(\omega_1) = \sum_{k=1}^{N} h_k(i\omega_1)\alpha_k(\omega_1)\{\beta_k^{+}(\omega_1)K_D(\omega_1)\alpha_\ell(\omega_1)\}$$

$$= h_\ell(i\omega_1)\alpha_\ell(\omega_1) \qquad \ldots(7)$$

That is, $\{\alpha_k(\omega_1)\}_{1\leq k\leq N}$ and $\{h_k(i\omega_1)\}_{1\leq k\leq N}$ are the eigenvectors and eigenvalues of $G(i\omega_1)K_D(\omega_1)$. In previous papers[1,2] physical considerations lead to the term modes for the vectors $\{\alpha_k(\omega_1)\}$ and, in view of equation (7), the choice of controller $K_D(\omega_1)$ is termed modal decoupling and $K_D(\omega_1)$ a decoupling matrix at the frequency ω_1.

Suppose that the desired values of the characteristic loci at $s = i\omega_1$ are $q_1, q_2, \ldots, q_N$. A suitable controller matrix $K(s,\omega_1)$ is obtained using modal-decoupling to be the dyadic transfer function matrix[2]

$$K(s,\omega_1) = K_D(\omega_1) \sum_{j=1}^{N} k_j(s,\omega_1)\alpha_j(\omega_1)\gamma_j^{+}(\omega_1) \qquad \ldots(8)$$

where $\{\gamma_j^{+}(\omega_1)\}_{1\leqslant j\leqslant N}$ are vectors such that, $1\leqslant j,\ \ell\leqslant N$

$$\gamma_j^{+}(\omega_1)\alpha_\ell(\omega_1) = \delta_{j\ell} \quad , \qquad \ldots(9)$$

$\{k_j(s,\omega_1)\}_{1\leqslant j\leqslant N}$ are scalar rational polynomial transfer functions such that

$$k_j(i\omega_1,\omega_1) = q_j/h_j(i\omega_1) \qquad \ldots(10)$$

and, for $1\leqslant j\leqslant N$, if $\overline{\alpha_j(\omega_1)} = \alpha_\ell(\omega_1)$ then

$$k_j(\bar{s},\omega_1) = \overline{k_\ell(s,\omega_1)} \text{ for all } s \qquad \ldots(11)$$

Equation (11), together with the invariance of $\{\alpha_k(\omega_1)\beta_k^{+}(\omega_1)\}_{1\leqslant k\leqslant N}$ under complex conjugation, ensures that $K(\bar{s},\omega_1) = \overline{K(s,\omega_1)}$ everywhere in the complex plane. Equations (7)-(10) indicate that

$$G(i\omega_1)K(i\omega_1,\omega_1)\alpha_\ell(\omega_1) = G(i\omega_1)K_D(\omega_1) \sum_{j=1}^{N} k_j(i\omega_1,\omega_1)\alpha_j(\omega_1)\{\gamma_j^{+}(\omega_1)\alpha_\ell(\omega_1)\}$$

$$= G(i\omega_1)K_D(\omega_1)\{q_\ell/h_\ell(i\omega_1)\}\alpha_\ell(\omega_1)$$

$$= q_\ell\alpha_\ell(\omega_1) \quad , \qquad 1\leqslant \ell\leqslant N \ , \qquad \ldots(12)$$

so that $\{q_\ell\}_{1\leqslant \ell\leqslant N}$ are the eigenvalues of $G(i\omega_1)K(i\omega_1,\omega_1)$ as required and the modes $\{\alpha_\ell(\omega_1)\}_{1\leqslant \ell\leqslant N}$ are the eigenvectors.

As the structure and orders of the compensation networks $\{k_j(s,\omega_1)\}_{1\leqslant j\leqslant N}$ are unspecified except for the constraints of equations (10) and (11), the matrix $K(s,\omega_1)$ is obviously non-unique. A technique for using these degrees of freedom in the compensation of the characteristic loci in the vicinity of $s = i\omega_1$ is described in the next section.

2.2 Manipulation of Characteristic Loci in a Frequency Interval

Although the manipulation of the characteristic loci at a given frequency is of some practical interest, the technique is limited without an assessment of the effect of a particular choice of $K(s,\omega_1)$ on the characteristic loci in the vicinity of $s = i\omega_1$, and a method for using this information to choose suitable forms for the compensation networks $\{k_j(s,\omega_1)\}_{1 \leq j \leq N}$. An approach to the solution of this problem is described in this section.

It is convenient to manipulate $G(s)K(s,\omega_1)$ into a more amenable form by defining

$$T(\omega_1) = [\alpha_1(\omega_1), \alpha_2(\omega_1), \ldots, \alpha_N(\omega_1)] \qquad \ldots(13)$$

and noting from equations (8) and (9) that

$$T^{-1}(\omega_1)G(s)K(s,\omega_1)T(\omega_1) = H(s,\omega_1)\mathrm{diag}\{k_1(s,\omega_1)..,k_N(s,\omega_1)\} \quad \ldots(14)$$

where

$$H(s,\omega_1) = T^{-1}(\omega_1)G(s)K_D(\omega_1)T(\omega_1) \qquad \ldots(15)$$

is a transfer function matrix with rational polynomial elements. The common denominator of the elements of $H(s,\omega_1)$ is simply the common denominator of elements of $G(s)$. In general the coefficients in the numerator polynomials of $H(s,\omega_1)$ are complex, but, if $\{\alpha_j(\omega_1)\}_{1 \leq j \leq N}$ are real vectors, all such coefficients are real numbers.

As eigenvalues are invariant under similarity transformation, the characteristic loci of $G(s)K(s,\omega_1)$ are identical to the characteristic loci of $H(s,\omega_1)\mathrm{diag}\{k_j(s,\omega_1)\}$. Also the eigenvalues of $H(s,\omega_1)\mathrm{diag}\{k_j(s,\omega_1)\}$ are equal to the eigenvalues of $\mathrm{diag}\{k_j(s,\omega_1)\}H(s,\omega_1)$. Using this information, an estimate of the effect of a particular set of compensation elements $\{k_j(s,\omega_1)\}$ on the characteristic loci can be obtained by applying Gershgorins theorem[9] to equation (14). That is, the eigenvalues of $G(s)K(s,\omega_1)$ lie in the union of the closed discs $B_j(s,\omega_1)$ of centre $H_{jj}(s,\omega_1)k_j(s,\omega_1)$ and radius $d_j(s,\omega_1)$ where

$$d_j(s,\omega_1) = |k_j(s,\omega_1)| \sum_{i \neq j} |H_{ji}(s,\omega_1)| \quad , \quad \text{(row estimate)} \qquad \ldots(16)$$

or

$$d_j(s,\omega_1) = |k_j(s,\omega_1)| \sum_{i \neq j} |H_{ij}(s,\omega_1)| \quad , \quad \text{(column estimate)} \ldots(17)$$

Also, from equations (7) and (15), it follows that $H(s,\omega_1)$ is diagonal at $s = i\omega_1$ with eigenvalues $H_{jj}(i\omega_1,\omega_1) = h_j(i\omega_1)$. Hence, from equations (14), (16) and (17) $G(i\omega_1)K(i\omega_1,\omega_1)$ has eigenvalues $H_{jj}(i\omega_1,\omega_1)k_j(i\omega_1,\omega_1)$ and

$$d_j(i\omega_1,\omega_1) = 0 \quad , \qquad 1 \leqslant j \leqslant N \qquad \ldots(18)$$

so that the Gershgorin circles at the point $s = i\omega_1$ have zero radius. A graphical representation of these ideas is given in Fig. 1 for a case of $N = 2$.

The above analysis intuitively suggests that, in the vicinity of the point $s = i\omega_1$ where (equation (18)) the Gershgorin circles are small, the diagonal terms $\{H_{jj}(s,\omega_1)k_j(s,\omega_1)\}_{1\leqslant j\leqslant N}$ will be reasonable approximations to the characteristic loci $\{t_j(s)\}_{1\leqslant j\leqslant N}$ of $G(s)K(s,\omega_1)$ in the sense that

$$t_j(s) \in B_j(s,\omega_1) \quad , \qquad 1 \leqslant j \leqslant N \quad , \qquad \ldots(19)$$

and the fractional error, $1\leqslant j\leqslant N$,

$$\frac{|t_j(s) - H_{jj}(s,\omega_1)k_j(s,\omega_1)|}{|H_{jj}(s,\omega_1)k_j(s,\omega_1)|} \leqslant \frac{d_j(s,\omega_1)}{|H_{jj}(s,\omega_1)k_j(s,\omega_1)|} \qquad \ldots(20)$$

is small. With these assumptions, equations (16) and (17) indicate that the fractional error is independent of the chosen compensators $\{k_j(s,\omega_1)\}_{1\leqslant j\leqslant N}$ and hence that the validity of the loci approximations depends only on the extent to which $H(s,\omega_1)$ is diagonally dominant in the vicinity of $s = i\omega_1$.

The intuitive approach described above can be applied directly to the assessment of the effect of a particular choice of $\{k_j(s,\omega_1)\}$ on the characteristic loci of $G(s)K(s,\omega_1)$ in the vicinity of $s = i\omega_1$ by an analysis of the products $H_{jj}(s,\omega_1)k_j(s,\omega_1)$, $1\leqslant j\leqslant N$. Also, bearing in mind that $H_{jj}(s,\omega_1)$, $1\leqslant j\leqslant N$, are rational polynomials in s, the approach provides an intuitive but systematic approach to the choice of compensation elements $\{k_j(s,\omega_1)\}$ by the application of single-variable techniques (e.g. pole-zero analysis) to each approximation $H_{jj}(s,\omega_1)$ in turn. An example illustrating the approach is given in section 3.

It is important to note that the characteristic loci of $G(s)K(s,\omega_1)$ do not, in general, satisfy relation (19). For example, consider the matrix

$$\begin{bmatrix} 10 & 10i \\ i & 1 \end{bmatrix} \qquad \ldots(21)$$

which has eigenvalues 8.7 and 2.3. This implies that the error bounds of equation (20) can be over optimistic. However, if the process of compensation in the vicinity of $s = i\omega_1$ is regarded as an attempt to equalize the loci $H_{jj}(s,i\omega_1)k_j(s,i\omega_1)$ in that region, a consideration of perfect compensation, i.e. $H_{\ell\ell}(s,i\omega_1)k_\ell(s,i\omega_1) = H_{jj}(s,i\omega_1)k_j(s,i\omega_1)$, $1 \leqslant j, \ell \leqslant N$, and application of Gershgorins theorem indicates that, for $1 \leqslant j \leqslant N$,

$$\frac{|t_j(s) - H_{jj}(s,\omega_1)k_j(s,\omega_1)|}{|H_{jj}(s,\omega_1)k_j(s,\omega_1)|} \leqslant \max_{1 \leqslant j \leqslant N} \frac{d_j(s,\omega_1)}{|H_{jj}(s,\omega_1)k(s,\omega_1)|} \qquad \ldots(22)$$

i.e. the uncertainty in the position of the actual characteristic loci can be reduced by the very act of system compensation. In such cases, relation (19) can be regarded as a feasible assumption for practical applications.

Finally the size of the frequency interval over which the approximation is sensibly valid is an important consideration in practical applications. This will vary with the choice of ω_1 and the complexity of the interaction structure of the system under consideration. However, the example of section 3 and previous experience with applications of the technique of dyadic approximation[1,2] imply that it can be large enough to make the approach a useful addition to already available multivariable control-systems-design aids.

2.3 An Approach to Multivariable-Control-Systems Design

Based on the analysis of sections 2.1 and 2.2, the following procedure is suggested for the systematic manipulation and compensation of open-loop system characteristic loci. The controller is based upon the form

$$K(s) = \sum_{p=1}^{\ell} K(s,\omega_p) \qquad \ldots(23)$$

where $\omega_1,\ldots,\omega_\ell$ are distinct frequencies at which it is desired to manipulate the system characteristic loci. For theoretical purposes it is assumed that the term $K(s,\omega_k)$ dominates the summation in the vicinity of $s = i\omega_k$ and hence that compensation achieved in this region is largely unaffected by subsequent design exercizes in other frequency intervals. For practical purposes the choice of $\ell = 2$, ω_1 and ω_2 as representative high and low frequencies respectively, and $K(s,\omega_1)$, $K(s,\omega_2)$ as lead-lag and integral terms respectively should satisfy this assumption.

STEP ONE: Choose distinct frequencies $\omega_1 > \omega_2 > .. > \omega_\ell$ at which compensation of the characteristic loci is required. Set $j = 1$.

STEP TWO: Using the results of sections 2.1 and 2.2 calculate the decoupling matrix $K_D(\omega_j)$ and transformation $T(\omega_j)$. Hence calculate the transfer function matrix (eqn.(15)) $H(s,\omega_j)$.

STEP THREE: Using the rational polynomials $\{H_{\ell\ell}(s,\omega_j)k_\ell(s,\omega_j)\}_{1\leqslant\ell\leqslant N}$ as approximations to the system characteristic loci in the vicinity of $s = i\omega_j$, use single-loop design concepts to choose compensation networks $k_p(s,\omega_j)$, $1\leqslant p\leqslant N$, so that the products $H_{pp}(s,\omega_j)k_p(s,\omega_j)$, $1\leqslant p\leqslant N$, have the required properties in that region. Calculate $K(s,\omega_j)$ from eqn.(8).

STEP FOUR: Compute the characteristic loci of $G(s)\sum_{p=1}^{j} K(s,\omega_p)$ to check that the desired loci characteristics have been obtained in the vicinity of $s = i\omega_j$ and also that compensation at the frequencies $\omega_1,\omega_2,..,\omega_{j-1}$ is unaffected.

STEP FIVE: If $j = \ell$ set the controller to be as in equation (22). If $j<\ell$, replace j by $j+1$ and go to step two.

The application of the above procedure is illustrated in the next section using an example.

3. Illustrative Example

Consider a system with input-output relations defined by the transfer function matrix,

$$G(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 32.6+16s+2.15s^2 & 9.4+4s+1.1s^2 \\ 6.2+4s+1.05s^2 & 3+4s+s^2 \end{bmatrix} \qquad \ldots(24)$$

where

$$\Delta(s) = (s+1)(s+2)(s+3) \qquad \ldots(25)$$

As interactive effects in the system transient response are dominated by the intermediate to high frequency response characteristics of the system[8], attention is initially focused on this region. Choosing $\omega_1 = 8.0$, and applying the procedure of section 2.3, the eigenvectors of $G(-i8)G^{-1}(i8)$ are

$$\alpha_1(8) = [1 \quad 0]^T, \qquad \alpha_2(8) = [1 \quad 1]^T \qquad \ldots(26)$$

The procedure of section 2.1 yields the decoupling matrix,

$$K_D(8) = \begin{bmatrix} -13.4 & 13.4 \\ 13.4 & 20.6 \end{bmatrix} \qquad \ldots(27)$$

and hence, by equations (13) and (15),

$$H(s,8) = \frac{1}{\Delta(s)} \begin{bmatrix} -268-160.8s-13.4s^2 & 153.8+2.4s^2 \\ -60.7-0.95s^2 & 102+136s+34s^2 \end{bmatrix} \qquad \ldots(28)$$

which is a matrix of rational polynomial transfer functions with real coefficients. Note that the off-diagonal terms are zero at $s = i8$, as expected.

The diagonal terms are

$$H_{11}(s,8) = \frac{-13.4(s+10)}{(s+1)(s+3)} \qquad \ldots(29)$$

and

$$H_{22}(s,8) = \frac{34.0}{(s+2)} \qquad \ldots(30)$$

Plots of the frequency responses of these transfer functions are shown in Figure 2, together with the actual characteristic loci of $G(s)K_D(8)$. It is seen that the diagonal terms are good approximations to the loci at high frequencies and quite acceptable approximations at low frequencies. Typical Gershgorin circles (based on row estimates) are also plotted in the frequency interval $1 \leq \omega < +\infty$ and it can be seen that the characteristic loci of $G(s)K_D(8)$ are contained within the band defined by the circles. The fractional errors (equation (20)) are less than 0.5 at all frequencies greater than 1.0 and hence we can have reasonable confidence in the effect of compensation elements in this frequency range.

Using the diagonal terms (eqns (29),(30)) as rational transfer function approximations to the characteristic loci, elementary single-variable concepts indicate that, to ensure stability of the closed-loop system for arbitrarily high gains, the loci $H_{11}(i\omega,8)$ must be rotated through 180 degrees. Introducing some phase advance into $H_{11}(s,8)$, the above considerations suggest compensation networks of the form

$$k_1(s,8) = -k_1 \frac{(s+1)}{(s+10)} \qquad \ldots(31)$$

and $k_2(s,8) = k_2$ ...(32)

where k_1 and k_2 are positive real numbers. That is

$$H_{11}(s,8)k_1(s,8) = \frac{13.4k_1}{(s+3)} \qquad \ldots(33)$$

$$H_{22}(s,8)k_2(s,8) = \frac{34.0k_2}{(s+2)} \qquad \ldots(34)$$

Comparing these forms with the analysis of section 6 indicates that, if responses of the approximate form $1-e^{-kt}$ are required in each loop, then an intuitive estimate of the gains k_1, k_2 can be obtained by solving the linear simultaneous equations,

$$3 + 13.4k_1 = k \qquad \ldots(35)$$

$$2 + 34.0k_2 = k \qquad \ldots(36)$$

Choosing k = 20, for example,

$$k_1 = 1.3 \qquad k_2 = 0.53 \qquad \ldots(37)$$

and hence the high frequency controller factor (equation (8)) is set equal to

$$K(s,8) = -1.3\,\frac{(s+1)}{(s+10)}\begin{bmatrix}-13.4\\ 13.4\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + 0.53\begin{bmatrix}0\\ 35.0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} \qquad \ldots(38)$$

The characteristic loci of $G(s)K(s,8)$ are given in Fig. 3 together with the frequencies responses $H_{jj}(s,8)k_j(s,8)$, $j = 1,2$, for comparison. The desired compensation of the characteristic loci has been achieved with high accuracy at high frequencies, and with quite acceptable accuracy for intermediate frequencies $\omega \geqslant 1.0$. The closed-loop system is, in fact, asymptotically stable and the design procedure could be terminated by the choice of controller $K(s) = K(s,8)$ (eqn (38)). However, the systems responses to step inputs shown in Fig. 4 indicate that the steady state errors and interaction effects are significant. Figure 3 indicates that these could be reduced by increasing the overall gain of the system, or equivalently increasing the value of k in equations (35) and (36). For example, taking k = 50,

$$k_1 = 3.5 \quad , \quad k_2 = 1.4 \qquad \ldots(39)$$

The system responses to step inputs for this case are shown in Fig. 5. The high frequency controller factor $K(s,8)$ now becomes,

$$K(s,8) = -3.5 \frac{(s+1)}{(s+10)} \begin{bmatrix} -13.4 \\ 13.4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + 1.4 \begin{bmatrix} 0 \\ 35.0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \ldots(40)$$

The residual steady state error and interaction effects can now be removed by the use of integral action. This can be achieved quite simply by applying the procedure of section 2.3 in a frequency interval representative of low frequency dynamics. However, low frequency closed-loop dynamic responses are insensitive to the precise form of the compensation elements provided high controller gains are used and the system is closed-loop stable[8]. An adequate integral action can be obtained by using the technique of section 2.1 to manipulate the characteristic loci at a representative low frequency. For example, taking $\omega_2 = 0.5$ and applying the procedures of section 2.1, the eigenvectors of $G(-i0.5)G^{-1}(i0.5)$ are

$$\alpha_1(0.5) = \begin{bmatrix} -0.23 \\ 0.97 \end{bmatrix}, \quad \alpha_2(0.5) = \begin{bmatrix} 0.99 \\ 0.17 \end{bmatrix} \qquad \ldots(41)$$

The decoupling matrix is

$$K_D(0.5) = \begin{bmatrix} 0.476 & -1.06 \\ -0.77 & 3.85 \end{bmatrix} \qquad \ldots(42)$$

and the eigenvalues of $G(i0.5)K_D(0.5)$ are

$$h_1(i0.5) = 1.0 + i0.075$$
$$h_2(i0.5) = 1.0 - i0.72 \qquad \ldots(43)$$

Choosing $k_1(s,0.5)$, $k_2(s,0.5)$ to be pure integrators k_3/s, k_4/s respectively, the gain of the eigenvalues of $G(s)K(s,0.5)$ is equalized by setting,

$$|k_3 h_1(i0.5)| = |k_4 h_2(i0.5)| \qquad \ldots(44)$$

i.e.

$$\left|\frac{k_3}{k_4}\right| = 1.5 \qquad \ldots(45)$$

The precise values of k_3, k_4 can now be chosen by a parametric analysis of the characteristic loci of $G(s)\{K(s,8) + K(s,0.5)\}$. However, bearing in mind that the integral term will have only a small effect on the high frequency loci of $G(s)K(s,8)$, the form of Fig. 3 and equations (43) indicate the desired low frequency compensation can only be achieved by using positive values of k_4 and k_3. From equations (8) and (45) the controller integral term becomes

$$K(s,0.5) = \frac{k_4}{s}\begin{bmatrix} 0.476 & -1.06 \\ -0.77 & 3.85 \end{bmatrix}\begin{bmatrix} 1.02 & -0.11 \\ -0.08 & 1.48 \end{bmatrix}$$

$$= \frac{k_4}{s}\begin{bmatrix} 0.57 & -1.62 \\ -1.09 & 5.78 \end{bmatrix} \qquad \ldots(46)$$

Trial and error simulation using a controller of the form (eqns (46),(40))

$$K(s) = K(s,8) + K(s,0.5) \qquad \ldots(47)$$

leads to a choice of $k_4 = 30.0$. The transient responses for this final design are given in figure 6.

In summary, the use of dyadic expansions provides an approach to the synthesis of a feedback controller for the given system, by a systematic manipulation of the characteristic loci at high and low frequencies. System compensation is achieved very easily at high frequencies by approximating the system characteristic loci by the diagonal elements of the transfer function matrix $H(s,8)$ (eqn 28) and applying well-known single-input compensation procedures. Low frequency control is obtained by using the method of section 2.1 to balance the gains of the characteristic loci at a frequency of $\omega = 0.5$. Fine tuning of the controller is then achieved by simulation studies.

4. Conclusions

The paper presents a practical approach to the systematic manipulation of system characteristic loci for application to multivariable feedback control systems design. The technique has the ability to exactly manipulate the loci at a given frequency point and to assess the effect of chosen compensation networks in the vicinity of that point. This is achieved by the use of rational transfer function approximations to the system characteristic loci which are exact at the frequency of interest and in error at other frequencies to an extent defined by Gershgorins theorem. This feature strengthens the link between single-input and multivariable control design procedures by releasing well-known single-variable compensation techniques for application to multivariable systems. The systematic nature of the approach is illustrated by an example where it is demonstrated that the use of rational approximations provides direct insight into suitable compensation networks.

As an aid to the analysis of section 3, an analytical solution is obtained in section 6 to a simple, but conceptually useful, class of multivariable control problems.

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6. Appendix

This appendix presents an approach to the design of a unity feedback control configuration for a system described by the NxN dyadic transfer function matrix[7]

$$G(s) = \sum_{j=1}^{N} g_j(s)\alpha_j\beta_j^+ \qquad \ldots(48)$$

where, $1 \leq j \leq N$,

$$g_j(s) = \frac{a_j}{s+b_j} \quad , \qquad \ldots(49)$$

$\{b_j\}_{1\leq j\leq N}$ is a set of real poles, $\{a_j\}_{1\leq j\leq N}$ is a set of non-zero real numbers and $\{\alpha_j\beta_j^+\}_{1\leq j\leq N}$ is a set of linearly independent, real, frequency independent dyads.

Applying the procedure of section 2, suppose that $K_D(\omega_1)$ is a decoupling matrix for the system, then

$$G(s)K_D(\omega_1) = \sum_{j=1}^{N} \frac{a_j}{s+b_j}\alpha_j\gamma_j^+ \qquad \ldots(50)$$

where $\{\gamma_j\}_{1\leq j\leq N}$ is a set of vectors such that

$$\gamma_j^+\alpha_k = \delta_{j,k} \qquad 1\leq j,\ k\leq N \qquad \ldots(51)$$

Noting[7] that $\alpha_j(\omega_1) = \alpha_j$, $1\leq j\leq N$, then equations (14), (15), (50) and (51) give

$$T^{-1}(\omega_1)G(s)K(s,\omega_1)T(\omega_1) = \mathrm{diag}\{k_j(s,\omega_1)a_j/(s+b_j)\}_{1\leq j\leq N} \qquad \ldots(52)$$

Suppose that responses of the approximate form $1-e^{-kt}$ are required from each channel and choose $K(s,\omega_1)$ to be a proportional controller where $k_j(s,\omega_1) = k_j$, $1 \leq j \leq N$ and

$$b_j + k_j a_j = k \quad , \quad 1 \leq j \leq N \qquad \ldots(53)$$

then the controller becomes (eqn 8)

$$K(s,\omega_1) = K_D(\omega_1) \sum_{j=1}^{N} \frac{k-b_j}{a_j} \alpha_j \gamma_j^{+} \qquad \ldots(54)$$

and (equations (52),(53)) the system step responses are represented by

$$(I + G(s)K(s,\omega_1))^{-1} G(s)K(s,\omega_1) \frac{1}{s} = \sum_{j=1}^{N} \frac{k-b_j}{s+k} \alpha_j \gamma_j^{+} \frac{1}{s}$$

$$= \left\{ \frac{1}{s} - \frac{1}{s+k} \right\} \sum_{j=1}^{N} \frac{k-b_j}{k} \alpha_j \gamma_j^{+} \qquad \ldots(55)$$

i.e. the closed-loop system is represented by a single-scalar transfer function multiplying a constant matrix (dependent upon the choice of k) which represents closed-loop interaction effects and steady-state error.

A suitable choice for k can be obtained by noting that

$$\lim_{k \to \infty} \sum_{j=1}^{N} \frac{k-b_j}{k} \alpha_j \gamma_j^{+} = I \quad \text{(the unit matrix)} \qquad \ldots(56)$$

so that interaction effects decrease as the desired speed of response increases. This implies that k should be much greater than the maximum of $|b_j|$, $1 \leq j \leq N$.

The physical relationship between $K(s,\omega_1)$ and the plant $G(s)$ can be obtained by noting that

$$K_D(\omega_1) = \left\{ \sum_{j=1}^{N} \alpha_j \beta_j^{+} \right\}^{-1} = \sum_{j=1}^{N} \psi_j \gamma_j^{+} \qquad \ldots(57)$$

where $\{\psi_j\}_{1\leq j\leq N}$ are vectors such that $\beta_j^+\psi_\ell = \delta_{j\ell}$ and hence that (eqns 54,51)

$$K(s,\omega_1) = \sum_{j=1}^{N} \frac{k-b_j}{a_j} \psi_j\gamma_j^+ \qquad \ldots(58)$$

That is,

$$K_\infty = \lim_{k\to\infty} k^{-1}K(s,\omega_1) = \sum_{j=1}^{N} a_j^{-1}\psi_j\gamma_j^+ \qquad \ldots(59)$$

Also, defining

$$G_\infty = \lim_{s\to\infty} s\,G(s) = \sum_{j=1}^{N} a_j\alpha_j\beta_j^+ \qquad \ldots(60)$$

then (eqn 59)

$$G_\infty^{-1} = \sum_{j=1}^{N} a_j^{-1}\psi_j\gamma_j^+ = K_\infty \qquad \ldots(61)$$

i.e., for large k, the designed controller tends to diagonalize the plant at high frequencies. Such a controller has been previously used intuitively[8]. The above analysis provides some theoretical justification for the approach.

In summary, a theoretical approach has been presented to the design of a simple, but conceptually useful, class of multivariable feedback control problems. As illustrated by the example in section 3, the analysis can be an intuitive aid to the design of systems which approximate to this form at high frequencies. Finally, the form of the transfer functions (eqn (49)) and the ease with which a controller can be designed imply that the system can be regarded as a multivariable generalization of the classical first order system. The analysis of more general structures could be a useful aid in practical application of multivariable control theory.

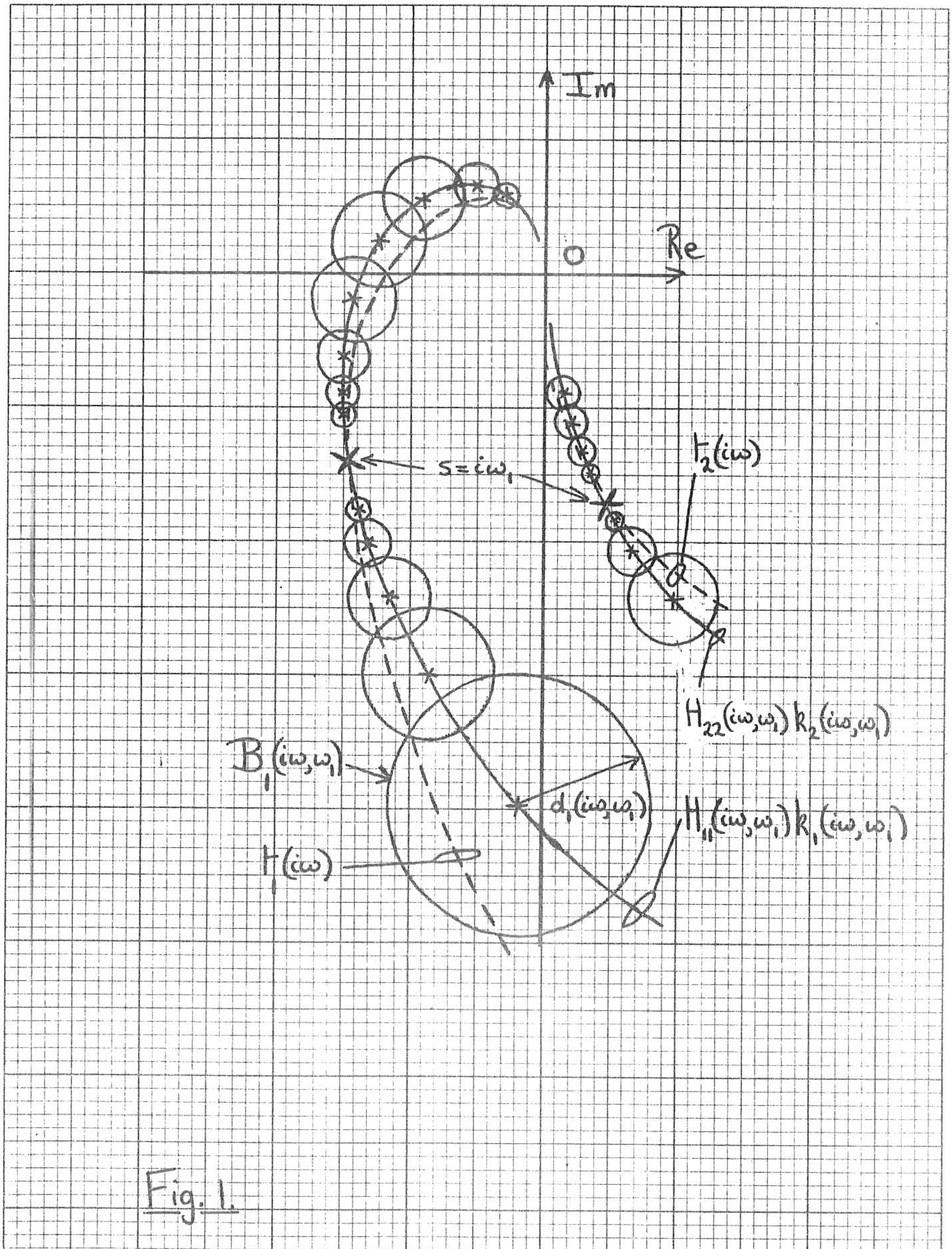
Im
Re
O
$s = i\omega_1$
$H_2(i\omega)$
$H_{22}(i\omega,\omega_1)\,k_2(i\omega,\omega_1)$
$B_1(i\omega,\omega_1)$
$d_1(i\omega,\omega_1)$
$H_{11}(i\omega,\omega_1)\,k_1(i\omega,\omega_1)$
$H_1(i\omega)$

Fig. 1.

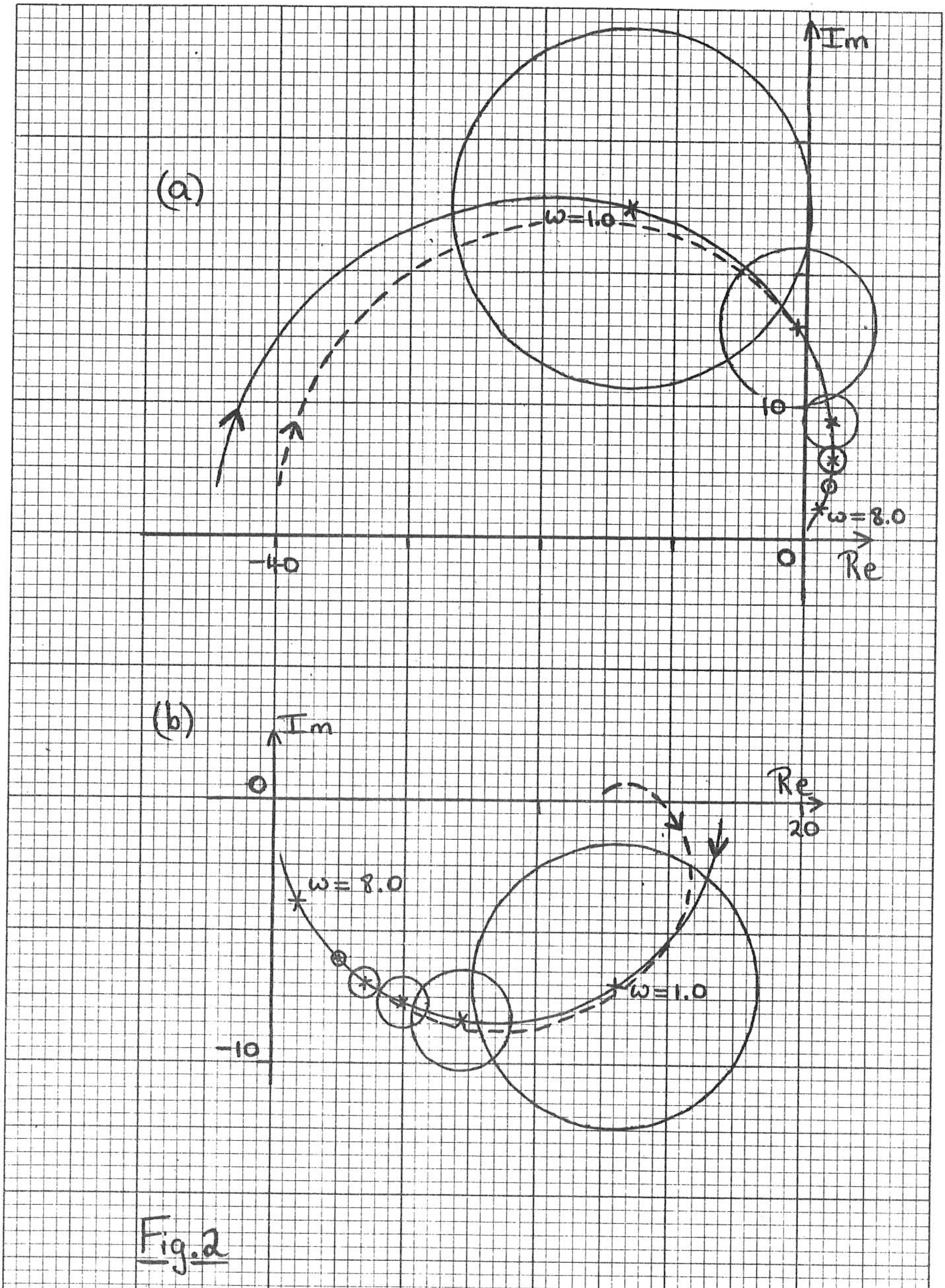

Fig.2

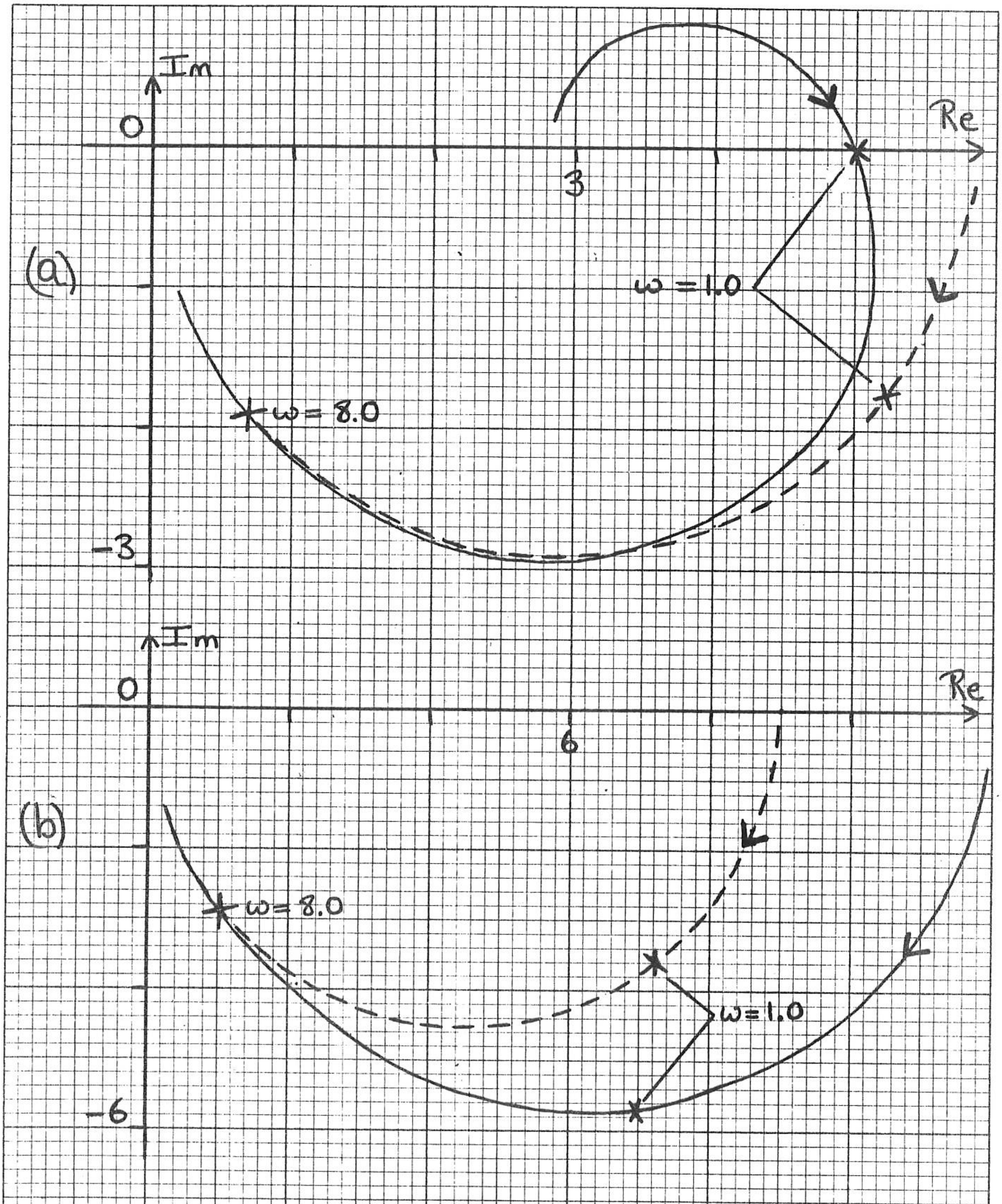

Fig. 3.

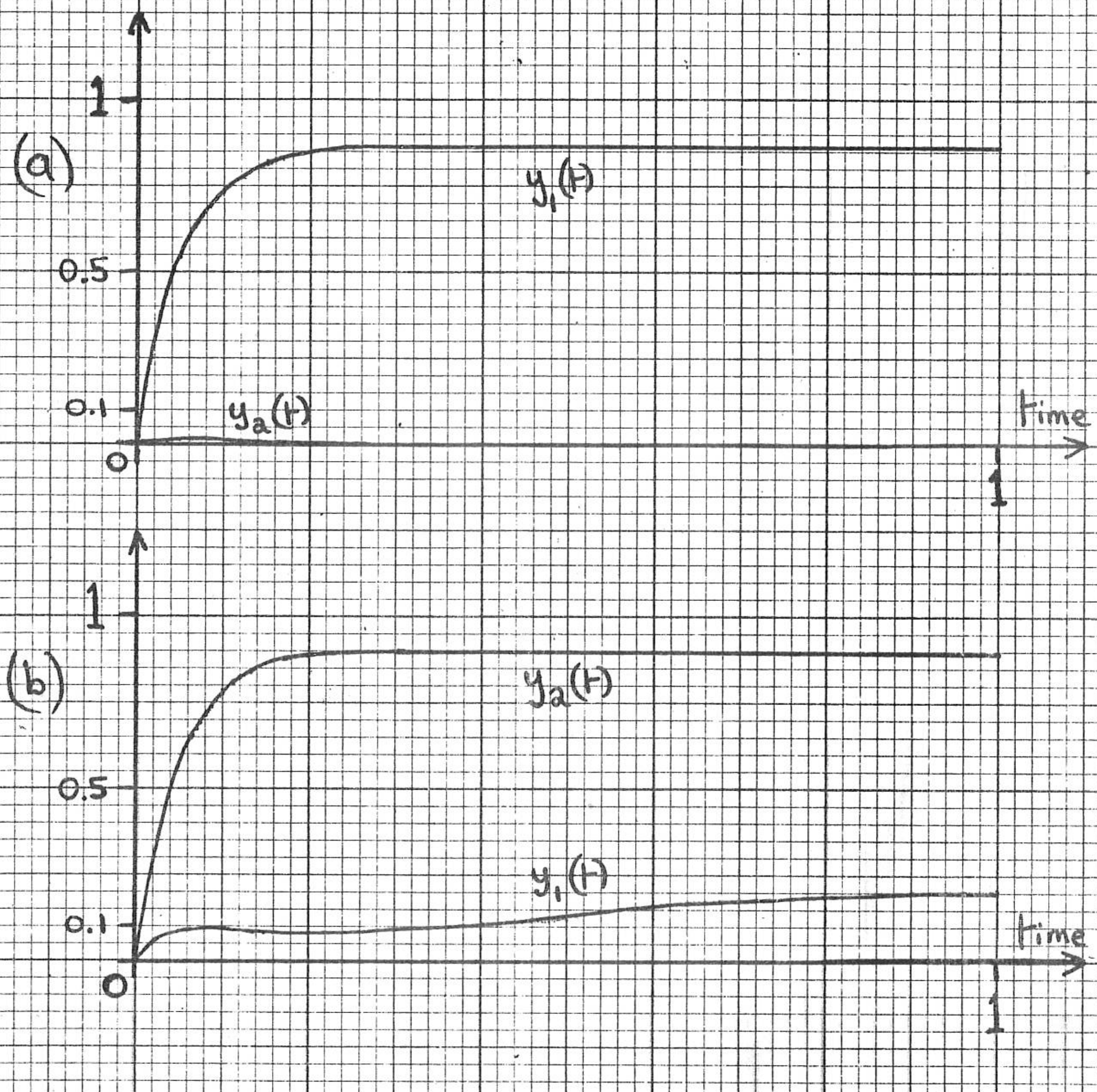

Fig. 4

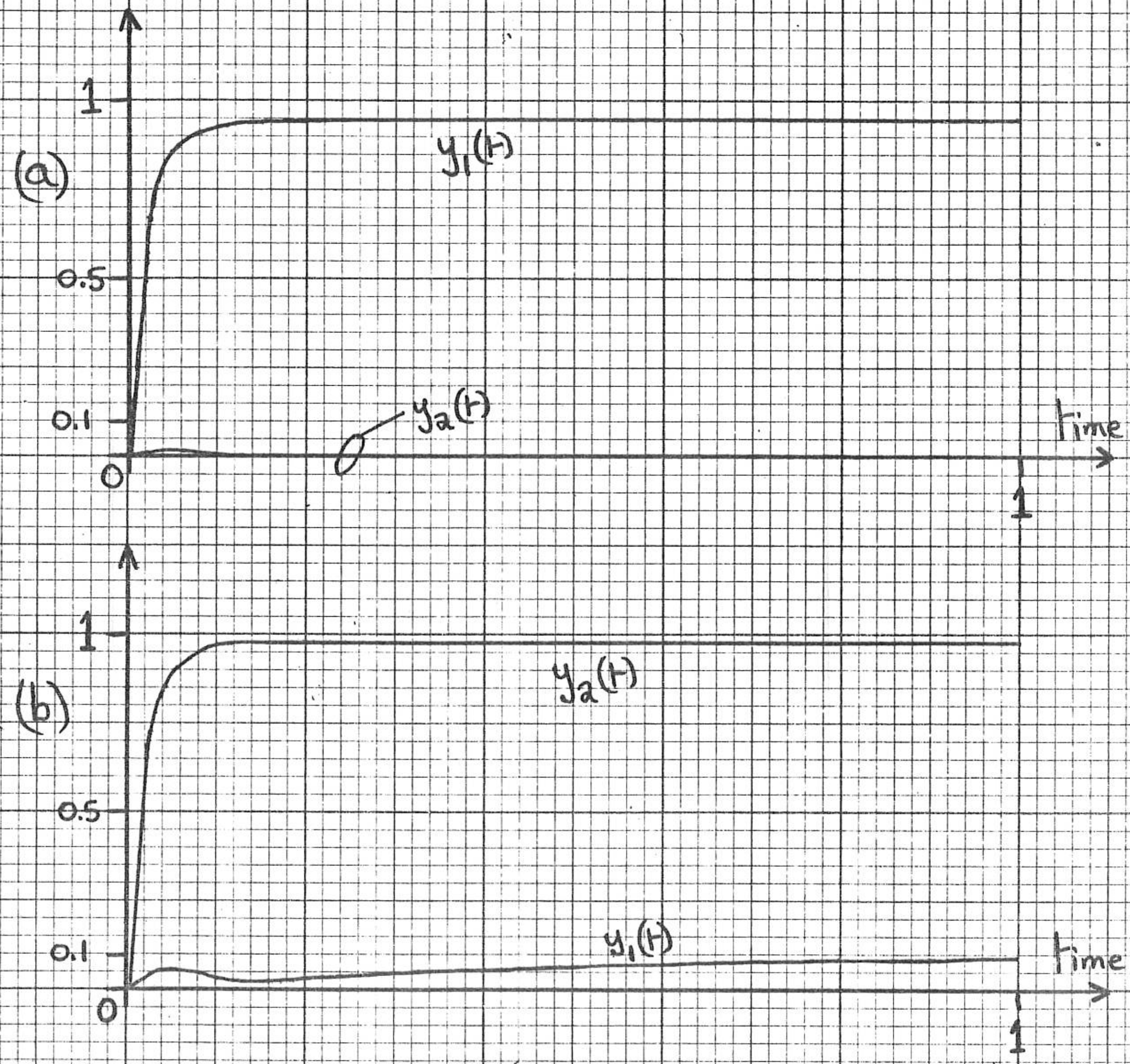

Fig. 5

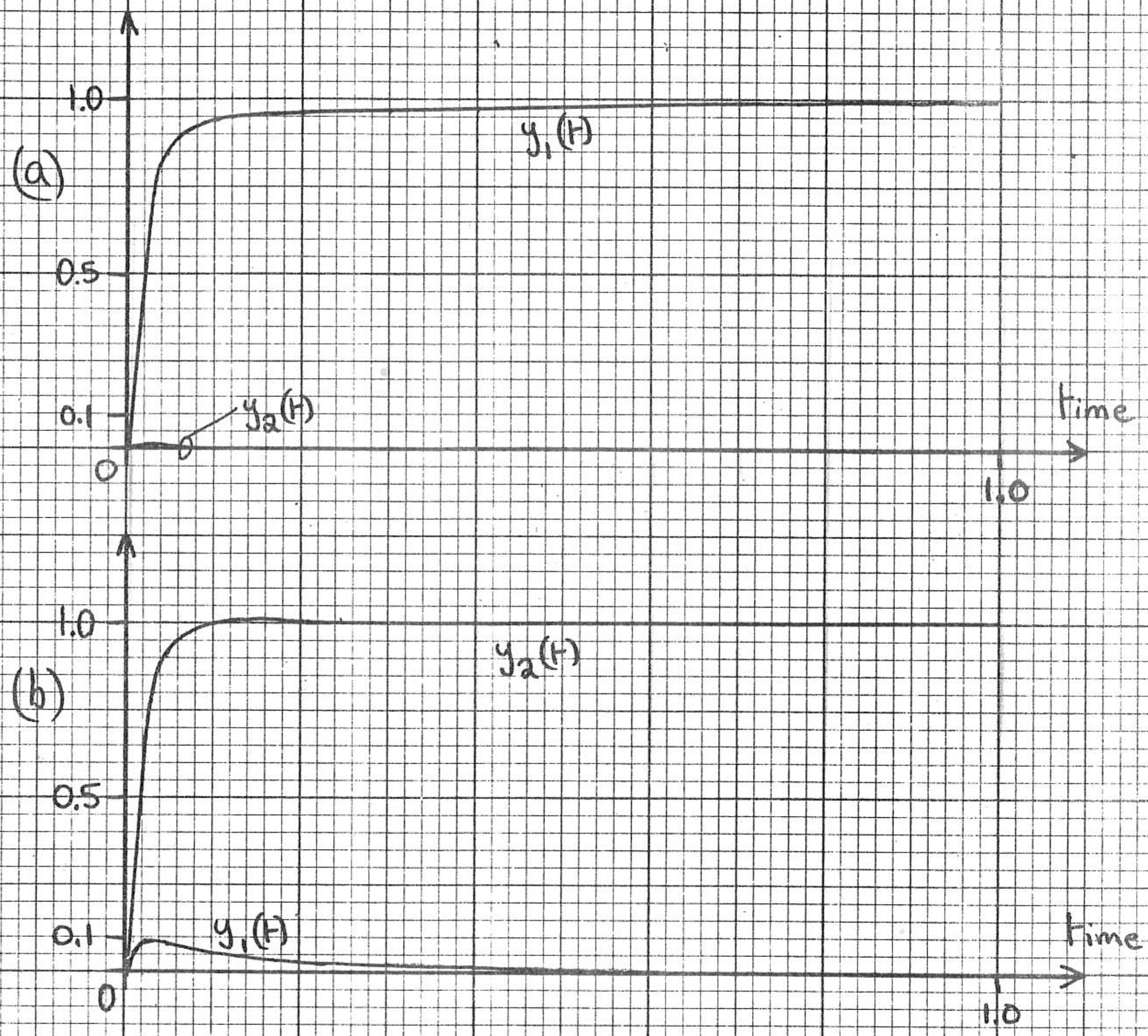

Fig. 6.