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Discrete Time subharmonic modelling and Analysis: Part I. MISO NARX modelling

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Discrete Time subharmonic modelling and Analysis: Part I. MISO NARX modelling

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Abstract: Traditionally the Volterra time and frequency domain analysis tools cannot be applied to severely nonlinear systems. In Part I of this paper, a new method of building a time-domain NARX MISO model for a class of severely SISO nonlinear systems that exhibit subharmonics is introduced and it is shown how this allows the Volterra time and frequency domain analysis to be extended to this class of nonlinear systems. The new approach is based on decomposing the original single input based on a Fourier analysis to provide a set of modified input signals which have the same period as the output signal. A MISO NARX model can then be constructed from the decomposed multiple inputs and the single output signal. The resulting MISO model is shown to meet the basic requirement for the existence of a Volterra series representation from which important frequency domain properties can be derived. In Part II of this paper, the frequency domain properties of the resulting MISO model will be explored in detail.

1. Introduction

Over the last few decades, the theory and techniques of nonlinear system identification, for both continuous time and discrete time models, have been studied and developed. The vast majority of these methods were developed for weakly or mildly nonlinear systems, where the underlying system can be represented by a Volterra series. A big advantage of the Volterra based representations is that they can be readily transformed into the frequency domain using Generalised Frequency Response Functions (GFRF's). The inherent features of the underlying nonlinear systems can be studied using the GFRF's(Bedrosian and Rice, 1971; Bussgang, et. al., 1974; Lang and Billings, 2000), and this provides an analogous theory to linear frequency response methods, which are so important for linear systems. Many nonlinear phenomena have been analysed and interpreted in terms of the GFRF's, including gain compression, intermodulation effects, harmonics and desensitisation.

A much more complex class of nonlinear systems is refered to as 'severely nonlinear systems', where dynamic behaviours such as limit cycles, subharmonics and chaos exist. This study will focus on a particular class of severely nonlinear systems that exhibit subharmonics. Subharmonic systems exist widely in the real world, for example, in medical science(Greenman et. al., 2004; Velazquez, et. al., 2003), in mechanical systems(Buhler and Frendi, 2004), in electric and magnetic systems(Tommaseo, et. al., 2004) and in marine science(Umar and Datta, 2003), etc. Unfortunately subharmonic systems cannot in general be represented by a finite Volterra series. Although this class of nonlinear systems can still be modelled using the NARMAX representation in the time domain, frequency domain analysis is very difficult and virtually no results exist in this field, despite the fact that subharmonics are essentially a frequency domain phenomenon. So far most analysis for subharmonic systems has been done using methods based on the bifurcation diagram, or Poincaré map, etc. It would however be highly desirable if the GFRF's, which are used to analyse weakly nonlinear systems, could be extended to apply to subharmonic systems to provide an analysis tool for a better understanding of the internal mechanism associated with subharmonic generation. Recently, Boaghe and Billings(2003) used a MISO Volterra modelling procedure for subharmonic systems, from which frequency domain properties can be acquired. The work of this paper is to try to establish an improved procedure for such a link between the subharmonic system and GFRF's.

The paper is divided into two parts. In Part I a procedure to build MISO NARX models that are potentially Volterra meaningful for subharmonic systems is introduced. This is done by decomposing the input signal such that the resulting decomposed individual input signals have the same period as the subharmonic. It is shown that these individual input signals can be considered as multiple input signals to the system which in turn allows the analysis of the system using GFRF's. This is basically an improved procedure based on Boaghe and Billings'(2003) method for time domain identification. The improvement is two-fold. First, a simple and straightforward input signal decomposition was adopted in Boaghe and Billings(2003). But the higher frequency components which result from the decomposition method may result in ill-parameterised models. A Fourier analysis based input signal decomposition is adopted in this paper to solve this problem. Another significant advantage of the new band-limited decomposition is that the quantitative analysis of the formation of subharmonics is now possible. This will be explained in detail in Part II. Second, Boaghe and Billings(2003) used discrete time Volterra models(NX models) in the MISO modelling. This had the advantage of converting the MISO model into two SISO sub-models, which simplifies the frequency domain analysis, but at the expense of using much larger model structures which can result in distorted Generalised Frequency Response Functions. Therefore MISO NARX model structures are adopted in this paper in a bid to obtain more accurate frequency domain estimation. A new frequency domain analysis will be performed based on the MISO model instead of on the SISO sub-models in Boaghe and Billings(2003). A simulation example is used to illustrate the new procedure.

In Part II of this paper, the frequency domain aspects of the MISO NARX modelling, including GFRF estimation from MISO NARX models, quantitative interpretation of the subharmonics, and validation of the accuracy of GFRF's by response synthesis will be considered. This provides, for the first time, a systematic analysis procedure for subharmonic systems in the frequency domain.

2. Volterra modelling and its limitations

One of the most studied representations of nonlinear systems is the Volterra series(Volterra,1930). The Volterra series is a nonlinear functional series that can be expanded as a polynomial functional series. This is a direct generalisation of the linear convolution integral, and can be expressed as

$$y(t)=\sum_{n=1}^{\infty} y_n(t) \tag{1.a}$$

where $y_n(t)$ is the 'n-th order output' of the system

$$y_n(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_i \qquad n>0 \tag{1.b}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is called the 'nth-order Kernel' or 'nth-order impulse response function'. If $n=1$, this reduces to the familiar linear convolution integral.

The SISO Volterra series representation (1) can easily be generalised to allow for multiple inputs. Thus for a multi-input system with a single output $y(t)$ and N inputs $\{u_1(t),\ldots,u_N(t)\}$, the MISO Volterra series is given by(Swain and Billings, 2001,)

$$y(t)=\sum_{n=1}^{\infty} y_n(t)=\sum_{n=1}^{\infty}\sum_{n_1=1}^{N}\sum_{n_2=n_1}^{N}\cdots\sum_{n_n=n_{n-1}}^{N} y_n^{u_{n_1},\ldots,u_{n_n}}(t) \tag{2.a}$$

where

$$y_n^{u_{n_1},\ldots,u_{n_n}}(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n^{u_{n_1},\ldots,u_{n_n}}(\tau_1,\ldots\tau_n)\prod_{i=1}^{n} u_i(t-\tau_i)d\tau_i \tag{2.b}$$

The discrete time domain counterpart of the continuous time domain SISO Volterra expression (1) is

$$y(k)=\sum_{n=1}^{\infty} y_n(k) \tag{3.a}$$

where

$$y_n(k)=\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(k-\tau_i) \qquad n>0, k\in Z \tag{3.b}$$

Similarly the discrete form of the MISO Volterra expression (2) is

$$y(k)=\sum_{n_1=0}^{\infty}\sum_{n_2}^{\infty}\cdots\sum_{n_N=1}^{\infty} y_{n_1,n_2,\ldots,n_N}(k) \tag{4.a}$$

where

$$y_{n_1,n_2,\ldots,n_N}(k)=\sum\cdots\sum h_{n_1,\ldots,n_N}(\tau_{11},\ldots\tau_{1n_1};\cdots;\tau_{N1},\ldots,\tau_{N_nN})\prod_{i=1}^{N}\prod_{j=1}^{n_i} u_i(k-\tau_{ji}) \tag{4.b}$$

Early research on nonlinear system identification using the Volterra and Wiener series have been thoroughly reviewed by Hung and Stark (1977), and Billings(1980). The direct application of the Volterra series representation in nonlinear system modelling has been reported, for example, in Schetzen(1980), Billings and Fadzil(1985), and Subba and Nunes(1985). The problems associated with this approach are the computational complexity involved, and an excessive parameter set which is often needed to describe simple nonlinear systems. This tends to inhibit practical applications.

There are many other methods which are available to model nonlinear systems, including neural networks, neuro fuzzy methods, etc. But the models obtained using these methods can be difficult to interpret. For example neural networks can find models that predict the data well, but it is very difficult to analyse the results, to relate them to the underlying system or to map them into the frequency domain. These methods therefore tend to be purely time domain approaches. The NARX model, which is the noise free part of the NARMAX model and has been extensively studied by Billings and co-workers(Billings et al.1988, Billings and Voon, 1983; Billings and Chen, 1989), can easily be mapped into the frequency domain to reproduce Generalised Frequency Response functions(GFRF's) (Billings and Tsang, 1989). This produces a powerful representation for a large class of nonlinear systems and the Volterra model can be regarded as a special case of the NARX model. Therefore, because of the excellent potential Volterra /frequency domain analysis features it is often of practical interest to construct NARX models wherever possible.

However, not all NARX models can be mapped into the frequency domain. For example, strongly nonlinear systems or systems that exhibit complex dynamics, such as limit cycles, subharmonics and chaos, which can be described by standard NARX models in the time domain, cannot generally be analysed systematically in the frequency domain.

In the following sections, the link between the NARX model and the Volterra/frequency representation for a class of severely nonlinear systems that exhibit subharmonics is investigated.

3. Subharmonic analysis and input signal decomposition

In the real world, it is usually assumed that the output signals are corrupted by noise. There are many ways available to remove the influence of noise in system identification so that the system analysis can be conducted based on noise free models. Here we can assume that an accurate noise free model, no matter in what format, e.g., neural network, polynomial or wavelets, etc, has been obtained by system identification techniques. Then the designated input signal, which would trigger the appearance of subharmonics, is fed into the noise free model and input-output signals are collected accordingly.

For linear systems, the steady-state output response always has the same frequency components as that of the input signal. For nonlinear systems driven by a sine wave, new frequency components can be produced in the steady-state response. When these new frequency components are at fractions of the frequency of the driven signal, they are referred to as subharmonics. The presence of subharmonics is an important practical problem in the dynamics of many nonlinear mechanical systems, and Feigenbaum(1980) has noticed that subharmonic generation is the first step on the route to chaos. Subharmonic systems however do not have a Volterra representation. This can be seen from the following theorem.

Theorem 1 (Boyd et al, 1984). [Periodic steady state theorem] If the input of a nonlinear system described by a Volterra series operator N is periodic with period T for $t \geq 0$, then the output *Nu* approaches a steady state, which is also periodic with period *T*.

It can be concluded from Theorem 1 that a Volterra/frequency domain equivalence of a direct SISO NARX model built upon the original input-output signals of a subharmonic system will not exist. This is because the period of the output signal of a subharmonic system will be an integer multiple greater than the period of the input signal. An alternative means is therefore needed to allow a frequency domain analysis and interpretation of systems that exhibit subharmonics.

It is therefore impossible to obtain a Volterra analysis directly from an ordinary input-output model of a subharmonic system. But if the given input signal could be modified to create input(s) with period(s) equal to the subharmonic oscillation, then the periodic steady state theorem could be satisfied(Boaghe and Billings, 2003). If the subharmonic under investigation has a period n times the original input period $T(=1/\omega)$, a modified input u_{mod} should have a period n times the real input period. Intuitively, this can be achieved by decomposing the original input signal $u(t)$ into an n dimensional signal, that is, $u_{\text{mod}}=[u_1;\ldots;u_i;\ldots;u_n]$, where

$$u_{\text{mod}}=\begin{cases} 0 & t\in[0;T) \\ \vdots & \\ u(t) & t\in[(i\text{-}1)T;iT) \\ 0 & t\in[iT;(i+1)T) \\ \vdots & \\ 0 & t\in[(n\text{-}1)T;nT) \end{cases} \tag{5}$$

It can easily be verified that

$$u_1(t)+\ldots+u_i(t)+\ldots+u_n(t)=u(t) \tag{6.a}$$

which has the property

$$u_i(t)=u_1(t-iT) \tag{6.b}$$

An illustrative example of the creation of a modified input for a second order subharmonic system is illustrated in Figure 1.

By modifying the input signal $u(t)$ as in (5), each individual component $u_i(t)$ has the same period as the subharmonic. It can be assumed that the original single harmonic input signal $u(t)$ is actually composed by those individual components $u_i(t)$, which can then be seen as multiple driven inputs of the subharmonic system. Theorem 1 can now be satisfied by the fact that the each input component $u_i(t)$ in u_{mod} and the output y(t) have the same period nT, and this in turn enables a Volterra representation based on a new MISO model using the multiple-input-single-output configuration $\{[u_1;\ldots;u_i;\ldots;u_n],y(t)\}$. The advantage of this approach is that now the Generalised Frequency Response Functions can be used, for the first time, to provide a unified frequency domain analysis of subharmonic systems.

It can be seen that the information required for MISO modelling is the order n of the subharmonic and the period T of the input signal $u(t)$. This information can be obtained by analysing the given system using a bifurcation diagram and a response spectrum map (Billings and Boaghe, 2001). However this naive way of decomposing the input may cause discontinuity effects, which can result in high frequency components, in each decomposed individual input signal. Usually the steady state output of a subharmonic system contains only a limited frequency bandwidth. Any excessive high frequency components may cause inaccuracies in the model predicted output(MPO), and large model coefficients. Therefore an improved decomposition technique has to be introduced.

Since the decomposed input signals $u_i(t)$ in (5) are periodic in nT, this can be represented by a simple trigonometric Fourier series as (Hsu, 1967)

$$u_i(t) = a_0 + \sum_{k=1}^{\infty}[a_k \cos(\tfrac{k\omega t}{n}) + b_k \sin(\tfrac{k\omega t}{n})] \tag{7}$$

with $\omega = 1/T$.

The Fourier coefficients a_0, a_k and b_k can be evaluated as

$$a_0 = \tfrac{1}{nT}\int_{-nT/2}^{nT/2} u_i(t)dt \tag{8}$$

$$a_k = \tfrac{2}{nT}\int_{-nT/2}^{nT/2} u_i(t)\cos(\tfrac{k\omega t}{n})dt \tag{9}$$

$$b_k = \tfrac{2}{nT}\int_{-nT/2}^{nT/2} u_i(t)\sin(\tfrac{k\omega t}{n})dt \tag{10}$$

The high frequency components in (7) may induce problems as discussed above. Therefore a modified decomposition input method is introduced where only a limited band-width is allowed by truncating the standard Fourier series expression (7)

$$u_i^{new}(t) = a_0 + \sum_{k=1}^{N}[a_k \cos(\tfrac{k\omega t}{n}) + b_k \sin(\tfrac{k\omega t}{n})] \tag{11}$$

This point is perhaps better understood through a simple example.

Assume the input signal to be analysed is

$$u(t) = 6\sin(\omega t), \qquad \omega = 1$$

as shown in Figure 1 (a).

Assume that there is a third order subharmonic, that is, $n = 3$.

Decomposing $u(t)$ using the rules in (5) yields $u_1(t), u_2(t)$ and $u_3(t)$ as in Figure 1 (b),(c) and (d).

u_i $(i = 1,2,3)$ can also be expressed as a Fourier series

$$\begin{aligned} u_1(t) &= a_{10} + \sum_{k=1}^{\infty}[a_{1k} \cos(\tfrac{k\omega t}{n}) + b_{1k} \sin(\tfrac{k\omega t}{n})] \\ &= 1.0741\cos(t/3) - 0.6201\sin(t/3) + 1.7187\cos(2t/3) + 0.9923\sin(2t/3) + 2.0\sin(t) \\ &\quad - 1.228\cos(4t/3) + 0.709\sin(4t/3) - 0.5374\cos(5t/3) - 0.3102\sin(5t/3) + \dots \end{aligned}$$

$$\begin{aligned} u_2(t) &= a_{20} + \sum_{k=1}^{\infty}[a_{2k} \cos(\tfrac{k\omega t}{n}) + b_{2k} \sin(\tfrac{k\omega t}{n})] \\ &= 1.2402\sin(t/3) - 1.9846\sin(2t/3) + 2.0\sin(t) - 1.4180\sin(4t/3) + 0.6204\sin(5t/3) + \dots \end{aligned}$$

$$\begin{aligned} u_3(t) &= a_{30} + \sum_{k=1}^{\infty}[a_{3k} \cos(\tfrac{k\omega t}{n}) + b_{3k} \sin(\tfrac{k\omega t}{n})] \\ &= -1.0741\cos(t/3) - 0.6201\sin(t/3) - 1.7187\cos(2t/3) + 0.9923\sin(2t/3) + 2.0\sin(t) \\ &\quad + 1.228\cos(4t/3) + 0.709\sin(4t/3) + 0.5374\cos(5t/3) - 0.3102\sin(5t/3) + \dots \end{aligned}$$

The modified u_i^{new} can now be obtained from the truncated u_i by removing all frequency components above 4/3 rad/sec, and this is shown in Figure 2.

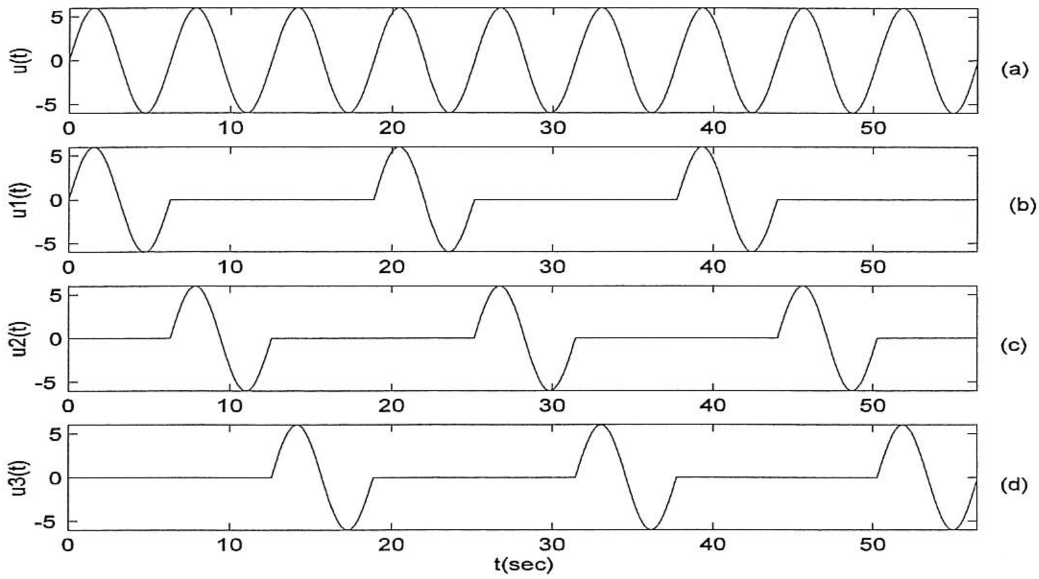

Figure 1. Original input and decomposed signal using rules (5)

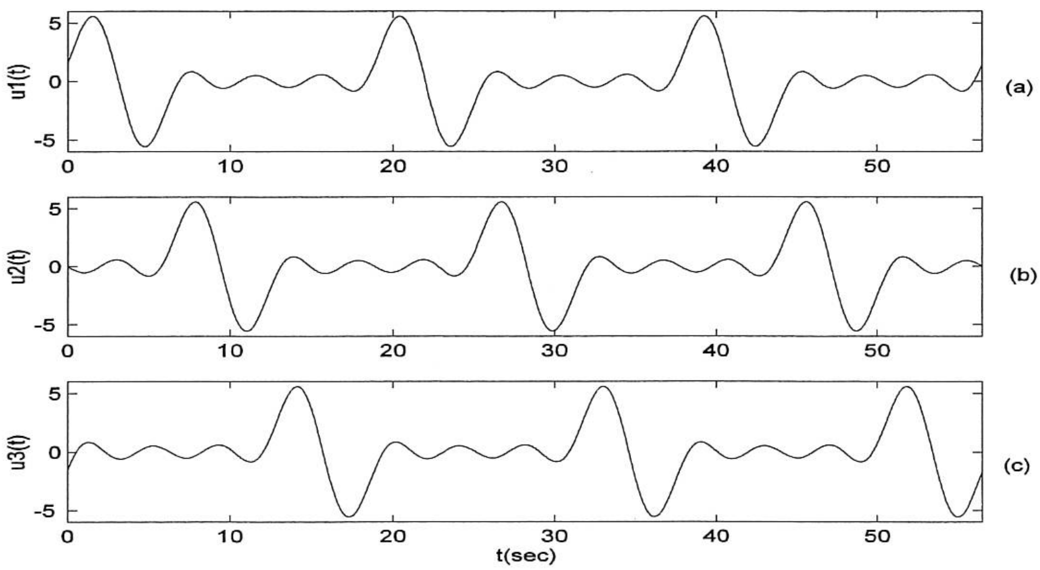

Figure 2. Fourier analysis based decomposition using rules (11)

It can also be observed that the newly decomposed signals u_i^{new} satisfy the relations

$$u_1^{new}(t) + u_2^{new}(t) + u_3^{new}(t) = u(t) \tag{12.a}$$

and $u_i^{new}(t) = u_1^{new}(t - iT)$ (12.b)

The above result in (12) can be generalised to the nth order subharmonics case, to give an equation similar to eqn (6) for the infinite frequency case:

$$u_1^{new}(t) + \ldots + u_i^{new}(t) + \ldots + u_n^{new}(t) = u(t) \tag{13.a}$$

$$\text{and } u_i^{new}(t) = u_1^{new}(t - iT) \qquad (13.b)$$

The relation in (13) is very important for the modification of the input signal decomposition because this means that the modification does not violate the basic fact that the original input is composed of all individually decomposed signals.

It can be seen that the new input decomposition is band-limited and smooth. Another advantage of using a Fourier analysis based decomposition is that a quantitative Volterra/frequency domain analysis is now possible. This will be illustrated later in Part II of this paper.

4. MISO NARX modelling

The decomposed individual signals u_i^{new} satisfying (13) can be regarded as multiple driven input signals to a system which can be modelled using a MISO NARX model as shown in Figure 3. This model consists of a time multiplexor M and a MISO system S. To simplify the modelling process, it is assumed that the final MISO NARX model does not contain cross-product terms of the type $u_i u_j, i \neq j$.

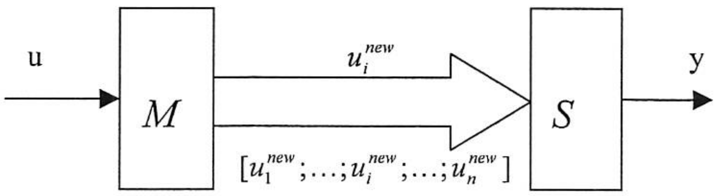

Figure 3. MISO modelling structure for a subharmonic system

The modelling procedure for nonlinear systems that exhibit subharmonics can therefore be summarised as

i). Compute the bifurcation diagram and response spectrum map from the input-output data,

ii). Identify the subharmonics of order n from the response spectrum map,

iii). Generate an n-dimensional decomposed input $u_{\text{mod}} = [u_1;\ldots;u_i;\ldots;u_n]$ using the rules in eqn (5),

iv). Perform a Fourier analysis on each individual decomposed input u_i, and determine the bandwidth of the final inputs u_i^{new} based on the spectrum of the output, and

v). Build a MISO NARX model based on the n-dimensional inputs u_i^{new} and the single output $y(t)$.

5. Simulation example

Consider a system described as

$$\ddot{y} + 0.2\dot{y} + y - 0.1y^2 = A\sin(\omega t) \qquad (14)$$

The Response Spectrum Map(RSM), introduced by Billings and Boaghe(2001), for system (14) which is excited at the frequency $\omega = 2$ rad/sec, is ploted in Figure 4.

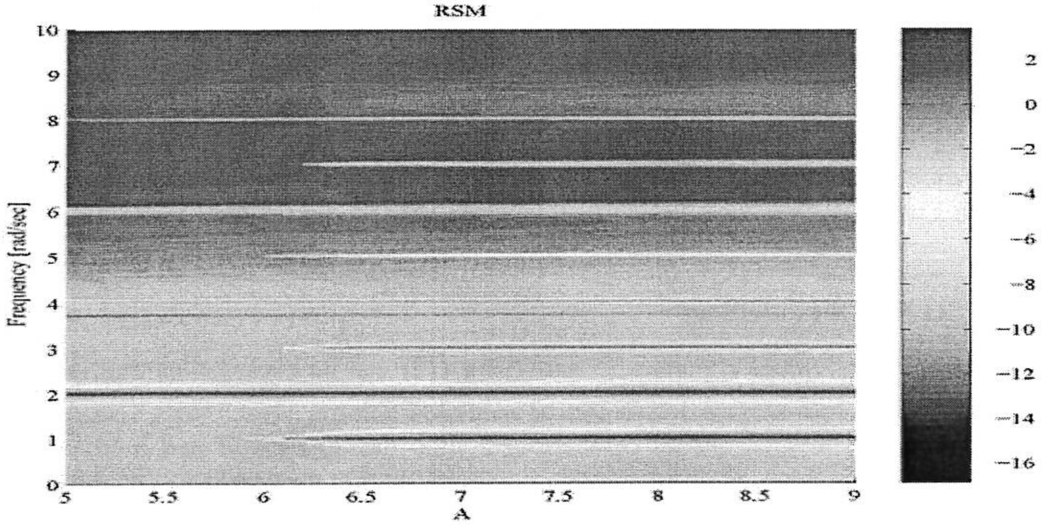

Figure 4. RSM for the continuous time system (14)

From the RSM in Figure 4, it can be seen that a second order subharmonic occurs at the amplitude range $A \in [6,10.8]$. In this example, the amplitude of the driven input is set as $A = 8$. The decomposition factor should be set as $n = 2$ because of the second order subharmonic present.

The system (14) was simulated using a Fourth order Runge-Kutta algorithm at a sampling frequency $f_s = 40/\pi$ with zero initial conditions. The Fourier analysis of the initial decomposed u_i $(i = 1,2)$ is:

$$u_1(t) = 3.3901\cos(t) + 4.0\sin(2t) - 2.0424\cos(3t) - 0.4903\cos(5t) + \dots$$

$$u_2(t) = -3.3901\cos(t) + 4.0\sin(2t) + 2.0424\cos(3t) + 0.4903\cos(5t) + \dots \quad (15)$$

In order to decide the appropriate truncation order of the input u_i, it is necessary to consider the spectrum of the resulting output, which is shown in Figure 5.

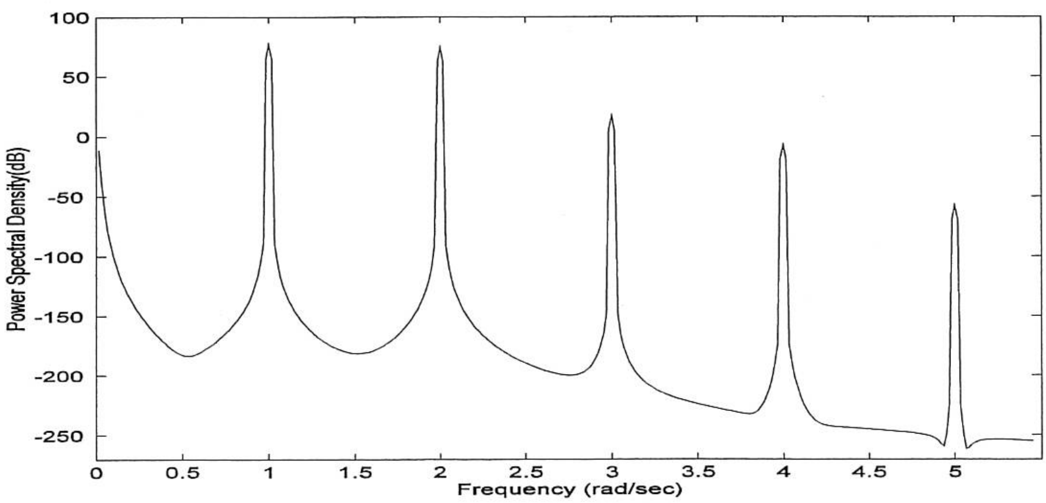

Figure 5. Power spectrum of the output $y(t)$ for eqn (14)

It can be seen from Figure 5 that the spectrum of the output y becomes negligibly small for the frequencies above (including) $\omega = 3$. This suggests that the frequency range in the input signals can be set up to $\omega = 3$. Therefore the truncation order N was chosen as N=3 in this example, which means that the truncated input signals will contain 3 frequency components $\omega = 1,2 \text{ and } 3$, which are incidentally also the dominant frequency components in u_i. The final modified input signals u_1^{new} and u_2^{new} which will be used in the MISO NARX modelling are shown in Figure 6.

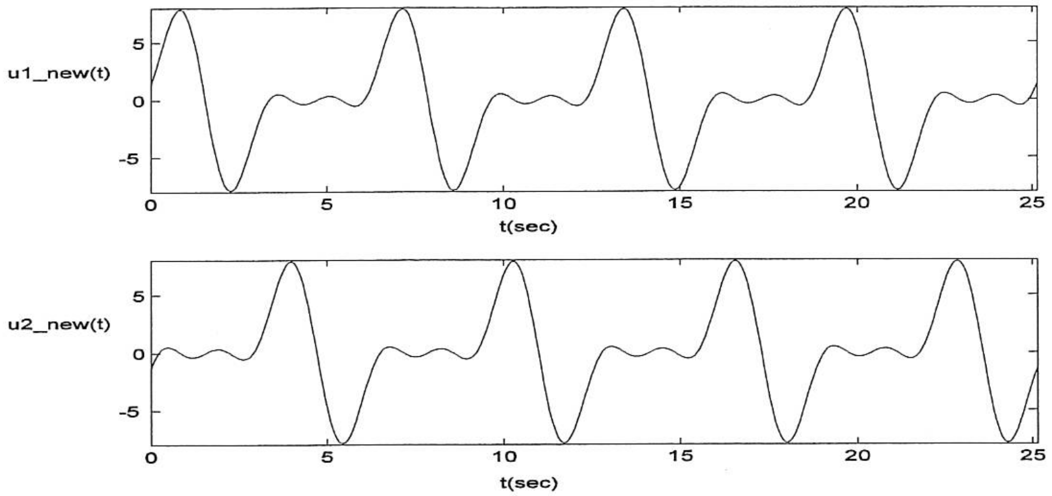

Figure 6. The finally decomposed input signals u_1^{new} and u_2^{new} used in the MISO modelling of eqn (16)

By using the decomposed 2-dimentional inputs $[u_1^{new} \quad u_2^{new}]$ and the output y, a second order MISO NARX model was identified as

$$
\begin{aligned}
y(k) = {} & 3.9187y(k\text{-}1) - 5.7870y(k\text{-}2) + 3.8163y(k\text{-}3) - 0.9482y(k\text{-}4) + 0.0032u_1(k\text{-}6) - 0.00794u_1(k\text{-}5) \\
& + 0.005331u_1(k\text{-}4) - 0.000614u_1(k\text{-}1) - 0.00439u_2(k\text{-}6) - 0.00840u_2(k\text{-}4) + 0.01135u_2(k\text{-}5) \\
& + 0.001437u_2(k\text{-}2) + 0.00126y^2(k\text{-}1) + 0.0000321y^2(k\text{-}4) - 0.002491y(k\text{-}1)y(k\text{-}2) \\
& + 0.000414y^2(k\text{-}2) + 0.000739y(k\text{-}2)y(k\text{-}4) + 0.00191y(k\text{-}1)y(k\text{-}3) - 0.001312y(k\text{-}2)y(k\text{-}3) \\
& - 0.000756y(k\text{-}1)y(k\text{-}4) + 0.000223y^2(k\text{-}3) + 0.0002209u_1^2(k\text{-}6) - 0.00102713u_1(k\text{-}5)u_1(k\text{-}6) \\
& + 0.001770u_1^2(k\text{-}5) + 0.0001113u_1(k\text{-}3)u_1(k\text{-}6) + 0.0003873u_1(k\text{-}3)u_1(k\text{-}5) \\
& - 0.001462u_1(k\text{-}4)u_1(k\text{-}5)
\end{aligned} \tag{16}
$$

The model predicted output, which is a perfect match to the original output(actually overlying the original output), is plotted in Figure 7.

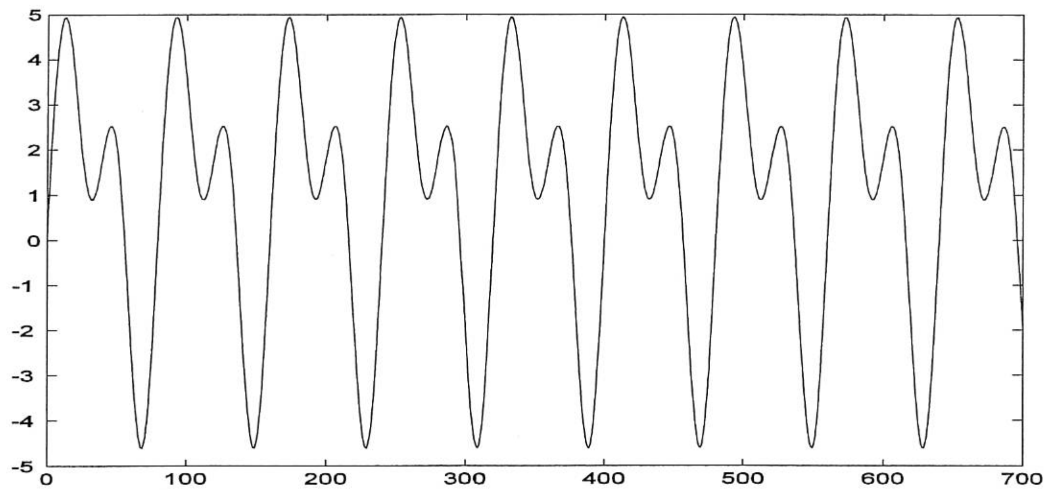

Figure 7. Model Predicted Output for the MISO model eqn (16) compared to the real output signal

6. Conclusions

NARX modelling has been extensively applied in model fitting for a large class of nonlinear systems and is often followed by frequency domain analysis. Unfortunately, the frequency domain analysis has until now been restricted to mildly (or weakly) nonlinear systems. While NARX time domain modelling is capable of fitting models to severely nonlinear systems that exhibit subharmonics and other complex behaviours, the frequency domain capacity is lost.

A novel MISO NARX modelling algorithm, designed for a special class of severely nonlinear systems that exhibit subharmonics, has been proposed in this study to overcome these problems. The key idea is to split the original input signal into n individual inputs(where n is the order of subharmonics), with each individual input having the same period as that of the output signal. To enable accurate and parsimonious modelling, a smooth/band-limited input signal decomposition is essential, and this can be done based on a Fourier analysis/truncation. In Part II of this paper, a comprehensive frequency domain analysis will be introduced to verify this idea and to provide a frequency domain interpretation of subharmonic systems.

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