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Discrete Time subharmonic modelling and Analysis: Part II. Frequency domain analysis

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Discrete Time subharmonic modelling and Analysis: Part II. Frequency domain analysis

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Abstract: In Part I of this paper(Li and Billings, 2004), a new method of constructing MISO NARX models was introduced for a class of severely nonlinear systems that exhibit subharmonics. In this, the second part, the frequency domain properties based on the results of the time domain MISO modeling in Part I are revealed, explained and discussed. First, the derivation of generalised frequency response functions(GFRF's) from time domain MISO NARX models is introduced. The steady state response synthesis problem using the input spectrum and the MISO GFRF's is then investigated in order to verify the effectiveness and accuracy of the MISO modeling approach for severely nonlinear systems. Finally a new frequency domain analysis method is introduced for systems that exhibit subharmonic oscillations.

1. Introduction

In part I of this paper(Li and Billings, 2004) a new way of constructing MISO NARX models for systems that exhibit subharmonics was presented based on a multiplexed input signal. In this, the second part of the paper, the frequency domain properties of the MISO NARX model, which is used to represent systems that exhibit subharmonics, will be introduced.

The frequency domain behaviour of nonlinear systems has been studied by several authors, including Bussgang, et. al.(1974) and Bedrosian and Rice(1971). More recently Kim and Powers(1988) and Billings and Peyton Jones(1990) investigated the estimation and analysis of generalised frequency response functions. However, this analysis is only valid for weakly nonlinear systems. In the second part of this paper new results are introduced to provide a frequency domain analysis of severely nonlinear systems which exhibit subharmonics. The frequency domain analysis of subharmonic behaviour is an important problem but surprisingly few results appear to be available, and the main objective of the current paper is to begin to address this oversight.

The paper begins in Section 2 with the determination of the generalised frequency response functions(GFRF's) for MISO nonlinear models. These are extended in Section 3 to generate the GFRF's based on a Dual-Input-Single-Output(DISO) NARX model and the interpretation of the frequency response behaviour is studied. In Section 4 a formulation for the frequency domain response synthesis of DISO systems using GFRF's is proposed, and applied to verify the validity of the results. New analysis results relating to subharmonic behaviour are also derived. Conclusions are presented in Section 5.

2. MISO Frequency Domain Volterra Analysis

For a SISO nonlinear system the Volterra functional series can be expressed as

$$y(t)=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is the 'nth-order Volterra Kernel'. The multi-dimensional Fourier transform of $h_n(\cdot)$ yields the 'nth-order frequency response function' or the Generalised Frequency Response Function (GFRF):

$$H_n(\omega_1,\cdots,\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{2}$$

In present study, the models that allow a frequency domain analysis for subharmonic systems are based on the MISO format arising from the new input-multiplexing technique, which was illustrated in detail in part I of this paper(Li and Billings, 2004). Therefore the GFRF's for SISO nonlinear systems will need to be extended to MISO nonlinear systems.

For a MISO nonlinear system with N inputs, the output can be expressed as(Swain and Billings, 2001)

$$y(t)=\sum_{n=1}^{\infty}y_n(t)=\sum_{n=1}^{\infty}\sum_{n_1=1}^{N}\sum_{n_2=n_1}^{N}\cdots\sum_{n_n=n_{n-1}}^{N}y_n^{u_{n_1},\ldots,u_{n_n}}(t) \tag{3.a}$$

where

$$y_n^{u_{n_1},\ldots,u_{n_n}}(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n^{u_{n_1},\ldots,u_{n_n}}(\tau_1,\ldots\tau_n)\prod_{i=1}^{n}u_i(t-\tau_i)d\tau_i \tag{3.b}$$

Provided that the infinite sum of homogeneous terms in a Volterra system representation is convergent, which usually requires a bound on $u(t)$ over a time interval, a finite Volterra series, sometimes called a polynomial Volterra series(Rugh, 1981), can be used in practice. Among polynomial Volterra series, usually only the first few terms are significant, and the higher order terms decay rapidly. In this discussion, the first three orders of the nonlinear output will be considered, since this has been found to be sufficient for most practical cases.

The simplest form of MISO model, that is, a dual-input-single-output(DISO) model, will be studied in detail. The results below are based on the DISO model identified for a system that exhibits second order subharmonic oscillations introduced in Part I of this paper(Li and Billings, 2004). The GFRF's of this model, up to third order, will be considered throughout the paper. This makes the whole analysis much simpler, more transparent and thus easier to understand.

From (3), the first three orders of the outputs for a DISO system are

$$y_1(t)=y_1^{u_1}(t)+y_1^{u_2}(t)=\sum_{n_1=1}^{2}\int_{-\infty}^{\infty}h_1^{u_{n_1}}(\tau)u(t-\tau)d\tau \tag{4.a}$$

$$\begin{aligned}y_2(t)&=y_2^{u_1u_1}(t)+y_2^{u_1u_2}(t)+y_2^{u_2u_2}(t)\\&=\sum_{n_1=1}^{2}\sum_{n_2=n_1}^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2^{u_{n_1}u_{n_2}}(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1d\tau_2\end{aligned} \tag{4.b}$$

$$\begin{aligned}y_3(t)&=y_3^{u_1u_1u_1}(t)+y_3^{u_1u_1u_2}(t)+y_3^{u_1u_2u_2}(t)+y_3^{u_2u_2u_2}(t)\\&=\sum_{n_1=1}^{2}\sum_{n_2=n_1}^{2}\sum_{n_3=n_2}^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_3^{u_{n_1}u_{n_2}u_{n_3}}(\tau_1,\tau_2,\tau_3)u(t-\tau_1)u(t-\tau_2)u(t-\tau_3)d\tau_1d\tau_2d\tau_3\end{aligned} \tag{4.c}$$

and the first three orders of GFRF's for this DISO system are

$$H_1^{u_{n_1}}(\omega_1) = \int_{-\infty}^{\infty} h_1^{u_{n_1}}(\tau_1)\exp(-j\omega_1\tau_1)d\tau_1 \qquad \text{with} \quad n_1 = \{1,2\} \tag{5.a}$$

$$H_2^{u_{n_1}u_{n_2}}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{u_{n_1}u_{n_2}}(\tau_1,\tau_2)\exp(-j(\omega_1\tau_1+\omega_2\tau_2))d\tau_1 d\tau_2 \tag{5.b}$$

with $[n_1, n_2] = \{[1,1],[1,2],[2,2]\}$

$$H_3^{u_{n_1}u_{n_2}u_{n_3}}(\omega_1,\omega_2,\omega_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3^{u_{n_1}u_{n_2}u_{n_3}}(\tau_1,\tau_2,\tau_3)\exp(-j(\omega_1\tau_1+\omega_2\tau_2+\omega_3\tau_3))d\tau_1 d\tau_2 d\tau_3$$

with $[n_1, n_2, n_3] = \{[1,1,1],[1,1,2],[1,2,2],[2,2,2]\}$

(5.c)

The self-kernel DISO GFRF's generated using the above probing scheme are not necessarily unique in the sense that changing the order of any two arguments generates a new function without changing the value of $y_n(t)$ in eqn (3). The symmetric version of $H_n^{u_{n_1},\ldots,u_{n_n}}(\cdot)$ is normally used because this is unique and has values that are independent of the order of the arguments. This is given as

$$H_{n_{sym}}^{u_{n_1},\ldots,u_{n_n}}(\omega_1,\cdots,\omega_n) = \frac{1}{n!}\sum_{\substack{\text{all permutations}\\ \text{of } \omega_1,\ldots,\omega_n}} H_n^{u_{n_1},\ldots,u_{n_n}}(\omega_1,\cdots,\omega_n) \tag{6}$$

The DISO Volterra model structure can then be visualized as illustrated in Figure 1.

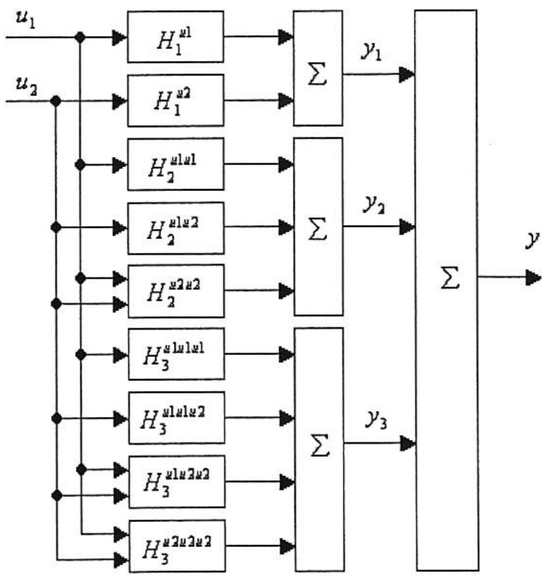

Figure 1. DISO frequency domain polynomial Volterra model

3. Generating the GFRF's for the MISO Model Built for Subharmonic Systems

Nonlinear generalized frequency response functions represent an inherent and invariant property of the underlying system. They are independent of external input-output signals and provide a powerful analysis tool for characterising nonlinear phenomena.

The GFRF's are essentially an extension of the widely applied gain and phase linear frequency response analysis method to the nonlinear case.

Non-parametric estimation methods for GFRF's of weakly nonlinear SISO systems have been studied by many authors(Brillinger, 1970; Lee and Schetzen, 1965, Koukoulas and Kalouptsidis, 1995; Kim and Powers,1988). The common limitations of these methods are the high complexity of the computations involved and the large data sets required, the restriction on the input signals, and the need for windowing and smoothing. Alternatively, GFRF's can be obtained using the so-called 'probing method' (Bedrosian and Rice, 1971;Billings and Peyton Jones, 1990), based on parametric models such as nonlinear differential/difference models. The latter parametric method of obtaining GFRF's will be used in this study.

For weakly SISO nonlinear systems, the SISO GFRF's provide comprehensive tools in frequency domain analysis. However, for severely nonlinear systems that exhibit subharmonics, this approach cannot be applied. To overcome this limitation, the SISO problem has to be transformed into a MISO problem, as studied in part I of this paper(Li and Billings, 2004). The parametric method of obtaining GFRF's for MISO systems(Swain and Billings, 2001), which is an extension of the SISO parametric method, will therefore be investigated below.

The frequency domain analysis of nonlinear systems based on MISO parametric models is generally complex and is not easily transparent. Hence to illustrate the approach consider the DISO nonlinear difference model from eqn (16) in part I(Li and Billings, 2004), for the subharmonic nonlinear system eqn (14) in Part I with a sampling frequency $f_s = 40/\pi$. Note that the analysis results are not restricted to the DISO case.

Eqn (16) from part I(Li and Billings, 2004) is rewritten below as eqn(7).

$$\begin{aligned} y(k) = {} & 3.9187y(k\text{-}1) - 5.7870y(k\text{-}2) + 3.8163y(k\text{-}3) - 0.9482y(k\text{-}4) + 0.0032u_1(k\text{-}6) - 0.00794u_1(k\text{-}5) \\ & + 0.005331u_1(k\text{-}4) - 0.000614u_1(k\text{-}1) - 0.00439u_2(k\text{-}6) - 0.00840u_2(k\text{-}4) + 0.01135u_2(k\text{-}5) \\ & + 0.001437u_2(k\text{-}2) + 0.00126y^2(k\text{-}1) + 0.0000321y^2(k\text{-}4) - 0.002491y(k\text{-}1)y(k\text{-}2) \\ & + 0.000414y^2(k\text{-}2) + 0.000739y(k\text{-}2)y(k\text{-}4) + 0.00191y(k\text{-}1)y(k\text{-}3) - 0.001312y(k\text{-}2)y(k\text{-}3) \\ & - 0.000756y(k\text{-}1)y(k\text{-}4) + 0.000223y^2(k\text{-}3) + 0.0002209u_1^2(k\text{-}6) - 0.00102713u_1(k\text{-}5)u_1(k\text{-}6) \\ & + 0.001770u_1(k\text{-}5)\hat{}2 + 0.0001113u_1(k\text{-}3)u_1(k\text{-}6) + 0.0003873u_1(k\text{-}3)u_1(k\text{-}5) \\ & - 0.001462u_1(k\text{-}4)u_1(k\text{-}5) \end{aligned} \tag{7}$$

Initially the system (7) is probed by setting $u_1(k) = e^{j\omega_1 k}$ and $u_2(k) = 0$. The output of the system is therefore

$$y_1^{u_1}(k) = H_1^{u_1}(\omega_1)e^{j\omega_1 k} \tag{8}$$

Substituting the values of $u_1(k)$, $u_2(k)$ and $y_1^{u_1}(k)$ into (7) and equating coefficients of $e^{j\omega_1 k}$ yields

$$H_1^{u_1}(\omega_1) = \frac{-0.0032e^{-j6\omega_1} - 0.00794e^{-j5\omega_1} + 0.00533e^{-j4\omega_1} - 0.000614e^{-j\omega_1}}{1\text{-}3.9187e^{-j\omega_1} + 5.7870e^{-j2\omega_1} \text{-} 3.8163e^{-j3\omega_1} + 0.9482e^{-j4\omega_1}} \tag{9}$$

Setting $u_1(k) = 0$ and $u_2(k) = e^{j\omega_1 k}$ as the probing input yields

$$H_1^{u_2}(\omega_1) = \frac{-0.00439e^{-j6\omega_1} - 0.00840e^{-j4\omega_1} + 0.01135e^{-j5\omega_1} + 0.001437e^{-j2\omega_1}}{1\text{-}3.9187e^{-j\omega_1} + 5.7870e^{-j2\omega_1} \text{-} 3.8163e^{-j3\omega_1} + 0.9482e^{-j4\omega_1}} \tag{10}$$

The second order frequency response functions are functions of the coefficients of the second order nonlinearities and the first order(linear) frequency response functions $H_1^{u_m}$. The derivation of the self-kernel transforms $H_2^{u_1u_1}$ and $H_2^{u_2u_2}$ is similar to that in the SISO case, setting $\{u_1(k)=e^{j\omega_1 k}+e^{j\omega_2 k}, u_2(k)=0\}$ and $\{u_1(k)=0, u_2(k)=e^{j\omega_1 k}+e^{j\omega_2 k}\}$ respectively in the probing procedure.

Notice that $H_2^{u_1u_1}$ has association with pure output AR terms(such as $y(k-i)y(k-k)$), pure u_1 MA terms(such as $u_1(k-i)u_1(k-k)$) and output-u_1 cross product terms(such as $y(k-i)u_1(k-k)$), and has nothing to do with any regressors involving u_2 (for example, $y(k-i)u_2(k-k), u_2(k-i)u_2(k-k)$).

Setting $u_1(k)=0, u_2(k)=e^{j\omega_1 k}+e^{j\omega_2 k}$, $H_2^{u_2u_2}$ can be derived which has a association with pure output AR terms, pure u_2 MA terms and output-u_2 terms, and has nothing to do with any regressors involving u_1.

For the current system (7), setting $u_1(k)=e^{j\omega_1 k}+e^{j\omega_2 k}, u_2(k)=0$, the output will be

$$\begin{aligned} y(k) &= y_1^{u_1}(k)+y_2^{u_1u_1}(k) \\ &= H_1^{u_1}(\omega_1)e^{j\omega_1 k}+H_1^{u_1}(\omega_2)e^{j\omega_2 k}+H_2^{u_1u_1}(\omega_1,\omega_2)e^{j(\omega_1+\omega_2)k} \\ &\quad +H_2^{u_1u_1}(\omega_2,\omega_1)e^{j(\omega_1+\omega_2)k}+H_2^{u_1u_1}(\omega_1,\omega_1)e^{j2\omega_1 k}+H_2^{u_1u_1}(\omega_2,\omega_2)e^{j2\omega_2 k} \end{aligned} \tag{11}$$

Substituting the values of $u_1(k)$, $u_2(k)$ and $y(k)$ into (7) and equating coefficients of $e^{j(\omega_1+\omega_2)k}$ yields

$$H_2^{u_1u_1}(\omega_1,\omega_2)$$
$$= \frac{\begin{aligned} &0.00126\, H_{11}^{u_1u_1} e^{-j(\omega_1+\omega_2)} + 3.21\text{e-}05\, H_{11}^{u_1u_1} e^{-j(4\omega_1+4\omega_2)} - 0.002492\, H_{11}^{u_1u_1} e^{-j(\omega_1+2\omega 2)} \\ &+0.000739\, H_{11}^{u_1u_1} e^{-j(2\omega_1+4\omega_2)} + 0.00191\, H_{11}^{u_1u_1} e^{-j(\omega_1+3\omega_2)} + 0.000414 H_{11}^{u_1u_1} e^{-j(2\omega_1+2\omega_2)} \\ &-0.00131 H_{11}^{u_1u_1} e^{-j(2\omega_1+3\omega_2)} - 0.000756 H_{11}^{u_1u_1} e^{-j(\omega_1+4\omega_2)} + 0.000223\, H_{11}^{u_1u_1} e^{-j(3\omega_1+3\omega_2)} \\ &+0.000221 e^{-j(6\omega_1+6\omega_2)} - 0.00103 e^{-j(5\omega_1+6\omega_2)} + 0.00177 e^{-j(5\omega_1+5\omega_2)} \\ &+0.0001113 e^{-j(3\omega_1+6\omega_2)} + 0.0003873 e^{-j(3\omega_1+5\omega_2)} - 0.001462 e^{-j(4\omega_1+5\omega_2)} \end{aligned}}{1\text{-}3.9187 e^{-j(\omega_1+\omega_2)} + 5.7870 e^{-j2(\omega_1+\omega_2)} \text{-}3.8163 e^{-j3(\omega_1+\omega_2)} + 0.9482 e^{-j4(\omega_1+\omega_2)}} \tag{12}$$

where $H_{11}^{u_1u_1}(\omega_1,\omega_2)=H_1^{u_1}(\omega_1)H_1^{u_1}(\omega_2)$.

Note that $H_2^{u_1u_1}(\omega_1,\omega_2)$ in (12) is not symmetric over $\{\omega_1,\omega_2\}$. A unique symmetric version of $H_2^{u_1u_1}(\omega_1,\omega_2)$ which is used in practical situations can be obtained by using (6).

Unlike the self-kernel transforms $H_2^{u_1u_1}$ and $H_2^{u_2u_2}$, the cross-kernel transforms $H_2^{u_1u_2}$ will be associated with pure output AR terms(such as $y(k-i)y(k-k)$), output-u_1 cross product terms(such as $y(k-i)u_1(k-k)$) and output-u_2 cross product terms(such as $y(k-i)u_2(k-k)$), but will have no relation to pure u_1 MA terms(such as $u_1(k-i)u_1(k-k)$) and pure u_2 MA terms(such as $u_2(k-i)u_2(k-k)$). By probing the system (7) using $u_1(k)=e^{j\omega_1 k}$ and $u_2(k)=e^{j\omega_2 k}$, the output of the system becomes

$$
\begin{aligned}
y(k) &= y_1^{u_1}(k) + y_1^{u_2}(k) + y_2^{u_1u_3}(k) \\
&= H_1^{u_1}(\omega_1)e^{j\omega_1 k} + H_1^{u_2}(\omega_2)e^{j\omega_2 k} + H_2^{u_1u_2}(\omega_1,\omega_2)e^{j(\omega_1+\omega_2)k} \\
&\quad + H_2^{u_2u_1}(\omega_2,\omega_1)e^{j(\omega_1+\omega_2)k}
\end{aligned}
\tag{13}
$$

Substituting the values of $u_1(k)$, $u_2(k)$ and $y(k)$ into (7) and equating coefficients of $e^{j(\omega_1+\omega_2)k}$ yields

$$
H_2^{u_1u_2}(\omega_1,\omega_2) = \frac{\begin{aligned}&0.00126\, H_{11}^{u_1u_2} e^{-j(\omega_1+\omega_2)} + 3.21\mathrm{e}-05\, H_{11}^{u_1u_2} e^{-j(4\omega_1+4\omega_2)} - 0.002492\, H_{11}^{u_1u_2} \mathrm{e}^{-j(\omega_1+2\omega 2)} \\ &+ 0.000739\, H_{11}^{u_1u_2} \mathrm{e}^{-j(2\omega_1+4\omega_2)} + 0.00191\, H_{11}^{u_1u_2} \mathrm{e}^{-j(\omega_1+3\omega_2)} + 0.000414 H_{11}^{u_1u_2} \mathrm{e}^{-j(2\omega_1+2\omega_2)} \\ &- 0.00131 H_{11}^{u_1u_2}\, \mathrm{e}^{-j(2\omega_1+3\omega_2)} - 0.000756 H_{11}^{u_1u_2}\, \mathrm{e}^{-j(\omega_1+4\omega_2)} + 0.000223\, H_{11}^{u_1u_2} \mathrm{e}^{-j(3\omega_1+3\omega_2)}\end{aligned}}{1-3.9187e^{-j(\omega_1+\omega_2)} + 5.7870e^{-j2(\omega_1+\omega_2)} - 3.8163e^{-j3(\omega_1+\omega_2)} + 0.9482e^{-j4(\omega_1+\omega_2)}}
\tag{14}
$$

where $H_{11}^{u_1u_2}(\omega_1,\omega_2) = H_1^{u1}(\omega_1)H_1^{u_2}(\omega_2)$.

Note that $H_2^{u_1u_2}$ does not have the symmetric property like $H_2^{u_1u_1}$ so that $H_2^{u_1u_2}$ cannot be symmetrized using (6). An alternative average cross-kernel transform was therefore defined in Swain and Billings(2001) to overcome this problem. This is defined as

$$
H_{2avg}^{u_1u_2}(\omega_1,\omega_2) = \tfrac{1}{2}[H_2^{u_1u_2}(\omega_1,\omega_2) + H_2^{u_2u_1}(\omega_2,\omega_1)]
\tag{15}
$$

The third order self-kernel frequency response functions $H_3^{u_1u_1u_1}$ and $H_3^{u_2u_2u_2}$ can be obtained by probing the system (7) by setting $\{u_1(k) = e^{j\omega_1 k} + e^{j\omega_2 k} + e^{j\omega_3 k}, u_2(k) = 0\}$ and $\{u_1(k) = 0, u_2(k) = e^{j\omega_1 k} + e^{j\omega_2 k} + e^{j\omega_3 k}\}$ respectively. The third order cross-kernel frequency response functions $H_3^{u_1u_1u_2}$ and $H_3^{u_1u_2u_2}$ can be obtained by probing the system (7) by setting $\{u_1(k) = e^{j\omega_1 k} + e^{j\omega_2 k}, u_2(k) = e^{j\omega_3 k}\}$ and $\{u_1(k) = e^{j\omega_1 k}, u_2(k) = e^{j\omega_2 k} + e^{j\omega_3 k}\}$ respectively. Like the second order cross-kernel transform $H_2^{u_1u_2}$, the third order cross-kernel GFRF's, $H_3^{u_1u_1u_2}$ and $H_3^{u_1u_2u_2}$, are not symmetric over $\{\omega_1,\omega_2,\omega_3\}$. However, $H_3^{u_1u_1u_2}(\omega_1,\omega_2,\omega_3)$ and $H_3^{u_1u_2u_2}(\omega_1,\omega_2,\omega_3)$ are still symmetric over $\{\omega_1,\omega_2\}$ and $\{\omega_2,\omega_3\}$ respectively.

$H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$ for the model eqn (7) are plotted in Figure 2.

It can be seen from Figure 2 that the difference in the gain plots between $H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$ is very small, but there is a significant phase difference. This difference is reflected in the higher order transfer functions. As will be seen in the next section, this difference is vital in the formation of subharmonics at a frequency of 1 rad/sec. This point will be revisited latter in section 4.2.

Inspection of figures 2-5 shows that the gain for every transfer function becomes very small for frequencies over 3 rad/sec. This reflects the fact that the output signal contains mainly frequency components at 1 rad/sec and 2 rad/sec. The interpretation of Figures 2-5 will be considered in more detail in section 4.2.

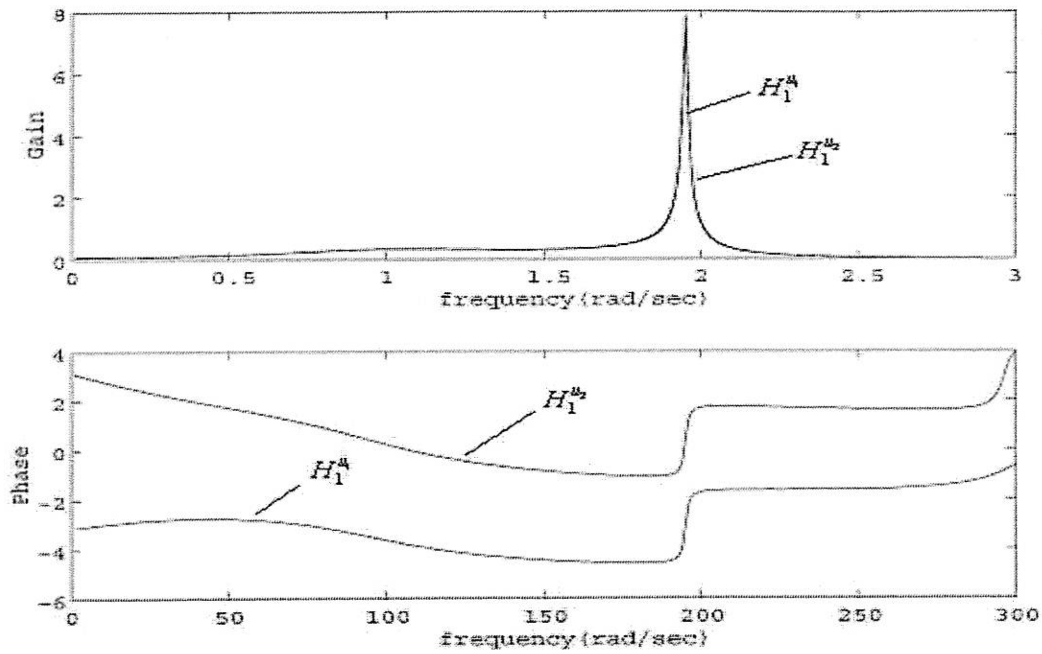

Figure 2. $H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$ for the DISO system (7): (a) Gain (b) Phase

The functions $H_2^{u_1u_1}, H_2^{u_2u_2}$ and $H_{2avg}^{u_1u_2}$ are plotted in Figures 3,4 and 5 respectively.

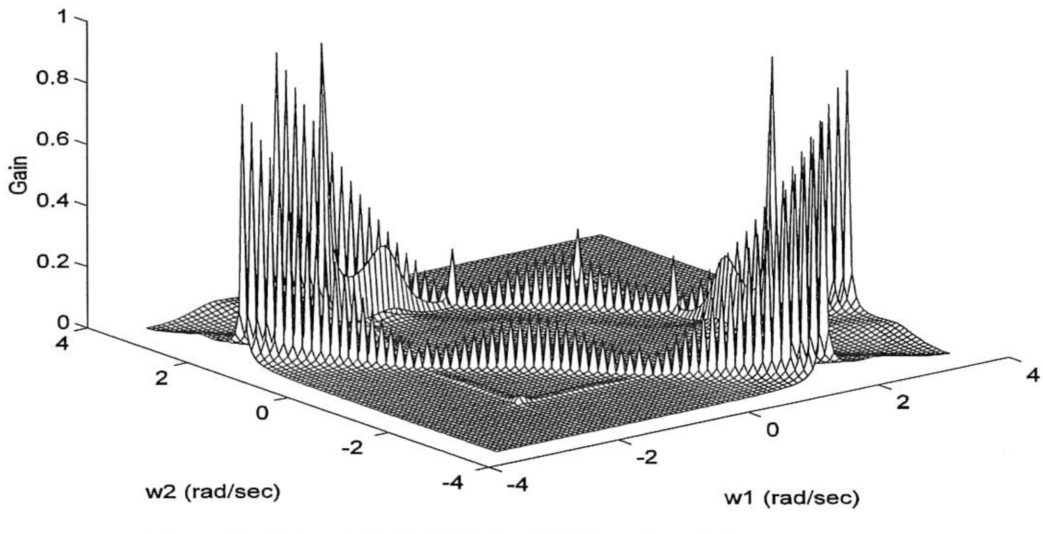

Figure 3. Gain plot of $H_2^{u_1u_1}$ for DISO system (7)

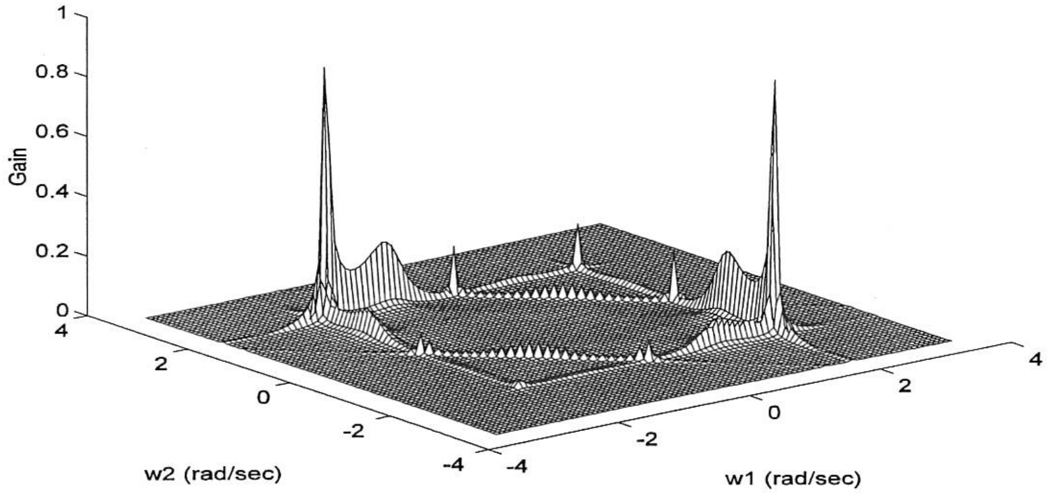

Figure 4. Gain plot of $H_2^{u_2u_2}$ for DISO system (7)

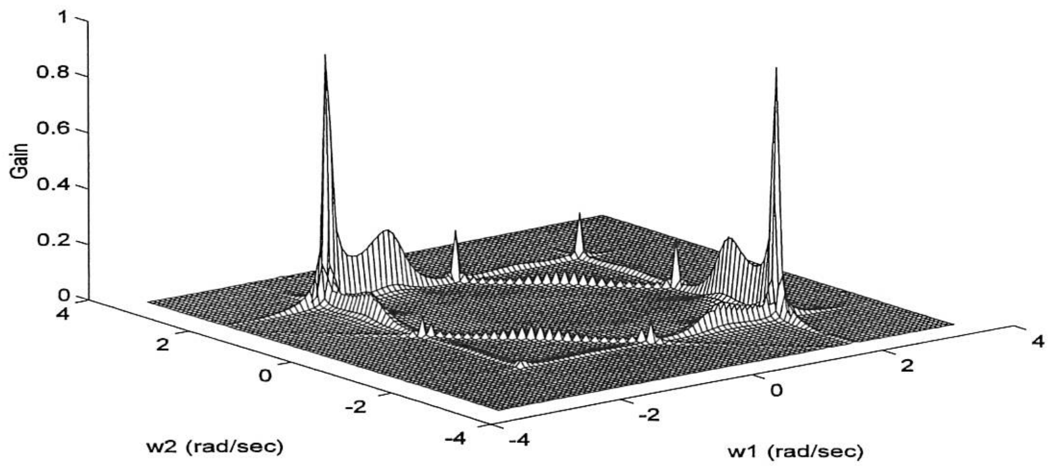

Figure 5. Gain plot of $H_2^{u_1u_2}$ for DISO system (7)

4. Frequency Domain Response Synthesis

The main limitation for the analysis of nonlinear systems using the GFRF's is that only harmonics $n\omega_i$ and intermodulations $m\omega_j + k\omega_l$, where ω_i, ω_j and ω_l are frequency components in the input and n, m, k are integers, can be represented by this type of analysis. Therefore while a SISO NARX model built using the original input-output signals of a subharmonic system can provide a very good time domain representation of the underlying system, an equivalent Volterra domain/frequency domain representation will not exist. This is because the GFRF's cannot represent subharmonics components ω_i/n, where n is integer. This is a severe limitation which precludes a full frequency domain analysis of subharmonic behaviour which of course is essentially a frequency domain effect. We have shown in Part I(Li and Billings, 2004) however that this limitation can be avoided by using a MISO NARX model and computing the GFRF's based on this MISO model. All the analysis in section 3 was based on the assumption that the DISO GFRF's are valid and accurate in the Volterra-kernel sense. The validity of this Volterra/frequency representation will be verified in this section.

A valid frequency domain/Volterra representation of a nonlinear system will exist if the steady-state response of the system under harmonic input(s) can be accurately recovered from the GFRF's. Another advantage of using the Fourier-based band-limited multiple inputs therefore becomes clear. Since the spectrum of each individual input is band-limited and fully known, the output spectrum can be obtained from the input spectrum and the DISO GFRF's. This can then be used to provide a quantitative measure related to the formation of subharmonics.

4.1 Response of a DISO nonlinear system to a sum of sinusoidal signals

A new method of calculating the output response of a DISO nonlinear system in the frequency domain based on the symmetric self-kernel transforms and the averaged cross-kernel transforms will be derived in this section and used to analyse the subharmonic response.

The output response will consist of two parts. The first part is related to the self-kernel transforms and the second part is related to the cross-kernel transforms. Derivation of the response relating to the self-kernel operators is quite similar to the SISO case, as both share the property of symmetry. For example, when calculating the response arising from $H_2^{u_1u_1}$, it can be assumed that the response is due to the sole input u_1, with $u_2 \equiv 0$. Up to third order nonlinear output responses for SISO GFRF's under multi-tone input signals are given in Bussgang, et al., (1974) and general expressions for all GFRF's were derived by Peyton Jones and Billings(1989). But a new approach has to be adopted for the asymmetric cross-kernel operators. The derivation of the response of the second and third order cross-kernel operators is given below.

Assume that each of u_1 and u_2 is a single-tone sinusoidal

$$\begin{aligned} u_1 &= \tfrac{A}{2}e^{j\omega^{u_1}t} + \tfrac{A^*}{2}e^{-j\omega^{u_1}t} = a + a^* \\ u_2 &= \tfrac{B}{2}e^{j\omega^{u_2}t} + \tfrac{B^*}{2}e^{-j\omega^{u_2}t} = b + b^* \end{aligned} \tag{16}$$

where * denotes complex conjugation.

The response for the second order cross-kernel operators will be

$$\begin{aligned} y_2^{12} &= H_2^{u_1u_2}[u_1u_2] + H_2^{u_2u_1}[u_1u_2] \\ &= 2H_{2\,avg}^{u_1u_2}[u_1u_2] \\ &= 2\{H_{2\,avg}^{u_1u_2}[\mathrm{ab}] + H_{2\,avg}^{u_1u_2}[\mathrm{a^*b^*}] + H_{2\,avg}^{u_1u_2}[\mathrm{a^*b}] + H_{2\,avg}^{u_1u_2}[\mathrm{ab^*}]\} \end{aligned} \tag{17}$$

The first term in (15) is

$$\begin{aligned} 2H_{2\,avg}^{u_1u_2}[\mathrm{ab}] &= \tfrac{AB}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{2\,avg}^{u_1u_2}(\tau_1,\tau_2)e^{j\omega^{u_1}(t-\tau_1)}e^{j\omega^{u_2}(t-\tau_2)}d\tau_1 d\tau_2 \\ &= \tfrac{AB}{2}H_{2\,avg}^{u_1u_2}(\omega^{u_1},\omega^{u_2})e^{j(\omega^{u_1}+\omega^{u_2})t} \end{aligned} \tag{18}$$

The second term in (15) is

$$\begin{aligned} 2H_{2\,avg}^{u_1u_2}[\mathrm{a^*b^*}] &= \tfrac{A^*B^*}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{2\,avg}^{u_1u_2}(\tau_1,\tau_2)e^{-j\omega^{u_1}(t-\tau_1)}e^{-j\omega^{u_2}(t-\tau_2)}d\tau_1 d\tau_2 \\ &= \tfrac{A^*B^*}{2}H_{2\,avg}^{u_1u_2}(-\omega^{u_1},-\omega^{u_2})e^{-j(\omega^{u_1}+\omega^{u_2})t} \end{aligned} \tag{19}$$

Eqns (18) and (19) are complex conjugates of each other, so the addition of the first two terms in (17) is given by

$$2\{H_{2\,avg}^{u_1u_2}[\mathrm{ab}]+H_{2\,avg}^{u_1u_2}[\mathrm{a}^*\mathrm{b}^*]\}=\mathrm{Re}[ABH_{2\,avg}^{u_1u_2}(\omega^{u_1},\omega^{u_2})e^{j(\omega^{u1}+\omega^{u2})t}] \quad (20)$$

Similarly, the third and fourth terms in (17) are complex conjugates of each other, hence

$$2\{H_{2\,avg}^{u_1u_2}[\mathrm{a}^*\mathrm{b}]+H_{2\,avg}^{u_1u_2}[\mathrm{ab}^*]\}=\mathrm{Re}[AB^*H_{2\,avg}^{u_1u_2}(\omega^{u_1},-\omega^{u_2})e^{j(\omega^{u1}-\omega^{u2})t}] \quad (21)$$

Therefore the final second order cross-kernel response of a DISO Volterra system for a single-tone input signal is

$$\begin{aligned} y_2^{12} &= \mathrm{Re}[ABH_{2\,avg}^{u_1u_2}(\omega^{u_1},\omega^{u_2})e^{j(\omega^{u1}+\omega^{u2})t}] \\ &\quad +\mathrm{Re}[AB^*H_{2\,avg}^{u_1u_2}(\omega^{u_1},-\omega^{u_2})e^{j(\omega^{u1}-\omega^{u2})t}] \end{aligned} \quad (22)$$

For the third order cross-kernel response under the operator $H_{3\,avg}^{u_1u_1u_2}(\omega_1,\omega_2,\omega_3)$, it is essential to notice that there is no longer overall symmetry over all three frequencies $\{\omega_1,\omega_2$ and $\omega_3\}$, but only symmetry over $\{\omega_1,\omega_2\}$. Hence

$$\begin{aligned} y_3^{112} &= H_3^{u_1u_1u_2}[u_1u_1u_2]+H_3^{u_1u_2u_1}[u_1u_1u_2]+H_3^{u_2u_1u_1}[u_1u_1u_2] \\ &= 3H_{3\,avg}^{u_1u_1u_2}[u_1u_1u_2] \\ &= 3H_{3\,avg}^{u_1u_1u_2}[a^2b]+3H_{3\,avg}^{u_1u_1u_2}[(a^*)^2b^*]+3H_{3\,avg}^{u_1u_1u_2}[a^2b^*]+3H_{3\,avg}^{u_1u_1u_2}[(a^*)^2b] \\ &\quad +6H_{3\,avg}^{u_1u_1u_2}[aa^*b]+6H_{3\,avg}^{u_1u_1u_2}[aa^*b^*] \end{aligned} \quad (23)$$

The first and second term in (23) are complex conjugates of each other, hence

$$3H_{3\,avg}^{u_1u_1u_2}[a^2b]+3H_{3\,avg}^{u_1u_1u_2}[(a^*)^2b^*]=\tfrac{3}{4}\mathrm{Re}[A^2BH_{3\,avg}^{u_1u_1u_2}(\omega^{u_1},\omega^{u_1},\omega^{u_2})e^{j(2\omega^{u1}+\omega^{u2})t}] \quad (24)$$

The third and fourth term in (23) are also complex conjugates of each other, hence

$$3H_{3\,avg}^{u_1u_1u_2}[a^2b^*]+3H_{3\,avg}^{u_1u_1u_2}[(a^*)^2b]=\tfrac{3}{4}\mathrm{Re}[A^2B^*H_{3\,avg}^{u_1u_1u_2}(\omega^{u_1},\omega^{u_1},-\omega^{u_2})e^{j(2\omega^{u1}-\omega^{u2})t}] \quad (25)$$

Because of the symmetric property over $\{\omega_1,\omega_2\}$, the last two terms in (23) can be re-expressed as

$$6H_{3\,avg}^{u_1u_1u_2}[aa^*b]+6H_{3\,avg}^{u_1u_1u_2}[aa^*b^*]=6H_{3\,avg}^{u_1u_1u_2}[aa^*b]+6H_{3\,avg}^{u_1u_1u_2}[a^*ab^*] \quad (26)$$

which will be complex conjugates of each other, hence,

$$6H_{3\,avg}^{u_1u_1u_2}[aa^*b]+6H_{3\,avg}^{u_1u_1u_2}[aa^*b^*]=\tfrac{3}{2}\mathrm{Re}[AA^*BH_{3\,avg}^{u_1u_1u_2}(\omega^{u_1},-\omega^{u_1},\omega^{u_2})e^{j\omega^{u2}t}] \quad (27)$$

Therefore the third order cross-kernel response of a DISO Volterra system for a single-tone input signal is

$$\begin{aligned} y_3^{112} &= \tfrac{3}{4}\mathrm{Re}[A^2BH_{3\,avg}^{u_1u_1u_2}(\omega^{u_1},\omega^{u_1},\omega^{u_2})e^{j(2\omega^{u1}+\omega^{u2})t}] \\ &\quad +\tfrac{3}{4}\mathrm{Re}[A^2B^*H_{3\,avg}^{u_1u_1u_2}(\omega^{u_1},\omega^{u_1},-\omega^{u_2})e^{j(2\omega^{u1}-\omega^{u2})t}] \\ &\quad +\tfrac{3}{2}\mathrm{Re}[AA^*BH_{3\,avg}^{u_1u_1u_2}(\omega^{u_1},-\omega^{u_1},\omega^{u_2})e^{j\omega^{u2}t}] \end{aligned} \quad (28)$$

In summary, assume that each of u_1 and u_2 is a two-tone sinusoidal

$$u_1 = \frac{1}{2}\sum_{i=1}^{2}[A_i exp(j\omega_i^{u_1}t) + A_i^* exp(-j\omega_i^{u_1}t)]$$

$$u_2 = \frac{1}{2}\sum_{i=1}^{2}[B_i exp(j\omega_i^{u_2}t) + B_i^* exp(-j\omega_i^{u_2}t)]$$

The output frequency components from the first and second order DISO GFRF's in response to these inputs are tabulated in Table 1, and the third order results are tabulated in Table 2.

Frequency	Amplitude	Type of Response
$\omega_i^{u_1}$ (i = 1,2)	$A_i H_1^{u_1}(\omega_i^{u_1})$	Linear
$\omega_i^{u_2}$ (i = 1,2)	$B_i H_1^{u_2}(\omega_i^{u_2})$	
$2\omega_i^{u_1}$ (i = 1,2)	$0.5A_i A_i H_2^{u_1u_1}(\omega_i^{u_1},\omega_i^{u_1})$	second order harmonics
$2\omega_i^{u_2}$ (i = 1,2)	$0.5B_i B_i H_2^{u_2u_2}(\omega_i^{u_2},\omega_i^{u_2})$	
$\omega_i^{u_1} - \omega_i^{u_1} = 0$(i = 1,2)	$0.5A_i A_i^* H_2^{u_1u_1}(\omega_i^{u_1},-\omega_i^{u_1})$	second order self inter - modulation terms
$\omega_1^{u_1} + \omega_2^{u_1}$	$0.5A_1 A_2 H_2^{u_1u_1}(\omega_1^{u_1},\omega_2^{u_1})$	
$\omega_1^{u_1} - \omega_2^{u_1}$	$0.5A_1 A_2^* H_2^{u_1u_1}(\omega_1^{u_1},-\omega_2^{u_1})$	
$\omega_i^{u_2} - \omega_i^{u_2} = 0$(i = 1,2)	$0.5B_i B_i^* H_2^{u_2u_2}(\omega_i^{u_2},-\omega_i^{u_2})$	
$\omega_1^{u_2} + \omega_2^{u_2}$	$0.5B_1 B_2 H_2^{u_2u_2}(\omega_1^{u_2},\omega_2^{u_2})$	
$\omega_1^{u_2} - \omega_2^{u_2}$	$0.5B_1 B_2^* H_2^{u_2u_2}(\omega_1^{u_2},-\omega_2^{u_2})$	
$\omega_i^{u_1} + \omega_i^{u_2} (i = 1,2)$	$A_i B_i H_{2avg}^{u_1u_2}(\omega_i^{u_1},\omega_i^{u_2})$	second order cross inter - modulation terms
$\omega_i^{u_1} - \omega_i^{u_2} (i = 1,2)$	$A_i B_i^* H_{2avg}^{u_1u_2}(\omega_i^{u_1},-\omega_i^{u_2})$	second order cross inter - modulation terms
$\omega_i^{u_1} + \omega_j^{u_2}, i, j = 1,2, i \neq j$	$A_i B_j H_{2avg}^{u_1u_2}(\omega_i^{u_1},\omega_j^{u_2})$	
$\omega_i^{u_1} - \omega_j^{u_2}, i, j = 1,2, i \neq j$	$A_i B_j^* H_{2avg}^{u_1u_2}(\omega_i^{u_1},-\omega_j^{u_2})$	second order cross inter - modulation terms

Table 1. First and second order DISO nonlinear system responses

Frequency	Amplitude	Type of Response
$3\omega_i^{u_1}$ (i = 1,2)	$0.25A_i^3 H_3^{u_1u_1u_1}(\omega_i^{u_1},\omega_i^{u_1},\omega_i^{u_1})$	third order harmonics
$3\omega_i^{u_2}$ (i = 1,2)	$0.25B_i^3 H_3^{u_2u_2u_2}(\omega_i^{u_2},\omega_i^{u_2},\omega_i^{u_2})$	
$\omega_i^{u_1}$ (i = 1,2)	$0.75A_i^2 A_i^* H_3^{u_1u_1u_1}(\omega_i^{u_1},\omega_i^{u_1},-\omega_i^{u_1})$	third order self inter - modulation terms
$\omega_i^{u_2}$ (i = 1,2)	$0.75B_i^2 B_i^* H_3^{u_2u_2u_2}(\omega_i^{u_2},\omega_i^{u_2},-\omega_i^{u_2})$	
$\omega_i^{u_1} + \omega_j^{u_1} - \omega_i^{u_1}, i, j = 1,2, i \neq j$	$1.5A_i A_j A_i^* H_3^{u_1u_1u_1}(\omega_i^{u_1},\omega_j^{u_1},-\omega_i^{u_1})$	
$\omega_i^{u_2} + \omega_j^{u_2} - \omega_i^{u_2}, i, j = 1,2, i \neq j$	$1.5B_i B_j B_i^* H_3^{u_2u_2u_2}(\omega_i^{u_2},\omega_j^{u_2},-\omega_i^{u_2})$	
$2\omega_i^{u_1} + \omega_j^{u_1}, i, j = 1,2, i \neq j$	$0.75A_i^2 A_j H_3^{u_1u_1u_1}(\omega_i^{u_1},\omega_i^{u_1},\omega_j^{u_1})$	
$2\omega_i^{u_2} + \omega_j^{u_2}, i, j = 1,2, i \neq j$	$0.75B_i^2 B_j H_3^{u_2u_2u_2}(\omega_i^{u_2},\omega_i^{u_2},\omega_j^{u_2})$	
$2\omega_i^{u_1} - \omega_j^{u_1}, i, j = 1,2, i \neq j$	$0.75A_i^2 A_j^* H_3^{u_1u_1u_1}(\omega_i^{u_1},\omega_i^{u_1},-\omega_j^{u_1})$	

$2\omega_i^{u_2} - \omega_j^{u_2}, i,j = 1,2, i \neq j$	$0.75B_i^2 B_j^* H_3^{u_2u_2u_2}(\omega_i^{u_2}, \omega_i^{u_2}, -\omega_j^{u_2})$	
$2\omega_i^{u_1} + \omega_i^{u_2} \, (i = 1,2)$	$0.75A_i^2 B_i H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, \omega_i^{u_1}, \omega_i^{u_2})$	third order cross inter - modulation terms
$\omega_i^{u_1} + 2\omega_i^{u_2} \, (i = 1,2)$	$0.75A_i B_i^2 H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, \omega_i^{u_2}, \omega_i^{u_2})$	
$\omega_i^{u_2} \, (i = 1,2)$	$1.5A_i A_i^* B_i H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, -\omega_i^{u_1}, \omega_i^{u_2})$	third order cross inter - modulation terms
$\omega_i^{u_2} \, (i = 1,2)$	$0.75A_i A_i B_i^* H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, \omega_i^{u_1}, -\omega_i^{u_2})$	
$\omega_i^{u_1} \, (\mathrm{i} = 1,2)$	$1.5A_i B_i^* B_i H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, -\omega_i^{u_1}, \omega_i^{u_2})$	
$\omega_i^{u_1} \, (\mathrm{i} = 1,2)$	$0.75A_i^* B_i B_i H_{3\,avg}^{u_1u_2u_2}(-\omega_i^{u_1}, \omega_i^{u_1}, \omega_i^{u_2})$	
$\omega_i^{u_1} - \omega_i^{u_1} + \omega_j^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i A_i^* B_j H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, -\omega_i^{u_1}, \omega_j^{u_2})$	
$\omega_i^{u_1} - \omega_j^{u_1} + \omega_j^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i A_j^* B_j H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, -\omega_j^{u_1}, \omega_j^{u_2})$	
$\omega_i^{u_1} + \omega_j^{u_1} - \omega_i^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i A_j B_i^* H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, \omega_j^{u_1}, -\omega_i^{u_2})$	
$\omega_i^{u_1} + \omega_j^{u_2} - \omega_j^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i B_j B_j^* H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, \omega_j^{u_2}, -\omega_j^{u_2})$	
$\omega_i^{u_1} + \omega_j^{u_2} - \omega_i^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i B_j B_i^* H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, \omega_j^{u_2}, -\omega_i^{u_2})$	
$-\omega_i^{u_1} + \omega_i^{u_2} + \omega_j^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i^* B_i B_j H_{3\,avg}^{u_1u_2u_2}(-\omega_i^{u_1}, \omega_i^{u_2}, \omega_j^{u_2})$	
$2\omega_i^{u_1} + \omega_j^{u_1}, i,j = 1,2, i \neq j$	$0.75A_i^2 B_i H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, \omega_i^{u_1}, \omega_j^{u_2})$	
$\omega_i^{u_1} + \omega_j^{u_1} + \omega_i^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i A_j B_i H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, \omega_j^{u_1}, \omega_i^{u_2})$	
$\omega_i^{u_1} + 2\omega_j^{u_2}, i,j = 1,2, i \neq j$	$0.75A_i B_j^2 H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, \omega_j^{u_2}, \omega_j^{u_2})$	
$\omega_i^{u_1} + \omega_i^{u_2} + \omega_j^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i B_i B_i H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, \omega_i^{u_2}, \omega_j^{u_2})$	
$2\omega_i^{u_1} - \omega_j^{u_2}, i,j = 1,2, i \neq j$	$0.75A_i^2 B_i H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, \omega_i^{u_1}, -\omega_j^{u_2})$	
$\omega_i^{u_1} - \omega_j^{u_1} + \omega_i^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i A_j^* B_i H_{3\,avg}^{u_1u_1u_2}(\omega_i^{u_1}, -\omega_j^{u_1}, \omega_i^{u_2})$	
$-\omega_j^{u_1} + 2\omega_j^{u_2}, i,j = 1,2, i \neq j$	$0.75A_j^* B_i^2 H_{3\,avg}^{u_1u_2u_2}(-\omega_j^{u_1}, \omega_i^{u_2}, \omega_i^{u_2})$	
$\omega_i^{u_1} + \omega_i^{u_2} - \omega_j^{u_2}, i,j = 1,2, i \neq j$	$1.5A_i B_i B_i^* H_{3\,avg}^{u_1u_2u_2}(\omega_i^{u_1}, \omega_i^{u_2}, -\omega_j^{u_2})$	

Table 2. Third order DISO nonlinear system responses

For example for the DISO nonlinear model in (7) where u_1 and u_2 consists of three tones each,

$$
\begin{aligned}
u_1 &= \tfrac{1}{2}\sum_{i=1}^{3}[A_i exp(j\omega_i^{u_1}t) + A_i^* exp(-j\omega_i^{u_1}t)] \\
u_2 &= \tfrac{1}{2}\sum_{i=1}^{3}[B_i exp(j\omega_i^{u_2}t) + B_i^* exp(-j\omega_i^{u_2}t)]
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
&A_1 = 3.390, A_2 = \text{-}\,4.0j, A_3 = \text{-}2.0424, \\
&B_1 = -3.390, B_2 = \text{-}\,4.0j, B_3 = 2.0424 \\
&\omega_1^{u_i} = 1\,rad/\sec, \omega_2^{u_1} = 2\,rad/\sec, \omega_3^{u_i} = 3\,rad/\sec \\
&\omega_1^{u_2} = 1\,rad/\sec, \omega_2^{u_2} = 2\,rad/\sec, \omega_3^{u_2} = 3\,rad/\sec
\end{aligned}
\tag{30}
$$

Substituting the values from (30) into the expressions in Tables 1 and 2 yields the first, up to second and up to third order nonlinear output responses, shown in Figure 6. It can be seen from Figure 6(b) that the response up to second order(first order plus second order response) provides a quite good estimation of the actual output response, and as the order of the GFRF's used to compute the response increases to third order,Figure 6(c), the estimation error reduces even further. An additional estimation error reduction can be expected if the order of the GFRF's were increased further.

A comparison of the original output spectrum and the synthesized response spectrum in Figure 7 reveals that the DISO Volterra system response from the derived GFRF's provides a very accurate estimation of the harmonics. This provides a confirmation of the validity in the analysis and the procedure developed to analyse severely nonlinear systems that exhibit subharmonics.

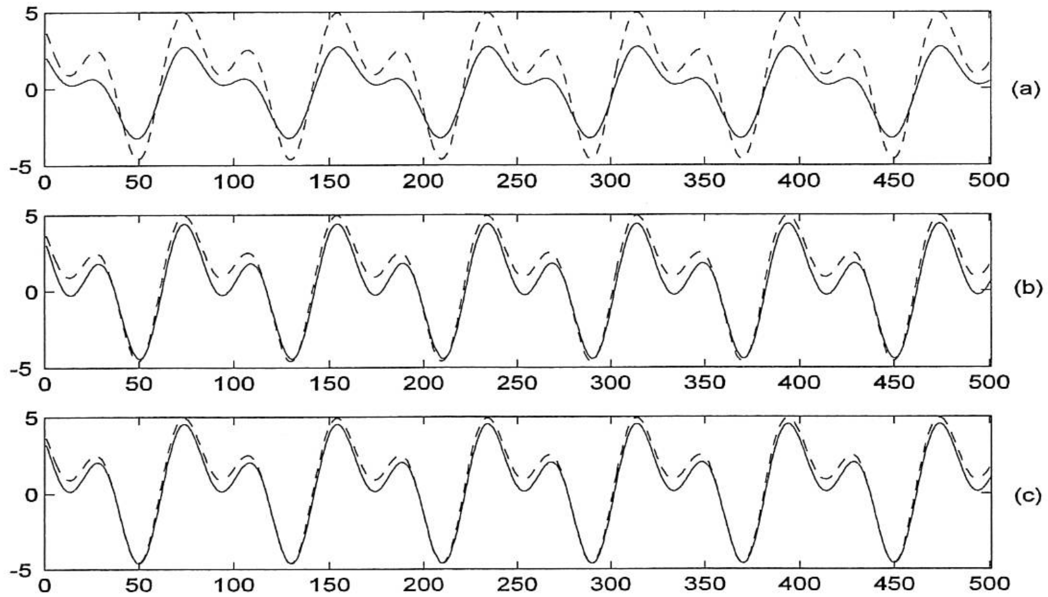

Figure 6. (a) First order output response, (b) up to the second order response and (c) up to third order response: Solid— synthesized output; Dashed-- original output

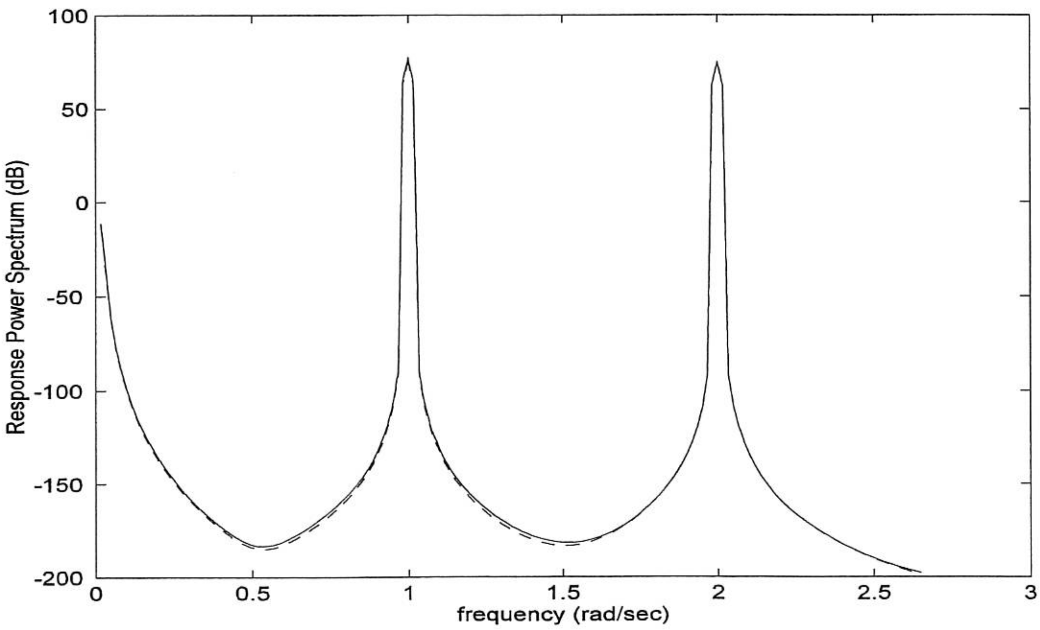

Figure 7. Output spectrum: Solid-- synthesized output; Dashed-- original output;

4.2 Discussion

A recently introduced method for detecting the presence of subharmonics is the Response Spectrum Map (RSM) (Billings and Boaghe, 2001). For the continuous time system under investigation(eqn (14), Part I of this paper(Li and Billings, 2004)), the Response Spectrum Map can be obtained by varying the input amplitude A, where the driving frequency is $\omega = 2$ rad/sec.

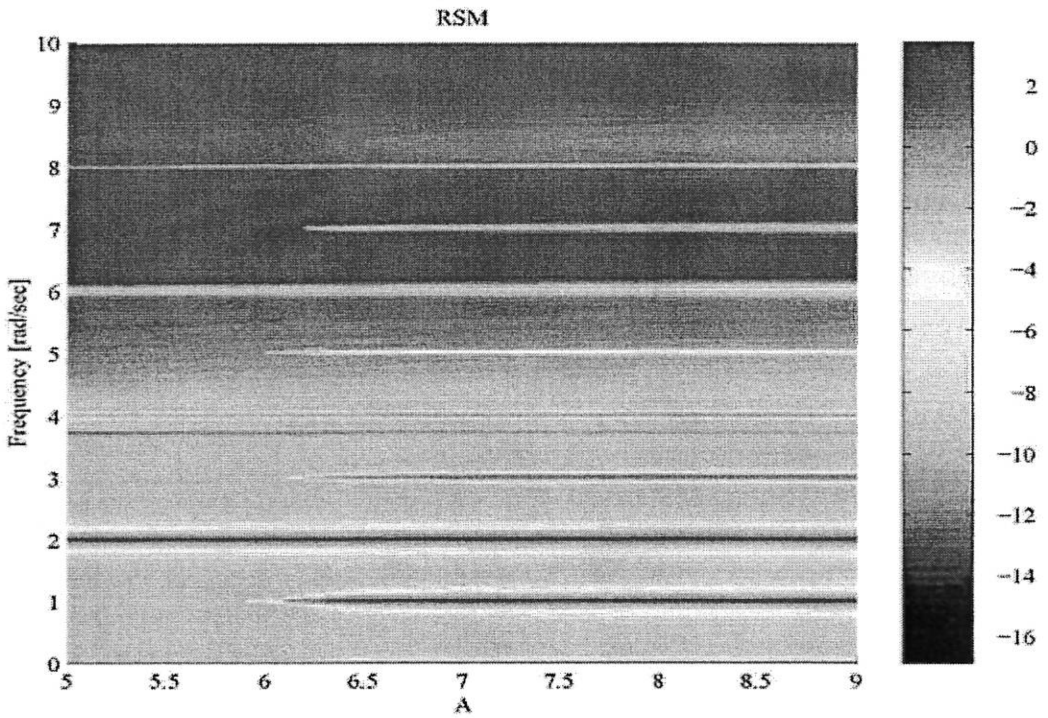

Figure 8. RSM for the continuous time system(eqn (14), Part I(Li and Billings, 2004))

The Response Spectrum Map for this system is illustrated in Figure 8, which gives a very clear insight into the dynamical regimes generated by varying the amplitude A. The darker the bars in the RSM, the more significant the frequency components are. For the amplitude range $A \in [0,6]$, only harmonic and superharmonic content is evident, although the superharmonic at $\omega = 4$ rad/sec is much less significant than the harmonic at $\omega = 2$ rad/sec. Around the neighborhood of $A>6$, a half subharmonic($\omega = 1$ rad/sec), harmonic($\omega = 2$ rad/sec), supersubharmonic ($\omega = 3$ rad/sec) and superharmonic($\omega = 4$ rad/sec) begin to appear. These effects peak in the amplitude range [7,10]. When $A>10.8$, the system becomes unstable.

Based on these results a global SISO NARX model for $A \in [0,10.8]$ is possible. Simulating the system (eqn (14) in Part I(Li and Billings, 2004)) and applying the NARX parametric estimation procedures produced the identified model

$$y(k) = 1.9783y(k-1) - 0.9844y(k-2) + 0.0006109y^2(k-1) + 0.006105u(k-1) \tag{31}$$

However, the RSM in Figure 8 shows that there is a difference in the system behaviour over the two amplitude regions $A \in [0,6]$ and $A \in [6,10.8]$. For $A \in [0,6]$, the GFRF's obtained from (31) will be meaningful since the system is clearly weakly nonlinear with a valid Volterra representation over this amplitude range. But for $A \in [6,10.8]$, the RSM shows that a simple Volterra representation will not be valid.

This indicates that, although the GFRF's are an inherent property of the underlying system, the existence of the GFRF's can depend on input signal properties, in this case the input amplitude. However, by using the results derived in the present paper a new and valid analysis based on the GFRF's can be obtained in the amplitude range $A \in [6,10.8]$, where subharmonics are present. This is achieved by expressing the time domain model as a multi-input system.

Before proceeding to the analysis of the formation of subharmonics for the input amplitude range $A \in [6,10.8]$ using the DISO GFRF's generated from model (7), consider initially why the global NARX model (31) will fail to explain the formation of subharmonics in the frequency domain.

The SISO GFRF's from (31) will be valid for the amplitude range $A \in [0,6]$ but will lose validity for the subharmonic situation with $A \in [6,10.8]$, since the output has frequency components which are half the value of the input frequency. Consider why simply splitting the input in the model eqn (31) will not work whereas the procedure introduced in section 3 will.

Define the input split as

$$u = [u_1; u_2] \text{ where } u_1(t) + u_2(t) = u(t) \tag{32}$$

Using (32), (31) can now be expressed as a DISO NARX model as

$$\begin{aligned} y(k) &= 1.9783y(k-1) - 0.9844y(k-2) + 0.0006109y^2(k-1) \\ &\quad + 0.006105u_1(k-1) + 0.006105u_2(k-1) \end{aligned} \tag{33}$$

The DISO model (33) now satisfies the Volterra requirement of period matching between the input-output signals. It is now therefore easy to see that from (33), the GFRF's will be

$$\begin{aligned} &H_1^{u_1}(\cdot) = H_1^{u_2}(\cdot) \\ &H_2^{u_1u_1}(\cdot) = H_2^{u_1u_2}(\cdot) = H_{2\,\text{avg}}^{u_2u_2}(\cdot) \\ &\ldots \end{aligned} \tag{34}$$

Consider the analysis of the subharmonic at $\omega_1^{u_1}, \omega_1^{u_2} = 1$ rad/sec for example. The corresponding linear part of the frequency domain formula relating to the generation of this subharmonic at $\omega_1^{u_1}, \omega_1^{u_2} = 1$ rad/sec, from Table 1, is

$$y_1^{\omega_1} = \text{Re}[A_1 H_1^{u_1}(\omega_1^{u_1})] + \text{Re}[B_1 H_1^{u_2}(\omega_1^{u_2})] \tag{35}$$

Note that from eqn (30) the amplitude at the subharmonic frequency($\omega_1^{u_1}, \omega_1^{u_2} = 1$ rad/sec) of the two inputs are opposite signs to one another, i.e., $A_1 = -B_1$. Therefore by substituting $A_1 = -B_1$ into (35) and noticing from (34) that $H_1^{u_1}(\omega_1^{u_1}) = H_1^{u_2}(\omega_1^{u_2})$, the result of (35) becomes

$$y_1^{\omega_1} \equiv 0 \tag{36}$$

this means that there is no subharmonic component which results from the linear part of the GFRF's from eqn (33). Actually, the frequency components other than those at the driven frequency $\omega_2^{u_1} = \omega_2^{u_2} = 2$ rad/sec from the linear transfer functions $H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$, will be cancelled out, as in the SISO case.

The contributions towards the formation of this subharmonic at $\omega_1^{u_1}, \omega_1^{u_2} = 1$ rad/sec by the second order transforms, according to Table 1, are from the three second order transforms, namely the two self-kernel transforms $H_2^{u_1u_1}(\cdot)$ and $H_2^{u_2u_2}(\cdot)$, and the cross-kernel transform $H_{2\text{avg}}^{u_1u_2}(\cdot)$. The components involving frequencies $\omega_3^{u_1}$, $\omega_3^{u_2} = 3$ rad/sec are relatively insignificant compared with the components involving the other two frequencies, therefore for simplicity only the first two frequencies in each input signals will be taken into account in the discussion. The exact formula, after omitting $\omega_3^{u_1}$ and ω_3^{u2}, is

$$y_2^{\omega_1} = y_2^{\omega_1}{}_{\text{self-kernel}} + y_2^{\omega_1}{}_{\text{cross-kernel}} \tag{37.a}$$

where

$$y_2^{\omega_1}{}_{\text{self-kernel}} = \text{Re}[\text{A}_1^* A_2 H_2^{u_1u_1}(-\omega_1^{u_1}, \omega_2^{u_1})] + \text{Re}[\text{B}_1^* B_2 H_2^{u_2u_2}(-\omega_1^{u_2}, \omega_2^{u_2})] \tag{37.b}$$

and

$$y_2^{\omega_1}{}_{\text{cross-kernel}} = \text{Re}[\text{A}_1^* B_2 H_{2\text{avg}}^{u_1u_2}(-\omega_1^{u_1}, \omega_2^{u_2})] + \text{Re}[\text{A}_2 B_1^* H_{2\text{avg}}^{u_1u_2}(\omega_2^{u_1}, -\omega_1^{u_2})] \tag{37.c}$$

By using the properties in (30) and (34) it can be seen that

$$y_2^{\omega_1}{}_{\text{self-kernel}} \equiv 0 \text{ and } y_2^{\omega_1}{}_{\text{cross-kernel}} \equiv 0$$

which means

$$y_2^{\omega_1} \equiv 0 \tag{38}$$

The same cancellation will also happen in the higher order GFRF's. Therefore overall, there are no subharmonic components resulting from the GFRF's derived from (33).

The above analysis indicates that GFRF's derived from the SISO-turned-MISO model (33) will play exactly the same role as the normal SISO GFRF's from (31), and hence will fail to explain the presence of subharmonics when $A \in [6,10.8]$ and therefore can not be seen as valid in the sense of the Volterra/frequency domain.

The new MISO NARX modeling procedure introduced in the present paper, however, avoids the above problems. The new approach is capable of generating valid frequency domain GFRF's but only if this is applied to the correct form of MISO model, eqn (7) in this case. The formation and analysis of this subharmonic in the frequency domain will be investigated using the new GFRF's results derived in section 3.

From Figure 2 it can be seen that although the gain plots of $H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$ are quite close, there is a significant phase difference between them. This phase difference plays an important role in this example for the generation of the subharmonic at $\omega_1^{u_1}$, $\omega_1^{u_2} = 1$ rad/sec. By substituting the amplitude values in (30) and the linear transfer function results in (9) and (10) into (35), the new subharmonic components due to linear transforms from the new DISO NARX model (7) can be obtained. From Figure 6(a) it can be seen that these subharmonic components from linear transforms make a significant contribution towards the formation of the overall subharmonic.

In this example it is the phase difference that leads to the generation of subharmonics. The results suggest that for general subharmonic systems, analysed using the new MISO modeling approach, there will be either a gain difference, or a phase difference, or both, in the linear transforms. Note that the higher order transfer functions, both the self-kernel and the cross-kernel transforms, are functions of the linear transfer functions $H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$, therefore the differences between $H_1^{u_1}(\cdot)$ and $H_1^{u_2}(\cdot)$ will be reflected in the higher order transfer functions. This means that there will be no cancellations when using (37) to compute the subharmonic components relating to second order transfer functions in (12) and (14). From figure 3 and 4, it can be seen that there are significant differences between $H_2^{u_1u_1}(\cdot)$ and $H_2^{u_2u_2}(\cdot)$. Therefore the resulting $y_{2\ \text{self-kernel}}^{\omega_1}$ will be quite significant. From Figure 5, the averaged cross-kernel transform $H_{2\text{avg}}^{u_1u_2}(\cdot)$ is nearly symmetric over $\{\omega_1, \omega_2\}$. This means that the resulting $y_{2\ \text{cross-kernel}}^{\omega_1}$ is relatively less significant.

This procedure of finding subharmonic components can be continued to include higher order transforms until an accurate estimation is achieved. If necessary this could be implemented using symbolic algebra software. From Figure 6(b) it can be seen that up to the second order frequency response has already provided a quite accurate estimation of the original output. Therefore the combination of the output response $y_1^{\omega_1}$ and $y_2^{\omega_1}$ provides a quantitative measure about the formation of the subharmonic $\omega_1^{u_1}, \omega_1^{u_2} = 1$ rad/sec.

5. Conclusions

It is well known that the generation of subharmonics is the first step towards chaos. Studying the internal mechanisms associated with subharmonic generation in the frequency domain is therefore very important both to study subharmonic behaviour and to provide insight into chaos. The frequency domain equivalence of the Volterra series representation, the GFRF's, have been extensively studied and have played a very important role in the analysis of mildly nonlinear systems. This powerful tool, however, can not be applied to nonlinear systems that exhibit subharmonics. In this study, for the first time, it has been shown that it is possible to associate GFRF's with subharmonics systems using a new MISO modeling procedure.

The MISO nonlinear response synthesis problem based on GFRF's has been studied based on a Dual-Input-Single-Output(DISO) illustrative example. The formulae for computing up to third order DISO nonlinear frequency responses has been derived and applied to a simulation example to demonstrate the validity of the Volterra/frequency representation. The formation of subharmonics using the new GFRF's analysis has also been discussed.

The application of this new procedure to real subharmonic systems should provide more insight and a better understanding of this complex phenomenon.

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