**Monograph:**

Owens, D.H. (1976) Asymptotic Root-Loci of Linear Multivariable Systems: Geometric Analysis. Research Report. ACSE Report 40 . Deoartment of Control Engineering, University of Sheffield, Mappin Street, Sheffield

# ASYMPTOTIC ROOT-LOCI OF LINEAR MULTIVARIABLE SYSTEMS: A GEOMETRIC ANALYSIS

by

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Research Report No.40

January 1976

## Abstract

Recent results on the asymptotic behaviour of the root-loci of a linear time-invariant system $S(A,B,C)$ are formulated in geometric terms and equivalent results obtained in two cases of practical interest in terms of the matrix coefficients in the expansion of $(A-pBC)^{\ell}$, $\ell \geq 1$.

## 1. Introduction

A recent paper (Shaked and Kouvaritakis, 1976) presented a theoretical analysis of the asymptotic behaviour of the eigenvalues of the linear, time-invariant system S(A,B,C)

$$\dot{x} = Ax(t)+Bu(t) \quad , \quad u(t) \in R^m \quad , \quad x(t) \in R^n$$
$$y(t) = Cx(t) \quad , \quad y(t) \in R^m \qquad \ldots(1)$$

when subject to unity negative feedback with scalar gain $p \geq 0$. The closed-loop system takes the form

$$\dot{x}(t) = \{A-pBC\}x(t)+Br(t)$$
$$y(t) = Cx(t) \qquad \ldots(2)$$

Previous work (Shaked and Kouvaritakis, 1976) used determinantal manipulation techniques to obtain explicit formula for the asymptotic directions and pivots of the root-locus. This paper describes some solutions of this problem by geometric analysis of the closed-loop eigenvalue equation

$$\{s(p)I_n - A+pBC\}x(p) = 0 \quad , \quad \|x(p)\| = 1 \quad , \quad p \geq 0 \qquad \ldots(3)$$

and the identification of the asymptotic directions and pivots in terms of the structural properties of the matrix coefficients in the expansion of $(A-pBC)^{\ell}$, $\ell \geq 1$.

## 2. Asymptotic Behaviour of Closed-loop Eigenvectors

Let $\{p_j\}_{j\geq 1}$ be an unbounded sequence of positive real numbers and $\{s(p_j)\}_{j\geq 1}$, $\{x(p_j)\}_{j\geq 1}$ a corresponding sequence of closed-loop eigenvalues and eigenvectors respectively. By extraction of a suitable subsequence, it is possible to assume that

$$\lim_{j\to\infty} x(p_j) = x_\infty \quad , \quad \|x_\infty\| = 1 \qquad \ldots(4)$$

If R(Q), N(Q) denote the range and null space of a matrix Q, then

Theorem 1

(a) If the sequence $\{|s(p_j)|\}_{j\geq 1}$ is unbounded, then $x_\infty \in R(B)$.

(b) If the sequence $\{s(p_j)\}_{j\geq 1}$ has a finite cluster point $\lambda$, then $x_\infty \in W_B$, where (Owens, 1975) $W_B$ is the maximal subspace of $N(C)$ satisfying the relation $AW_B \subset W_B + R(B)$.

Proof

To prove (a), divide equation (3) by $s(p)$, from which
$x_\infty = \lim_{j\to\infty} (s(p_j))^{-1} p_j BCx(p_j) \in R(B)$.

To prove (b), equation (3) implies that $(\lambda I_n - A)x_\infty = \lim_{j\to\infty} p_j BCx(p_j) \in R(B)$ ie $x_\infty \in W_B$.

Q.E.D.

## 3. Asymptotic Root-loci for Uniform-rank Multivariable Systems

To illustrate the general structure of the geometric relationships defining the asymptotic form of the system root-locus, consider the case of an open-loop system (equation (1)) satisfying the relations

$$CA^{j-1}B = 0 \qquad j < k$$
$$|CA^{k-1}B| \neq 0 \qquad \ldots(5)$$

Equivalently, if $G(s) = C(sI_n - A)^{-1}B$ is the open-loop system transfer function matrix, then

$$G_\infty^{(k)} \triangleq \lim_{s\to\infty} s^k G(s) = CA^{k-1}B \qquad \ldots(6)$$

exists, is nonsingular and $|G(s)| \not\equiv 0$. On intuitive grounds, the above relations imply that each loop has a dynamic behaviour analogous to a classical rank k transfer function and, as such, G(s) will be termed a uniform rank transfer function matrix and S(A,B,C) a uniform rank system.

The following theorem defines the asymptotic form of the root-locus plot in terms of the expansion of the matrix $(A-pBC)^\ell$. For convenience, define

$$\Gamma_o = 0 \;, \quad \Gamma_\ell = \sum_{j=1}^{\ell} A^{j-1}BCA^{\ell-j} \;, \quad \ell \geq 1 \qquad \ldots(7)$$

and note from condition (5) that

$$(A-pBC)^{\ell} = A^{\ell} - p\Gamma_{\ell} \quad , \quad 1 \leqslant \ell \leqslant k \qquad \ldots(8)$$

and, from equation (7), by induction,

$$\Gamma_o = 0 \qquad \ldots(9)$$

$$\Gamma_{j+1} = A\Gamma_j + BCA^j$$
$$= \Gamma_j A + A^j BC \quad , \quad j \geqslant 0 \qquad \ldots(10)$$

Theorem 2

With the above notation, and $S(A,B,C)$ of uniform rank, the closed-loop system $S(A-pBC,B,C)$ has km unbounded poles of the form, $1\leqslant j\leqslant m$, $1\leqslant \ell\leqslant k$

$$\mu_{j\ell}(p) = p^{\frac{1}{k}}\eta_{j\ell} + \alpha_j + \varepsilon_{j\ell}(p) \qquad \ldots(11)$$

where $\eta_{j\ell}$, $1\leqslant \ell\leqslant k$ are the kth roots of $\lambda_j$ where $\lambda_j$ is a non-zero solution of

$$\{\lambda_j I_n + BCA^{k-1}\}x_\infty = 0 \quad , \quad \|x_\infty\| = 1 \qquad \ldots(12)$$

Also,

$$\lim_{p\to\infty} \varepsilon_{j\ell}(p) = 0$$

and if,

$$N(\lambda_j I_n + BCA^{k-1}) \cap R(\lambda_j I_n + BCA^{k-1}) = \{0\} \quad , \qquad \ldots(13)$$

then, the pivot $\alpha_j$ is a solution of the relation

$$\{k\alpha_j I_n - A\}x_\infty \in R(\lambda_j I_n + BCA^{k-1}) \qquad \ldots(14)$$

The remaining n-km poles tend to the zeros of $S(A,B,C)$.

Proof

Equation (3) implies that

$$\{(s(p))^{\ell} I_n - (A-pBC)^{\ell}\}x(p) = 0 \quad , \quad \ell \geqslant 1 \qquad \ldots(15)$$

or, by equation (8), for $1\leqslant \ell\leqslant k$,

$$\{\frac{(s(p))^{\ell}}{p} I_n - p^{-1}A^{\ell} + \Gamma_{\ell}\}x(p) = 0 \qquad \ldots(16)$$

In an analogous manner to section 2, suppose that the family $\{x(p)\}$ has a cluster point $x_\infty$ ( $\|x_\infty\| = 1$) then

$$\lim_{p\to\infty} \frac{s(p)^\ell}{p} x_\infty = \Gamma_\ell x_\infty \qquad , \qquad 1\leq\ell\leq k \qquad \ldots(17)$$

Considering only unbounded eigenvalues, then (Theorem 1) $x_\infty \in R(B)$ so that (equations (5),(7))

$$\lim_{p\to\infty} \frac{s(p)^\ell}{p} = 0 \qquad , \qquad \ell<k \qquad \ldots(18)$$

and,

$$\Gamma_k x_\infty = BCA^{k-1} x_\infty \neq 0 \qquad \ldots(19)$$

i.e (equation (17)) $\lambda_j \triangleq \lim_{p\to\infty} p^{-1}(s(p))^k$ exists, is non-zero and is a solution of the eigenvalue equation (12). Write $s(p) = p^{\frac{1}{k}}\eta_{j\ell}+\psi_{j\ell}(p)$ where $\eta_{j\ell}^k = \lambda_j$ and

$$\lim_{p\to\infty} p^{-\frac{1}{k}} \psi_{j\ell}(p) = 0 \qquad \ldots(20)$$

It follows that

$$\lim_{p\to\infty} p^{\frac{1}{k}} \{\frac{s(p)^k}{p} - \lambda\}x(p) = -\lim_{p\to\infty} p^{\frac{1}{k}} \{\lambda_j+\Gamma_\ell\}x(p) \qquad \ldots(21)$$

Writing (equation (10)) $\Gamma_k = A\Gamma_{k-1}+BCA^{k-1}$ and noting (equation (16)) that

$$\lim_{p\to\infty} p^{\frac{1}{k}} A\Gamma_{k-1}x(p) = -\lambda_j^{\frac{k-1}{k}} Ax_\infty \qquad \ldots(22)$$

then

$$\{\lim_{p\to\infty} p^{\frac{1}{k}} \{\frac{s(p)^k}{p} -\lambda_j\}I-\lambda_j^{\frac{k-1}{k}} A\}x_\infty = \lim_{p\to\infty} p^{\frac{1}{k}} \{\lambda_j+BCA^{k-1}\}x(p) \qquad \ldots(23)$$

Using equation (13) and noting that $x_\infty \in N(\lambda_j I_n+BCA^{k-1})$, it follows that $\lim_{p\to\infty} p^{\frac{1}{k}} (\frac{s(p)^k}{p} - \lambda_j)$ exists, so that (equation (20))

$$\lim_{p\to\infty} p^{\frac{1}{k}} \left(\frac{s(p)^k}{p} - \lambda_j\right) = \lim_{p\to\infty} k\,\lambda^{\frac{k-1}{k}}\,\mu_{j\ell}(p)$$
$$= k\,\lambda^{\frac{k-1}{k}}\,\alpha_{j\ell} \qquad \ldots(24)$$

for some finite scalar $\alpha_{j\ell}$. It is easily seen that $\alpha_{j\ell}$ is independent of $\ell$ and writing $\alpha_{j\ell} = \alpha_j$, equation (23) implies that

$$(k\alpha_j I_n - A)x_\infty \in R(\lambda_j I_n + BCA^{k-1}) \qquad \ldots(25)$$

as required.

Finally, it can be shown (Owens, 1975) that S(A,B,C) has n-km zeros, each of which (Shaked and Kouvaritakis, 1976) attracts a pole at high gain.

Q.E.D.

The above theorem provides explicit geometrical conditions for the construction of the asymptotes of the root locus plot. For purposes of calculation, write $x_\infty = Bz$, then (equation (12)) as rank $B = m$ (equation (5)), $\lambda_j$ is the solution of the eigenvalue problem,

$$\begin{aligned} 0 &= \{\lambda_j I_m + CA^{k-1}B\}\alpha_j \\ &= \{\lambda_j I_m + G_\infty^{(k)}\}\alpha_j \quad , \qquad \alpha_j \neq 0 \end{aligned} \qquad \ldots(26)$$

Condition (13) is equivalent to the requirement that $G_\infty^{(k)}$ has a complete set of eigenvectors. To calculate the pivot, suppose that $u_1,\ldots,u_\ell$ are linearly independent eigenvectors of $BCA^{k-1}$ spanning the eigenspace corresponding to the eigenvalue $\lambda_j$, and let $v_1^+,\ldots,v_\ell^+$ be the corresponding dual eigenvectors satisfying $v_j^+ u_\ell = \delta_{j,k}$, then, if $M_j$ is the $\ell x \ell$ matrix with elements

$$(M_j)_{rq} = k^{-1}\, v_r^+ A u_q \quad , \qquad 1<r,\ q<\ell \qquad \ldots(27)$$

it follows that $\alpha_j$ is a solution of the eigenvalue equation

$$\{\alpha_j I_\ell - M\}\beta_j = 0 \qquad \beta_j \neq 0 \qquad \ldots(28)$$

## 4. Asymptotic Root-loci for Non-uniform-rank Multivariable Systems

In more general situations (Shaked and Kouvaritakis, 1976) $S(A-pBC,B,C)$ will have unbounded poles of various orders as $p\to+\infty$. The main result of this section (Theorem 3) provides a geometric characterization of the asymptotes of the root-locus in certain situations of practical interest, using relations analogous to those of Theorem 2.

Write, $\ell\geq 1$

$$(A-pBC)^{\ell} = \sum_{j=o}^{\ell} (-1)^j p^j B_{j,\ell} \qquad \ldots(29)$$

from which, by induction,

$$B_{o1} = A \quad , \quad B_{11} = BC \qquad \ldots(30)$$

and, for $\ell\geq 1$,

$$\begin{aligned} B_{o,\ell+1} &= AB_{o,\ell} \\ B_{j,\ell+1} &= AB_{j,\ell} + BCB_{j-1,\ell} \quad , \quad 1\leq j\leq \ell \\ B_{\ell+1,\ell+1} &= BCB_{\ell,\ell} \end{aligned} \qquad \ldots(31)$$

so that (equations (9),(10),(30)), for $\ell\geq 1$,

$$B_{o,\ell} = A^{\ell} \quad , \quad B_{1,\ell} = \Gamma_{\ell} \quad , \quad B_{\ell,\ell} = (BC)^{\ell} \qquad \ldots(32)$$

In a similar manner to equation (5), let k be the uniquely defined integer such that

$$CA^{j-1}B = 0 \quad , \quad j<k \quad , \quad CA^{k-1}B \neq 0 \qquad \ldots(33)$$

then (equation (8))

$$BC(A-pBC)^{\ell} = \begin{cases} BCA^{\ell} & , \quad 0\leq\ell<k \\ BCA^{k-1}(A-pBC) & , \quad \ell = k \end{cases} \qquad \ldots(34)$$

so that, $k\leq\ell\leq 2k-1$,

$$BC(A-pBC)^{\ell} = BCA^{k-1}\{A^{\ell+1-k}-p\Gamma_{\ell+1-k}\} \qquad \ldots(35)$$

Defining

$$V_o \triangleq R(B) \quad , \quad V_\ell \triangleq R(B) \cap \bigcap_{j=1}^{\ell} N(BCA^{j-1}) \equiv \bigcap_{j=1}^{\ell} N(\Gamma_j) \cap R(B) \quad , \quad \ell \geq 1 \qquad \ldots(36)$$

then, if $|G(s)| \neq 0$, there exists an integer $\hat{k}$ such that

$$V_\ell \neq \{0\} \qquad (\ell < \hat{k}) \quad , \qquad V_\ell = \{0\} \qquad (\ell \geq \hat{k}) \qquad \ldots(37)$$

for, if $x \in V_\ell$ for all $\ell \geq 1$, then $A^\ell x \in N(c)$ for all $\ell \geq 0$ ie (theorem 1) $x \in W_B$ and the proposition is proved by noting (Owens, 1975) that $|G(s)| \neq 0$ implies that $W_B \cap R(B) = \{0\}$. It is easily shown that

$$k \leq \hat{k} \qquad \ldots(38)$$

Defining, $k-1 \leq \ell \leq 2k-1$,

$$W_\ell \triangleq R(BCA^{k-1}\Gamma_{\ell+1-k}) \quad , \quad X_\ell \triangleq \bigcap_{j=k-1}^{\ell} N(BCA^{k-1}\Gamma_{j+1-k}) \qquad \ldots(39)$$

the following theorem is proved below,

<u>Theorem 3</u>

With the above notation, $|G(s)| \neq 0$, and $\hat{k} \leq 2k$, then, if

$$V_\ell \cap W_\ell = \{0\} \quad , \qquad k-1 \leq \ell \leq \hat{k}-1$$
$$\{\Gamma_{\ell+1} V_\ell\} \cap W_\ell = \{0\} \quad , \qquad k-1 \leq \ell \leq \hat{k}-1 \qquad \ldots(40)$$

the closed-loop system $S(A-pBC,B,C)$ possesses unbounded poles of the form, $k \leq \ell \leq \hat{k}$,

$$s(p) = p^{\frac{1}{\ell}}\eta + f(p) \qquad \ldots(41)$$

where, if $\lambda$ is a non-zero solution of the relation

$$\{\lambda I_n + BCA^{\ell-1}\}x_\infty \in W_{\ell-1} \qquad \ldots(42)$$
$$0 \neq x_\infty \in V_{\ell-1}$$

then $\eta$ is an $\ell$th root of $\lambda$, $p^{\frac{1}{\ell}}$ is the positive real $\ell$th root of $p$ and

$$\lim_{p\to\infty} p^{-\frac{1}{\ell}} f(p) = 0 \qquad \ldots(43)$$

Moreover, if, for $k \leq \ell \leq \hat{k}$,

$$\{\{\lambda I_n + BCA^{\ell-1}\}X_{\ell-1} + W_{\ell-1}\} \cap V_{\ell-1} \cap \{\lambda I_n + BCA^{\ell-1}\}^{-1} W_{\ell-1} = \{0\} \qquad \ldots(44)$$

then f(p) takes the form

$$f(p) = \alpha + \varepsilon(p) \qquad \ldots(45)$$

where $\lim_{p\to\infty} \varepsilon(p) = 0$, and the 'pivot' $\alpha$ is a constant, finite solution of the relation

$$\{\ell\alpha I_n - A\}x_\infty \in \{\lambda I_n + BCA^{\ell-1}\}X_{\ell-1} + W_{\ell-1} \qquad \ldots(46)$$

Proof

Equation (3) implies that

$$\{(s(p))^\ell I_n - (A-pBC)^\ell\}x(p) = 0 \quad , \quad \|x(p)\| = 1 \ , \ \ell \geq 1 \qquad \ldots(47)$$

Using equations (8),(29),(32), this takes the form

$$\{p^{-1}(s(p))^\ell - p^{-1}A^\ell + \Gamma_\ell\}x(p) = \begin{cases} 0 & ; \quad \ell \leq k \\ \sum_{j=2}^{\ell} (-1)^j p^{j-1} B_{j,\ell}x(p) & ; \quad \ell > k \end{cases} \qquad \ldots(48)$$

Taking the case of $\ell = k$, it follows that $p^{-1}s(p)$ is bounded $(p\to\infty)$ from which $\lim_{p\to\infty} p^{-1}(s(p))^k$ exists for every closed-loop pole. If $\lim_{p\to\infty} p^{-1}(s(p))^k = \lambda \neq 0$, then, if $\lim_{p\to\infty} x(p) = x_\infty$ ($\|x_\infty\| = 1$), $\Gamma_j x_\infty = 0$, $j<k$, so that $x_\infty \in V_{k-1}$ and

$$\{\lambda I_n + BCA^{k-1}\}x_\infty = 0 \in W_{k-1} \qquad \ldots(49)$$

proving (42) in the case of $\ell = k$. Using induction, suppose that $\lim_{p\to\infty} p^{-1}(s(p))^r = 0$, $k \leq r < \ell \leq \hat{k}$, and $x_\infty \in V_{\ell-1}$, so that (equation (40)),

$$\lim_{p\to\infty} \sum_{j=2}^{r} (-1)^j p^{j-1} B_{j,r}x(p) = 0 \quad , \quad k \leq r < \ell \qquad \ldots(50)$$

Using equation (31),(32),(35), equation (50) becomes

$$\lim_{p\to\infty} \{A \sum_{j=2}^{r-1} (-1)^j p^{j-1} B_{j,r-1}x(p) - BC \sum_{j=1}^{r-1} (-1)^j p^j B_{j,r-1}x(p)\}$$

$$= \lim_{p\to\infty} pBCA^{k-1}\Gamma_{r-k}x(p) = 0 \quad , \quad k \leq r < \ell \qquad \ldots(51)$$

In a similar manner, it can be shown that

$$\lim_{p\to\infty} \{p^{-1}(s(p))^{\ell} I_n + \Gamma_\ell\}x(p) = \lim_{p\to\infty} pBCA^{k-1}\Gamma_{\ell-k}x(p) \in W_{\ell-1} \qquad \ldots(52)$$

Equation (40) implies that $p^{-1}(s(p))^{\ell}$ can only have a finite cluster point $\lambda$. If $\lambda = 0$, then $\Gamma_\ell x_\infty \in W_{\ell-1}$ or (equation (40)) $x_\infty \in N(\Gamma_\ell) \cap V_{\ell-1} = V_\ell$ and $\lim_{p\to\infty} pBCA^{k-1}\Gamma_{\ell-k}x(p) = 0$. Alternatively, if $\lambda \neq 0$, it is a solution of the relation

$$\{\lambda I_n + BCA^{\ell-1}\}x_\infty = BCA^{k-1}\Gamma_{\ell-k}z \in W_{\ell-1} \qquad \ldots(53)$$

for some vector $z \in R^n$, proving equation (42). Note that, if $\lim_{p\to\infty} p^{-1}(s(p))^{\ell} = 0$, $1 \leq \ell \leq \hat{k}$, then, from the definition of $\hat{k}$, $x_\infty \notin R(B)$ ie (theorem 1) $s(p)$ has a finite limit.

Finally, if $p^{-1}(s(p))^{\ell} \to \lambda \neq 0$ $(p\to\infty)$ and $\ell = k$, equation (40) follows directly from theorem 2. Alternatively, if $\ell > k$, rewrite equation (48) in the form,

$$p^{\frac{1}{\ell}} \{p^{-1}(s(p))^{\ell} - \lambda\}x(p)$$

$$= p^{\frac{1}{\ell}} \{-\{\lambda I_n + \Gamma_\ell\} + p^{-1}A^{\ell} + \sum_{j=2}^{\ell} (-1)^j p^{j-1} B_{j,\ell}\}x(p)$$

$$= p^{\frac{1}{\ell}} \{-\{\lambda I_n + \Gamma_\ell\} + p^{-1}A^{\ell} + A\sum_{j=2}^{\ell-1} p^{j-1}(-1)^j B_{j,\ell-1} - BC\sum_{j=1}^{\ell-1} (-1)^j p^j B_{j,\ell-1}\}x(p) \qquad \ldots(54)$$

From (48) replacing $\ell$, by $\ell-1$,

$$\lim_{p\to\infty} p^{\frac{1}{\ell}} A\{\sum_{j=2}^{\ell-1} p^{j-1}(-1)^j B_{j,\ell-1} - \Gamma_{\ell-1}\}x(p) = \eta^{\frac{\ell-1}{\ell}} Ax_\infty \qquad \ldots(55)$$

so that, using equation (10), equation (54) takes the form

$$\lim_{p\to\infty} \{p^{\frac{1}{\ell}} \{p^{-1}(s(p))^{\ell} - \lambda\}I_n - \eta^{\frac{\ell-1}{\ell}} A\}x_\infty$$

$$= -\lim_{p\to\infty} p^{\frac{1}{\ell}} \{\lambda I_n + BCA^{\ell-1} + BC\sum_{j=1}^{\ell-1} (-1)^j p^j B_{j,\ell-1}\}x(p)$$

$$= -\lim_{p\to\infty} p^{\frac{1}{\ell}} \{\lambda I_n + BCA^{\ell-1} - pBCA^{k-1}\Gamma_{\ell-k}\}x(p) \qquad \ldots(56)$$

Write $x(p) = x_1(p)+x_2(p)$, $x_1(p) \in X_{\ell-1}$, $x_2(p) \in X_{\ell-1}^{\perp}$, then the relations (equations (51)-(53))

$$\lim_{p\to\infty} p\, BCA^{k-1}\Gamma_{r-k}x(p) = 0 \qquad , \qquad k \leq r<\ell$$

$$\lim_{p\to\infty} p\, BCA^{k-1}\Gamma_{\ell-k}x(p) = BCA^{k-1}\Gamma_{\ell-k}z \qquad \text{(finite)} \qquad \ldots(57)$$

imply that $\lim_{p\to\infty} p^{\frac{1}{\ell}} x_2(p) = 0$ ie

$$\lim_{p\to\infty} \{p^{\frac{1}{\ell}} \{p^{-1}(s(p))^{\ell}-\lambda\}I_n-\eta^{\frac{\ell-1}{\ell}} A\}x_\infty$$

$$= -\lim_{p\to\infty} p^{\frac{1}{\ell}} \{\{\lambda I_n+BCA^{\ell-1}\}x_1(p)-pBCA^{k-1}\Gamma_{\ell-k}x(p)\} \qquad \ldots(58)$$

Condition (44) implies that $\lim_{p\to\infty} p^{\frac{1}{\ell}} \{(s(p))^{\ell}p^{-1}-\lambda\}$ exists or (equation (41),(43))

$$\lim_{p\to\infty} p^{\frac{1}{\ell}} \{p^{-1}(p^{\frac{1}{\ell}} \eta+f(p))^{\ell}-\lambda\} = \lim_{p\to\infty} \ell\eta^{\frac{\ell-1}{\ell}} f(p) \triangleq \ell\eta^{\frac{\ell-1}{\ell}} \alpha \qquad \ldots(59)$$

for one finite constant $\alpha$. Substituting into (58) yields the relation

$$\{\ell\alpha I_n-A\}x_\infty \in \{\lambda I_n+BCA^{\ell-1}\}X_{\ell-1} + W_{\ell-1} \qquad \ldots(60)$$

which prove the result.

Q.E.D.

It is easily shown that theorem 3 reduces to theorem 2 if S(A,B,C) is of uniform rank.

## 5. Illustrative Example

Consider the non-uniform-rank system defined by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

from which, $|G(s)| \neq 0$, $k = 1$

$$\Gamma_o = 0 \quad , \quad \Gamma_1 = BC = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad , \quad (BC)^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$BCA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \Gamma_2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \ldots(62)$$

so that

$$V_o = \text{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} , \quad V_1 = \text{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} , \quad V_2 = \{0\} \qquad \ldots(63)$$

ie $\hat{k} = 2 = 2k$. Also,

$$W_o = \{0\} \quad , W_1 = \text{span}\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \} , \quad X_o = R^3 \quad , \quad X_1 = \text{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \} \ldots(64)$$

so that $W_o \cap V_o = W_1 \cap V_1 = \{0\}$, $\Gamma_1 V_o \cap W_o = \{0\}$, $\Gamma_2 V_1 \cap W_1 = \{0\}$.

To calculate the first order asymptote, solve the equation

$$\{\lambda I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \} x_\infty \in W_o \quad , \quad x_\infty \in V_o \qquad \ldots(65)$$

ie $\lambda = -1$ and $x_\infty = \{0,-1,1\}^T \in R(B)$. The corresponding pivot is the solution of the relation

$$\{\alpha I_3 - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} z \quad , \quad z \in R^3 \qquad \ldots(66)$$

ie $\alpha = 0$, and the asymptote takes the form $-p$; and passes through the origin of the complex plane. Considering now the second order type asymptote,

$$\{\lambda I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\} \; x_\infty \in W_1 \quad , \quad x_\infty \in V_1 \qquad \ldots(67)$$

ie $\lambda = -2$ and $x_\infty = \{0, 1, 0\}^T$. The corresponding pivot is the solution of the relation

$$\{2\alpha I_3 - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + z_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad \ldots(68)$$

ie $\alpha = 1$ and the second order asymptote takes the form $\pm p^{\frac{1}{2}}\sqrt{-2} + 1$.

## 6. Conclusions

The geometric characterization of the asymptotes of multivariable root-loci of a linear system has been discussed for two cases of practical interest. The work augments the analysis of Shaked and Kouvaritakis (1976) and illustrates the fact that (i) two integer parameters $k, \hat{k}$, derived from geometric considerations, play a fundamental role in the description of the root-locus, and (ii) both the asymptotic directions and pivots are described by inclusion relationships in the state space of the form

$$\{\xi I_n + F\}x \in Q \quad , \quad z \in P \qquad \ldots(69)$$

where F is a nxn matrix and P,Q are well-defined subspaces of the state space. A glance at theorems 2,3 will indicate that the matrices A, $BCA^{j-1}$, $\Gamma_j$, $k \leq j \leq \hat{k}$ play a fundamental role in the root-locus theory. Writing,

$$\Gamma_j = [B, AB, \ldots, A^{j-1}B] \begin{bmatrix} CA^{j-1} \\ CA^{j-2} \\ \vdots \\ CA \\ C \end{bmatrix} \qquad \ldots(70)$$

it is seen that the controllability and observability matrices play an important role in determining the structure of the root-locus. Further work could relate the structure of the root-loci to parameters defining controllability and observability, provide valuable insight into difficulties occurring in pole allocation and suggest new algorithms for the calculation of the system asymptotes.

## References

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Submitted to Intl J. Control