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ANALYSIS OF ESTIMATION ERRORS IN THE IDENTIFICATION OF NONLINEAR SYSTEMS

by

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Abstract

Estimation errors introduced in the identification of nonlinear systems are analysed. The influence of record length, mean level, power and bandwidth of the input excitation, the effects of input and output noise and errors introduced by the decomposition techniques are investigated. Experimental design and input selection are discussed and the results are illustrated by simulated examples.

1. Introduction

Although there have been several applications of nonlinear identification techniques based on either the Volterra or Wiener series most of the authors assume that only the first two terms in the functional expansions are present or that higher order terms can be neglected. This situation is in many ways inevitable because of the large number of data points and excessive computations associated with the identification of third and higher terms using correlation analysis[1]. The problem is particularly difficult in the Volterra series case because whilst it is easy to decouple the identification of the first two kernels when there is no contribution due to higher order terms the existence of significant higher order terms invalidates this approach and multilevel inputs have to be applied to isolate the contribution of each kernel prior to identification[2]. This problem does not arise in the Lee and Schetzen[3] procedure because of the orthogonality of the functionals. However it has recently been suggested[4] that third and higher order terms cannot be accurately estimated using this algorithm in continuous time because of fundamental errors associated with this approach. Other authors have considered alternative expansions of the input-output map and algorithms have been developed for systems which can be represented by interconnections of linear dynamic and static nonlinear elements[5,6]. The advantage of this approach is that even systems containing very violent nonlinearities can be identified in a manner which preserves the system structure and provides a concise description of the process. Because practical identification based on the Volterra or Wiener series is often restricted computationally to systems containing just the first two kernels most authors have applied these techniques to systems which can be represented by the block orientated models discussed above.

Analysis of the estimation errors associated with the Lee and Schetzen procedure using CSRS inputs have been investigated by Marmarelis[7-9]. Similar studies have revealed anomalies associated with pseudorandom inputs and correlation analysis based on the Volterra series[10,11].

In the present study an analysis of the estimation errors associated with the identification of a class of nonlinear systems is considered. The results are derived for an algorithm based on the theory of separable processes[12] and the general model illustrated in Fig.1 but apply equally to the identification of the first and second order Volterra kernels assuming higher order terms can be neglected. Errors introduced into the estimation of the first and second order correlation function estimates are investigated by analysing the effects of record length, internal noise, finite bandwidth and variance of the Gaussian input. The selection of inputs and errors introduced in the estimation of the individual elements of the general model are considered and simulations are included to illustrate the results obtained.

2. Identification Algorithms

Consider a nonlinear system which can be described by a truncated Volterra series consisting of just the first two terms

$$z(t) = \int_{-\infty}^{\infty} g_{1V}(\tau_1)u(t-\tau_1)d\tau_1 + \iint_{-\infty}^{\infty} g_{2V}(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)\, d\tau_1 d\tau_2 \qquad (1)$$

If u(t) is a zero mean white Gaussian noise signal where $\phi_{uu}(t) = \lambda\delta(t)$, estimates of the first and second order Volterra kernels can be obtained by computing the first and second correlation functions to yield

$$\phi_{uz'}(\sigma) = \alpha_1 \hat{g}_{1V}(\sigma) \qquad (2)$$

$$\phi_{uuz'}(\sigma_1,\sigma_2) = \alpha_2 \hat{g}_{2V}(\sigma_1,\sigma_2) \qquad (3)$$

where $\alpha_1 = \lambda$, $\alpha_2 = 2\lambda^2$, eqn (3) represents the symmetrical second order kernel and the superscript ' is used throughout to indicate a zero mean process.

Alternatively, if the system has the structure of the general model illustrated in Fig.1 a similar identification procedure can be used to determine estimates of the individual component subsystems. By considering the theory of separable processes[5,12] it can readily be shown that for an input u(t)+b where u(t) is a zero mean separable white Gaussian process and b is a nonzero mean level

$$\phi_{uz'}(\sigma) = \alpha_1 C_F \int_{-\infty}^{\infty} h_1(\tau_1)h_2(\sigma-\tau_1)d\tau_1 = \alpha_1 C_F \hat{g}_{1g}(\sigma) \qquad (4)$$

$$\phi_{uuz'}(\sigma_1,\sigma_2)\Big|_{\sigma_1=\sigma_2=\sigma} = \alpha_2 C_{FF} \int_{-\infty}^{\infty} h_2(\tau_1)h_1(\sigma_1-\tau_1)h_1(\sigma_2-\tau_1)d\tau_1$$

$$= \alpha_2 C_{FF} \hat{g}_{2g}(\sigma,\sigma) \qquad (5)$$

If the nonlinear element can, in theory, be represented by a polynomial $y(t) = \sum_{i=1}^{k} \gamma_i q^i(t)$ then provided $h_1(t)$ is stable, boundedinputs bounded-outputs, C_F and C_{FF} are constant scale factors

$$C_F = \frac{1}{\lambda} \sum_{i=1}^{k} \gamma_i \sum_{r=o}^{\frac{N-1}{2}} \binom{i}{2r+i-N} \mu_x^{(2r+i-N)} \frac{(2P)!}{2^P P!} \left(\lambda \int_{-\infty}^{\infty} h_1^2(\theta)d\theta\right)^{P-1} \quad (6)$$

$$C_{FF} = \frac{1}{2\lambda^2} \sum_{i=2}^{k} \gamma_i \sum_{r=o}^{\frac{M-1}{2}} \binom{i}{2r+i-M} \mu_x^{(2r+i-M)} \frac{(2q)!}{2^{(q-1)}(q-1)!} \left(\lambda \int_{-\infty}^{\infty} h_1^2(\theta)d\theta\right)^{q-1} \quad (7)$$

where $\mu_x = b\int h_1(\theta)d\theta$, $P = \frac{N-2r+1}{2}$, $q = \frac{M-2r}{2}$, $N = \begin{cases} i & , i \text{ odd} \\ i-1, & i \text{ even} \end{cases}$,

$M = \begin{cases} i & , i \text{ even} \\ i-1 & , i \text{ odd} \end{cases}$ and

$$z'(t) = \int_{-\infty}^{\infty} h_2(\theta) \sum_{i=1}^{k} \gamma_i \sum_{r=o}^{i} \binom{i}{r} \mu_x^{r} \int_{o}^{\infty}..\int h_1(\tau_1)..h_1(\tau_{i-r}) \left(u(t-\tau_1-\theta)\ldots u(t-\tau_{i-r}-\theta) - \overline{u(t-\tau_i-\theta)\ldots u(t-\tau_{i-r}-\theta)}\right) d\tau_1\ldots d\tau_{i-r} \quad (8)$$

The estimates of eqn (4) and eqn (5) with $\sigma_1 = \sigma_2 = \sigma$ can be decomposed to yield estimates of the linear subsystems $\mu_1 h_1(t)$ and $\mu_2 h_2(t)$ where μ_1 and μ_2 are constants.[13] A polynomial, a series of straight line segments or any other suitable function can then be fitted to the nonlinear element to complete the identification. Notice that because the identification of the linear and nonlinear components are completely decoupled systems with very violent nonlinearities can be identified using the above algorithm. An input with a mean level b is used to ensure that all terms in C_F and C_{FF} exist.

Comparison of eqns (2) and (3), (4) and (5) respectively shows that the identification procedure for the Volterra system eqn (1) and the general model are identical except for scale factors. This is because the results for the general model yield estimates of the first two Volterra kernels scaled by the constants C_F and C_{FF} even though the Volterra expansion for this system may contain numerous higher order terms. The effects of the higher order terms reduce under the theory of separable processes to the form of eqns (4) and (5) for this model structure[5,12].

The identification algorithm for the general model can be applied to feedforward, feedback and multiplicative systems composed of linear dynamic and static nonlinear elements with only slight modification[5]. This provides a unified identification procedure for systems within this class and avoids the difficulty of isolating the contribution of each kernel which would be necessary if the identification were based on the Volterra series expansion.

3. Error Analysis

The application of correlation analysis to the identification of either the Volterra system eqn (1) or the general model Fig.1, introduces various errors which are primarily due to finite record length and finite stimulus bandwidth. The effect of these, and other errors due to internal noise on the estimates of eqns (2), (3) and (4) and (5) are analysed in the following sections.

3.1 Errors due to finite record length

The first and second order correlation functions associated with the estimates of eqns (2), (3) and (4), (5) must inevitably be computed from a finite data record of length R

$$\hat{\phi}_{uz'}(\sigma) = \frac{1}{R}\int_0^R z'(t)u(t-\sigma)dt \qquad (9)$$

$$\hat{\phi}_{uuz'}(\sigma_1,\sigma_2) = \frac{1}{R}\int_0^R z'(t)u(t-\sigma_1)u(t-\sigma_2)dt \qquad (10)$$

The effect of the finite record length R on these estimates can be investigated by computing the variance of the estimates.

3.1.1 The Volterra model

Consider the variance of the estimate of the first order Volterra kernel

$$\mathrm{var}\left[\hat{g}_{1v}(\sigma)\right] = E\{\left[\frac{1}{\lambda}\hat{\phi}_{uz'}(\sigma)\right]^2\} - \{E\left[\frac{1}{\lambda}\hat{\phi}_{uz'}(\sigma)\right]\}^2$$

$$= \frac{1}{\lambda^2 R^2}\int_0^R\int_0^R (E\left[z'(\mu)u(\mu-\sigma)z'(\nu)u(\nu-\sigma)\right]$$

$$- E\left[z'(\mu)u(\mu-\sigma)\right]E\left[z'(\nu)u(\nu-\sigma)\right]d\mu d\nu \qquad (11)$$

Substituting from eqn (1)

$$\mathrm{var}\left[\hat{g}_{1v}(\sigma)\right] = \frac{1}{R}\left[\int_{-T}^{T}\{\int_{-\infty}^{\infty} g_{1v}(\tau_2-\xi)g_{1v}(\tau_2)d\tau_2\right.$$

$$+ g_{1v}(\sigma-\xi)g_{1v}(\xi+\sigma)\}d\xi + \lambda\{2\iint_{-\infty}^{\infty} g_{2v}{}^2(\tau_3,\tau_4)d\tau_3 d\tau_4$$

$$+ 4\int_{-T}^{T}\int_{-\infty}^{\infty} g_{2v}(\tau_3-\xi,\sigma)g_{2v}(\tau_3,\sigma)d\tau_3 d\xi +$$

$$3\int_{-T}^{T}\int_{-\infty}^{\infty} g_{2v}(\tau_3-\xi,\sigma-\xi)g_{2v}(\tau_3,\sigma+\xi)d\tau_3 d\xi$$

$$\left.+\int_{-T}^{T}\int_{-\infty}^{\infty} g_{2v}(\sigma-\xi,\tau_4+\xi)g_{2v}(\sigma+\xi,\tau_4)d\tau_4 d\xi\}\right]$$

$$= \frac{1}{R}(c_1 + \lambda c_2) \qquad (12)$$

where c_1 and c_2 are constants which depend upon the first two Volterra kernels of the system under study. Thus the variance of the estimate for the first order kernel is directly proportional to λ, the variance of the input excitation and inversely proportional to the record length R.

Consider the variance of the estimate of the second order kernel

$$\mathrm{var}\left[\hat{g}_{2v}(\sigma_1,\sigma_2)\right] = \frac{1}{4\lambda^4R^2}\int_0^R\int_0^R \{E\left[z'(\mu)u(\mu-\sigma_1)u(\mu-\sigma_2)\right.$$

$$\left.z'(\nu)u(\nu-\sigma_1)u(\nu-\sigma_2)\right] - E\left[z'(\mu)u(\mu-\sigma_1)u(\mu-\sigma_2)\right]$$

$$E\left[z'(\nu)u(\nu-\sigma_1)u(\nu-\sigma_2)\right]\}d\mu d\nu \qquad (13)$$

Substituting eqn (1) into eqn (13) and simplifying yields

$$\mathrm{var}\left[\hat{g}_{2v}(\sigma_1,\sigma_2)\right] = \frac{1}{4R}\{k_1 + \frac{k_2}{\lambda} + \frac{k_3\varepsilon}{\lambda^2} + \frac{k_4\varepsilon}{\lambda^3}\} \quad \text{for } \sigma_1 \neq \sigma_2 \qquad (14)$$

where $\varepsilon = \int_{-\infty}^{\infty} \phi^2_{uu}(t)dt$, and k_i, $i = 1,..4$ are constants which depend upon the first two Volterra kernels of the system under study and are defined in Appendix 9.1. Assuming that $\phi_{uu}(t)$ can be approximated by a triangular function centred on the origin with height $\phi_{uu}(0)$ and base width $2\Delta t$, then

$$\lambda = \phi_{uu}(0)\Delta t \qquad (15)$$

$$\varepsilon = \frac{2\phi^2_{uu}(\theta)\Delta t}{3} \qquad (16)$$

and eqn (14) can be expressed as

$$\mathrm{var}\left[\hat{g}_{2v}(\sigma_1,\sigma_2)\right] = \frac{1}{4R}\left\{k_1 + \frac{k_2}{\phi_{uu}(0)\Delta t} + \frac{2k_3}{3\Delta t^2} + \frac{2k_4}{\phi_{uu}(0)\Delta t^2}\right\}$$

$$= \frac{1}{4R}\left\{K_1 + \frac{K_2}{\phi_{uu}(0)}\right\} \qquad \sigma_1 \neq \sigma_2 \qquad (17)$$

where $K_1 = k_1 + \frac{2}{3}\frac{k_3}{\Delta t^2}$ and $K_2 = \frac{k_2}{\Delta t} + \frac{2}{3}\frac{k_4}{\Delta t^2}$

When $\sigma_1 = \sigma_2 = \sigma$

$$\mathrm{var}\left[\hat{g}_{2v}(\sigma,\sigma)\right] = \frac{1}{4R}\left\{\frac{\alpha_1}{\lambda} + \frac{\alpha_2\phi_{uu}(0)}{\lambda^2} + \frac{\alpha_3\phi_{uu}^2(0)}{\lambda^3} + \frac{\alpha_4\varepsilon}{\lambda^3} + \alpha_5 + \frac{\alpha_6\phi_{uu}(0)}{\lambda} + \frac{\alpha_7\phi_{uu}^2(0)}{\lambda^2} + \frac{\alpha_8\varepsilon}{\lambda^2}\right\} \qquad (18)$$

where α_i, $i = 1\ldots8$ are constants defined in Appendix 9.2. Combining eqns (15), (16) and (18)

$$\mathrm{var}\left[\hat{g}_{2v}(\sigma,\sigma)\right] = \frac{1}{4R}\left\{\frac{\alpha_1}{\phi_{uu}(0)\Delta t} + \frac{\alpha_2}{\phi_{uu}(0)\Delta t^2} + \frac{\alpha_3}{\phi_{uu}(0)\Delta t^3} + \frac{2\alpha_4}{\phi_{uu}(0)\Delta t^2} + \alpha_5 + \frac{\alpha_6}{\Delta t} + \frac{\alpha_7}{\Delta t^2} + \frac{2\alpha_8}{3\Delta t}\right.$$

$$= \frac{1}{4R}\left\{\frac{\beta_1}{\phi_{uu}(0)} + \beta_2\right\} \qquad \text{for } \sigma_1 = \sigma_2 = \sigma \qquad (19)$$

where $$\beta_1 = \frac{\alpha_1}{\Delta t} + \frac{\alpha_2}{\Delta t^2} + \frac{\alpha_3}{\Delta t^3} + \frac{2\alpha_4}{3\Delta t^2} \qquad (20)$$

and $$\beta_2 = \alpha_5 + \frac{\alpha_6}{\Delta t} + \frac{\alpha_7}{\Delta t^2} + \frac{2\alpha_8}{3\Delta t} \qquad (21)$$

The variance of the estimate of the second order kernel is therefore inversely proportional to the record length and to the variance of the input signal.

3.1.2 The General Model

Consider initially the general model assuming $\gamma_1, \gamma_2 = 1$ and $\gamma_i = 0$ for $i>2$ such that eqns (4) and (5) reduce to

$$\hat{g}_{1g}(\sigma) = \int h_2(\theta) h_1(\sigma-\theta) d\theta = \frac{1}{\lambda(1+2b)} \phi_{uz'}(\sigma) \qquad (22)$$

$$\hat{g}_{2g}(\sigma,\sigma) = \int h_2(\theta) h_1^2(\sigma-\theta) d\theta = \frac{1}{2\lambda^2(1+3bp)} \phi_{u^2z'}(\sigma) \qquad (23)$$

where $p = \int h_1(\tau) d\tau$, for a non-zero mean input $u_1(t) = u(t)+b$. Substituting eqn (22) in eqn (11) yields the variance of the first order correlation function

$$\text{var}\left\{\frac{1}{\lambda(1+2b)} \phi_{uz'}(\sigma)\right\} = \frac{1}{(1+2b)R} (c_1+\lambda c_2+b^2 c_3) \qquad (24)$$

where the constants c_1 and c_2 are defined by eqn (12) and

$$c_3 = 4\iiint g_{2g}(\tau_1,\tau_2) g_{2g}(\tau_1,\tau_4) d\tau_1 d\tau_2 d\tau_4$$

$$+ 4\iiint g_{2g}(\sigma-\xi,\tau_1) g_{2g}(\sigma+\xi,\tau_2) d\tau_1 d\tau_2 d\xi \qquad (25)$$

Similarly for the second order correlation function, substituting eqn (23) in (13) gives

$$\text{var}\left\{\frac{1}{2\lambda^2(1+3bp)}\,\phi_{uuz'}(\sigma,\sigma)\right\} = \frac{1}{4R(1+3bp)}\left\{\frac{\beta_1'}{\phi_{uu}(0)} + \beta_2\right\} \qquad (26)$$

where $\beta_1' = \beta_1 + \frac{b^2}{\Delta t}\beta_3$ and

$$\beta_3 = \frac{4}{\Delta t^2}\int_{-T}^{T}\iiint_{-\infty}^{\infty} g_{2g}(\tau_1-\xi,\tau_2)g_{2g}(\tau_1,\tau_3)d\tau_1 d\tau_2 d\tau_3 d\xi + \frac{8}{\Delta t}\Big(\tfrac{1}{3}\iiint g_{2g}(\tau_1,\tau_2)$$

$$g_{2g}(\tau_3,\tau_4)d\tau_1 d\tau_2 d\tau_3 d\tau_4 + \iiint g_{2g}(\sigma,\tau_1)g_{2g}(\sigma+\xi,\tau_2)d\tau_1 d\tau_2 d\xi$$

$$+ \iiint g_{2g}(\sigma-\xi,\tau_1)g_{2g}(\sigma,\tau_2)d\tau_1 d\tau_2 d\xi + 2\iiint g_{2g}(\sigma-\xi,\tau_1)g_{2g}(\sigma+\xi,\tau_2)d\tau_1 d\tau_2 d\xi\Big)$$

$$+ 16\iint g_{2g}(\sigma,\tau_1)g_{2g}(\sigma,\tau_2)d\tau_1 d\tau_2$$

For the case when $\gamma_i \neq 0$ for $i = 1,\ldots k$ and $b \neq 0$ then

$$\text{var}\left\{\frac{1}{\lambda C_F}\,\phi_{uz'}(\sigma)\right\} = \frac{1}{R}\,(f_1+\lambda f_2+\lambda^2 f_3+\ldots) \qquad (27)$$

$$\text{var}\left\{\frac{1}{2\lambda^2 C_{FF}}\,\phi_{uuz'}(\sigma,\sigma)\right\} = \frac{1}{R}\left(\frac{\ell_1}{\phi_{uu}(0)} + \ell_2+\phi_{uu}(0)\ell_3 + \phi_{uu}^2(0)\ell_4 + \ldots\right) \qquad (28)$$

where $f_i = F_1(\gamma_i,\gamma_{i+1},\ldots\gamma_k,b,h_1,h_2)$

$\ell_i = F_2(\gamma_i,\gamma_{i+1},\ldots\gamma_k,b,h_1,h_2)$

are in general nonlinear functions for $i = 1,2\ldots k$.

Inspection of eqns (12), (17) and (19) for the Volterra model and eqns (27), (28) for the general model shows that the estimates are consistent and the rms error decreases as $1/\sqrt{R}$. The variance of the estimates is a function of the record length R, the input signal variance, the mean level of the input b and the specific system under identification. Notice that in general the estimates of the Volterra kernels for $\sigma_1 = \sigma_2$ will have a larger variance than estimates of the non-diagonal points ($\sigma_1 \neq \sigma_2$) because of the presence of low order integral terms. Comparison of eqns (12) and (24), (19) and (26) respectively shows that it is very difficult to formulate a simple procedure to determine an optimum value for the mean level of the input b without considering specific systems. However the simulation results indicate that suitable selection of b can result in a significant reduction in the variances.

3.2 Errors due to the Gaussian white noise stimulus

Since band-limited Gaussian white noise must be used in the identification algorithms this inevitably introduces various errors in the estimates. The two main sources of error introduced in this manner are convolution and statistical errors which are investigated below for the general model.

3.2.1 Convolution errors

The autocorrelation function of a white Gaussian process of limited bandwidth B_o and spectral density of λ watts/cycle is given by

$$\phi_{uu}(\tau_1,\tau_2) = 2B_o\lambda \frac{\sin(2\pi B_o(\tau_1-\tau_2))}{2\pi B_o(\tau_1-\tau_2)} \qquad (29)$$

Convolution errors can be determined by inserting eqn (29) in the expected value of the estimate of $\hat{g}_{1g}(\sigma)$

$$E[\hat{g}_{1g}(\sigma)] = \frac{1}{\lambda} E[\iint h_1(\tau_1)h_2(\theta)\hat{\phi}_{uu}(\sigma-\theta-\tau_1)d\theta d\tau_1]$$

$$= \frac{1}{\lambda} \iint h_1(\tau_1)h_2(\theta)(2B_o\lambda) \frac{\sin 2\pi B_o(\sigma-\theta-\tau_1)}{2\pi B_o(\sigma-\theta-\tau_1)} d\theta d\tau_1 \qquad (30)$$

Expanding $h_1(\tau_1)$ in the neighbourhood of $(\sigma-\theta)$ yields

$$h_1(\tau_1) = h_1(\sigma-\theta) + h_1^{(1)}(\sigma-\theta)(\tau_1-\sigma-\theta)$$

$$+ \frac{h_1^{(2)}(\sigma-\theta)}{2}(\tau_1-\sigma+\theta)^2 + \ldots \qquad (31)$$

where the bracketed superscripts indicate the order of derivative. Since $\phi_{uu}(\cdot)$ is an even function, the integration over a symmetric interval around $(\sigma-\theta)$ will eliminate the odd order terms in the Taylor expansion. Thus

$$E[\hat{g}_{1g}(\sigma)] \simeq \frac{1}{\pi} \int h_2(\theta) \int_{(\sigma-\theta-\sigma)}^{(\sigma-\theta+\sigma)} h_1(\tau_1) \frac{\sin 2\pi B_o(\sigma-\theta-\tau_1)}{(\sigma-\theta-\tau_1)} d\theta d\tau_1$$

and substituting from (31)

$$E[\hat{g}_{1g}(\sigma)] \simeq \frac{1}{\pi} \int h_2(\theta) \sum_{n=o}^{\infty} \frac{h_1^{(2n)}(\sigma-\theta)}{(2n)!} \int_{-r}^{r} \xi^{2n-1} \sin(2\pi B_o\xi)d\xi d\theta \qquad (32)$$

$$\simeq \{\int (h_2(\theta)h_1(\sigma-\theta) + \frac{2}{\pi} \sum_{n=1}^{\infty} b_n \frac{h_1^{(2n)}(\sigma-\theta))}{(2n)!} d\theta \} \qquad (33)$$

where $r >> 1/B_o$ is the settling time of the sinc function, $\xi = \sigma-\theta-\tau$ and

$$b_n = \int_0^r \xi^{2n-1} \sin(2\pi B_o \xi)\, d\xi$$

$$= - \sum_{j=o}^{2n-1} j! \binom{2n-1}{j} \frac{r^{2n-j-1}}{(2\pi B_o)^{j+1}} \cos(2\pi B_o r + \frac{j\pi}{2}) \qquad (34)$$

From (33) and (34) as the input bandwidth increases the influence of the second term in eqn (33) is reduced and the estimate tends towards an unbiased estimate.

Following a similar derivation for the estimate of $\hat{g}_{2g}(\sigma,\sigma)$ yields

$$E[\hat{g}_{2g}(\sigma,\sigma)] = \frac{1}{\lambda^2}\iiint h_2(\theta) h_1(\tau_1) h_1(\tau_2) \phi_{uu}(\sigma-\theta-\tau_1)$$

$$\phi_{uu}(\sigma-\theta-\tau_2)\, d\tau_1 d\tau_2 d\theta$$

$$= \frac{1}{\lambda^2}\iiint h_2(\theta) h_1(\tau_1) h_1(\tau_2) \frac{(2B_o\lambda)^2}{4\pi^2 B_o^2} \sin \frac{2\pi B_o(\sigma-\theta-\tau_1)}{(\sigma-\theta-\tau_1)}$$

$$. \sin \frac{2\pi B_o(\sigma-\theta-\tau_2)}{(\sigma-\theta-\tau_2)} \, d\tau_1 d\tau_2 d\theta$$

$$= \frac{1}{\pi^2} \int h_2(\theta) \left(\sum_{n=o}^{\infty} \frac{h_1^{(2n)}(\sigma-\theta)}{(2n)!} \int_{-r}^{r} \xi_1^{2n-1} \sin(2\pi B_o \xi_1)\, d\xi_1 \right)$$

$$\left(\sum_{m=o}^{\infty} \frac{h_1^{(2m)}(\sigma-\theta)}{(2m)!} \int_{-r}^{r} \xi_2^{2m-1} \sin(2\pi B_o \xi_2)\, d\xi_2 \right) d\theta \qquad (35)$$

$$= \{\int h_2(\theta)h_1^{\,2}(\sigma-\theta)d\theta + \Big[\frac{1}{\pi}\int h_2(\theta)h_1(\sigma-\theta)\Big(\sum_{m=1}^{\infty}\frac{h_1^{(2m)}(\sigma-\theta)}{(2m)!}$$

$$\int_{-r}^{r}\xi_2^{\,2m-1}\sin(2\pi B_o\xi_2)d\xi_2\Big)d\theta + \frac{1}{\pi}\int h_2(\theta)h_1(\sigma-\theta)$$

$$\Big(\sum_{n=1}^{\infty}\frac{h_1^{(2n)}(\sigma-\theta)}{(2n)!}\int_{-r}^{r}\xi_1^{\,2n-1}\sin(2\pi B_o\xi_1)d\xi_1\Big)$$

$$+ \int h_2(\theta)\frac{4}{\pi^2}\Big(\sum_{n=1}^{\infty}\frac{1}{(2n)!}h_1^{(2n)}(\sigma-\theta)b_n\Big)$$

$$\Big(\sum_{m=1}^{\infty}\frac{h_1^{(2m)}(\sigma-\theta)}{(2m)!}\cdot b_n\Big)d\theta\Big]\} \qquad (36)$$

where $\xi_1 = \sigma-\theta-\tau_1$, $\xi_2 = \sigma-\theta-\tau_2$.

Inspection of eqns (33) and (36) shows that convolution errors can be reduced by increasing the bandwidth of the input stimulus such that the first terms in eqns (33) and (36) dominate and the estimates tend to become unbiased. Unfortunately increasing the input bandwidth increases the statistical errors in the estimates as shown below and a compromise in the selection of B_o must be sought.

3.2.2 Statistical errors

To evaluate the statistical errors in the estimates consider initially $\hat{g}_{1g}(\sigma)$ evaluated by taking the average of a finite number M of samples $v(t_i,\sigma) = z'(t_i)u(t_i-\sigma)$ for all samples of $v(t_i,\sigma)$ which are independent such that the mean square error in the estimate is given by

$$E\left[(\hat{g}_{1g}(\sigma)-g_{1g}(\sigma))^2\right]$$

$$= \frac{1}{\lambda^2 C_F{}^2 M^2} E\left[\left\{\sum_{i=1}^{M} (z'(t_i)u(t_i-\sigma)-\phi_{uz'}(\sigma))\right\}^2\right]$$

$$= \frac{1}{\lambda^2 C_F{}^2 M^2} E\left[\left\{\sum_{i=1}^{M} (v(t_i,\sigma)-\phi_{uz'}(\sigma))\right\}^2\right]$$

$$= \frac{1}{\lambda^2 C_F{}^2 M} \left\{E\left[v^2(t,\sigma)\right] - (E\left[v(t,\sigma)\right])^2\right\} \tag{37}$$

$$= \frac{\sigma_v{}^2}{\lambda^2 C_F{}^2 M} \tag{38}$$

where $\sigma_v{}^2$ is the variance of $v(t,\sigma)$. But from eqn (8)

$$(z'(t)u(t-\sigma))^2 = \iint_0^\infty h_2(\theta_1)h_2(\theta_2) \sum_{i=1}^{k} \sum_{j=1}^{k} \gamma_i \gamma_j \sum_{r_1=o}^{i} \sum_{r_2=o}^{j} \binom{i}{r_1}\binom{j}{r_2}$$

$$\mu_x{}^{r_1} \mu_x{}^{r_2} \underbrace{\int_0^\infty \cdot \cdot \cdot \int_0^\infty}_{(i+j-r_1-r_2) \text{ integrals}} h_1(\tau_1)..h_1(\tau_{i-r_1})h_1(\lambda_1)..h_1(\lambda_{j-r_2})$$

$$\left[u(t-\tau_1-\theta_1)\ldots u(t-\tau_{i-r_1}-\theta_1)-\overline{u(t-\tau_1-\theta_1)\ldots u(t-\tau_{i-r_1}-\theta_1)}\right]$$

$$\cdot \left[u(t-\lambda_1-\theta_2)\ldots u(t-\tau_{j-r_2}-\theta_2)-\overline{u(t-\lambda_1-\theta_2)\ldots u(t-\tau_{j-r_2}-\theta)}\right]$$

$$u^2(t-\sigma)d\tau_1 d\tau_2 \ldots d\tau_{i-r_1} d\lambda_1 \ldots d\lambda_{j-r_2} d\theta_1 d\theta_2 \tag{39}$$

and combining this with eqn (37) gives

$$E\left[(\hat{g}_{1g}(\sigma)-g_{1g}(\sigma))^2\right]$$

$$= \frac{1}{\lambda^2 C_F^{\ 2} M} \ \{ \iint_0^\infty h_2(\theta_1)h_2(\theta_2)\gamma_1^{\ 2}\left[\iiint_0^\infty h_1(\tau_1)h_1(\lambda_1)\right.$$

$$(\phi_{uu}(0)\phi_{uu}(\tau_1+\theta_1+\lambda_1-\theta_2)+\phi_{uu}(\sigma-\tau_1-\theta)\phi_{uu}(\sigma-\lambda_1-\theta_2)d\tau_1 d\lambda_1$$

$$+ 2\gamma_1\gamma_2\mu_x\iiint_0^\infty h_1(\tau_1)h_1(\lambda_1)h_1(\lambda_2)(\phi_{uu}(\sigma-\lambda_1-\theta_1)\phi_{uu}(\sigma-\lambda_2-\theta_2)$$

$$\left. + \phi_{uu}(0)\phi_{uu}(\tau_1+\theta_1-\lambda_1-\theta_2))d\tau_1 d\lambda_1 d\lambda_2 + \ldots\right] d\theta_1 \qquad (40)$$

For a band limited Gaussian white input

$$S_{uu}(\omega) = \begin{cases} 1 & 0<\omega<\omega_o \\ 0 & \omega>\omega_o \end{cases}$$

and
$$\phi_{uu}(t) = \frac{\omega_o}{\pi} \sin\frac{(\omega_o t)}{(\omega_o t)} \qquad (41)$$

where $\omega_o = 2\pi B_o$. Inserting eqn (41) in eqn (40)

$$E\left[(\hat{g}_{1g}(\sigma)-g_{1g}(\sigma))^2\right]$$

$$= \frac{1}{\lambda^2 C_F^{\ 2} M}\left\{\frac{\omega_o}{4\pi^3}\ (\gamma_1^{\ 2}+2\gamma_1\gamma_2\mu_x \int h_1(\lambda)d\lambda+\ldots)\right.$$

$$\cdot \int_{-\omega_o}^{\omega_o} |G(j\omega)|^2 d\omega + \frac{1}{4\pi^2}\ (\gamma_1^{\ 2}+2\gamma_1\gamma_2\mu_x\int h_1(\lambda)d\lambda$$

$$\left. + \ldots)\ \left|\int_{-\omega_o}^{\omega_o} G(j\omega)e^{j\omega\sigma}d\omega\right|^2+\ldots\right\}$$

$$= \frac{C}{\lambda^2 C_F^2 M} \{ \frac{\omega_o}{\pi} \int_{-\omega_o}^{\omega_o} |G(j\omega)|^2 d\omega$$

$$+ | \int_{-\omega_o}^{\omega_o} G(j\omega) e^{-j\omega\sigma} d\omega + \dots \} \quad (42)$$

where $G(j\omega) = H_1(j\omega)H_2(j\omega)$ and

$$C = \frac{1}{4\pi^2} (\gamma_1^2 + 2\gamma_1\gamma_2\mu_x \int h_1(\lambda)d\lambda + \dots)$$

Similarly, the mean square error for $\hat{g}_{2g}(\sigma,\sigma)$ can be evaluated to yield

$$E\left[(\hat{g}_{2g}(\sigma,\sigma) - g_{2g}(\sigma,\sigma))^2\right]$$

$$= \frac{1}{4\lambda^2 C_{FF}^2 M} \{ \frac{3\omega_o^2}{4\pi^4} \gamma_1^2 \int_{-\omega_o}^{\omega_o} |G(j\omega)| d\omega +$$

$$\frac{12\omega_o \gamma_1^2}{4\pi^3} | \int_{-\omega_o}^{\omega_o} G(j\omega) e^{j\omega\sigma} d\omega |^2 + \dots \} \quad (43)$$

Equations (42) and (43) clearly indicate that the bandwidth of the Gaussian white stimulus must be as small as possible and the number of independent samples must be large in order to reduce the statistical error. The former requirement is however in direct conflict with the need to increase the input bandwidth to reduce convolution errors. The experimenter must therefore seek a compromise and select the input bandwidth to ensure that this just covers the system bandwidth such that the input excites all the system modes. The sampling interval of the data is closely related to the bandwidth of the input and should be selected to ensure that aliasing does not occur.

4. Errors due to noise

Inevitably in any identification experiment data recorded will include some amount of noise. In practice several noise sources may co-exist and in the present investigation the effects of noise at the input and output of the system are studied. To simplify the analysis the effect of each noise source is considered separately assuming that the other error sources do not exist.

4.1 Noise at the input

Noise at the input of the system can be induced by various factors. If the input signal u(t) deviates from a true Gaussian signal because of experimental limitations this can be regarded as an input noise source which is transmitted through the system as illustrated in Fig.2(i). Measurement errors associated with the input signal can also be considered as an input noise source as illustrated in Fig.2(ii), and both these cases are investigated below.

Consider the system depicted in Fig.2(i) where the system output $z^*(t)$ can be expressed as

$$z^*(t) = \sum_{i=1}^{k} \int\ldots\int g_i(\tau_1,\ldots\tau_i)(u(t-\tau_1)+n(t-\tau_1)+b)\ldots$$

$$\ldots(u(t-\tau_i)+u(t-\tau_i)+b)d\tau_1\ldots d\tau_i \qquad (44)$$

and the first order correlation function is given by

$$\begin{aligned}\phi_{u^*z^{*\prime}}(\sigma) &= \overline{\left[z^*(t)-\overline{z^*(t)}\right](u(t-\sigma)+n(t-\sigma))} \\ &= \phi_{z^{*\prime}u}(\sigma) + \phi_{z^{*\prime}n}(\sigma) \\ &= \phi_{z'u}(\sigma) + \sum_{i=1}^{k} E_i(\sigma) \qquad (45)\end{aligned}$$

where $E_i(\sigma)$, i = 1,2...k are the errors due to the input noise n(t). For the Volterra model eqn (1) where k = 2, b = O the errors are defined by

$$E_1(\sigma) = \int_0^\infty g_{1v}(\tau_1)(\phi_{un}(\sigma-\tau_1)+\phi_{nn}(\sigma-\tau_1))d\tau_1$$

$$E_2(\sigma) = \iint_0^\infty g_{2v}(\tau_1,\tau_2)(\phi_{(uu)'n}(\sigma-\tau_1,\sigma-\tau_2)+$$

$$\phi_{unu}(\sigma-\tau_1,\sigma-\tau_2)+\phi_{(nn)'u}(\sigma-\tau_1,\sigma-\tau_2)$$

$$+ 2\phi_{unn}(\sigma-\tau_1,\sigma-\tau_2)+\phi_{(nn)'n}(\sigma-\tau_1,\sigma-\tau_2)d\tau_1 d\tau_2 \qquad (46)$$

If u(t) and n(t) are independent zero mean processes then

$$E_1(\sigma) = \int_0^\infty g_{1v}(\tau_1)\phi_{un}(\sigma-\tau_1)d\tau_1$$
$$E_2(\sigma) = 0 \qquad (47)$$

For the general model, assuming that n(t) and u(t) are zero mean and independent, the errors are

$$E_1(\sigma) = \gamma_1 \int_0^\infty h_2(\theta)\int h_1(\tau_1)\phi_{un}(\sigma-\tau_1-\theta)d\tau_1 d\theta$$

$$E_2(\sigma) = 2\mu_x\gamma_2 \int_0^\infty h_2(\theta)\int h_1(\tau_1)\phi_{nn}(\sigma-\tau_1-\theta)d\tau_1 d\tau_2 d\theta$$

etc. (48)

Similarly for the second order correlation function

$$\phi_{z*'u*u*}(\sigma_1,\sigma_2) = \phi_{z*'uu}(\sigma_1,\sigma_2)+2\phi_{z*'un}(\sigma_1,\sigma_2)+\phi_{z*'nn}(\sigma_1,\sigma_2)$$

$$= \phi_{z'uu}(\sigma_1,\sigma_2) + \sum_{i=1}^{k} EE_i(\sigma_1,\sigma_2) \qquad (49)$$

where $EE_i(\sigma_1,\sigma_2)$, i = 1,2..k are the error terms which, assuming n(t) and u(t) are zero mean and independent, are given by

$$EE_1(\sigma_1,\sigma_2) = 0$$

$$EE_2(\sigma_1,\sigma_2) = \iint g_{2v}(\tau_1,\tau_2)\{ 4\phi_{uu}(\sigma_1-\tau_1)\phi_{un}(\sigma_2-\tau_2) - \phi_{(nn)'nn}(\sigma_1-\tau_1,\sigma_2-\tau_2)\}d\tau_1 d\tau_2 \quad (50)$$

for the Volterra model, and

$$EE_1(\sigma_1,\sigma_2) = 0$$

$$EE_2(\sigma_1,\sigma_2) = \gamma_2\int h_2(\theta)\iint h_1(\tau_1)h_1(\tau_2)\{4\phi_{uu}(\sigma_1-\tau_1-\theta) \cdot\phi_{nn}(\sigma_2-\tau_2-\theta)+\phi_{(nn)'nn}(\sigma_1-\tau_1-\theta,\sigma_2-\tau_2-\theta)\}d\tau_1 d\tau_2 d\theta \quad (51)$$

for the general model.

If the input noise is a measurement noise on u(t), as illustrated in Fig.2(ii) it is easy to show that the errors $\varepsilon_i(\sigma)$, $\varepsilon\varepsilon_i(\sigma_1,\sigma_2)$ are zero when u(t) and n(t) are zero mean and independent.

As expected input noise which is transmitted through the system will result in biased estimates where the bias is given by eqns (47), (48) and (50) and (51). The rms error induced by this bias can be shown to decrease with the square root of the record length and if the noise cannot be reduced by improved experimental conditions the record length should be as long as possible to minimise the influence of this bias.

where $EE_i(\sigma_1,\sigma_2)$, $i = 1,2..k$ are the error terms which, assuming $n(t)$ and $u(t)$ are zero mean and independent, are given by

$$EE_1(\sigma_1,\sigma_2) = 0$$

$$EE_2(\sigma_1,\sigma_2) = \iint g_{2v}(\tau_1,\tau_2)\{ 4\phi_{uu}(\sigma_1-\tau_1)\phi_{un}(\sigma_2-\tau_2)$$

$$- \phi_{(nn)'nn}(\sigma_1-\tau_1,\sigma_2-\tau_2)\}d\tau_1 d\tau_2 \qquad (50)$$

for the Volterra model, and

$$EE_1(\sigma_1,\sigma_2) = 0$$

$$EE_2(\sigma_1,\sigma_2) = \gamma_2\int h_2(\theta)\iint h_1(\tau_1)h_1(\tau_2)\{4\phi_{uu}(\sigma_1-\tau_1-\theta)$$

$$\cdot\phi_{nn}(\sigma_2-\tau_2-\theta)+\phi_{(nn)'nn}(\sigma_1-\tau_1-\theta,\sigma_2-\tau_2-\theta)\}d\tau_1 d\tau_2 d\theta \qquad (51)$$

for the general model.

If the input noise is a measurement noise on $u(t)$, as illustrated in Fig.2(ii) it is easy to show that the errors $\varepsilon_i(\sigma)$, $\varepsilon\varepsilon_i(\sigma_1,\sigma_2)$ are zero when $u(t)$ and $n(t)$ are zero mean and independent.

As expected input noise which is transmitted through the system will result in biased estimates where the bias is given by eqns (47), (48) and (50) and (51). The rms error induced by this bias can be shown to decrease with the square root of the record length and if the noise cannot be reduced by improved experimental conditions the record length should be as long as possible to minimise the influence of this bias.

4.2 Noise at the output

Any additive noise $n_o(t)$ which is induced on the output measurements $z(t)$ will introduce the error terms $\phi_{un_o}(\sigma)$ and $\phi_{uun_o}(\sigma_1,\sigma_2)$ on the estimates of the first and second order correlation functions respectively. Providing the noise $n_o(t)$ and $u(t)$ are zero mean and independent these terms will tend to zero and the estimates will be unbiased.

The effects of internal noise can be investigated in an analogous manner and will result in biased estimates similar to the case of input noise which is transmitted through the process.

5. Errors Introduced by Decomposition

If the identification is based on the general model representation estimates of the individual component subsystems $\mu_1 h_1(t)$, $\mu_2 h_2(t)$ and $F[\cdot]$ can be obtained by decomposing[13] the information contained in the first and second order correlation functions. The decomposition techniques used in this procedure and the errors which are introduced are briefly studied in the following sections.

5.1 Errors introduced by the multistage least squares procedure

The estimates of eqns (4) and (5) can be decomposed using a multistage least squares algorithm[13] to yield estimates of the individual linear component subsystems $\mu_1 h_1(t)$ and $\mu_2 h_2(t)$ where μ_1 and μ_2 are unknown constants. It has recently been shown[14] however that this direct approach tends to magnify the errors inherent in the first and second order correlation functions. These errors can be reduced considerably by minimising[16] thr criterion

$$J = \sum_{i=o}^{M} \left[(\hat{\phi}_{uz'}(i)-\psi_1(i))^2 + (\hat{\phi}_{uuz'}(i,i)-\psi_2(i))^2 \right] \quad (52)$$

to estimate $\mu_1 h_1(i)$ and $\mu_2 h_2(i)$ where

$$\psi_1(i) = \mu_1\mu_2 \sum_{j=1}^{i} \hat{h}_1(j)\hat{h}_2(i-j)$$

$$\psi_2(i) = \mu_1^2\mu_2 \sum_{j=1}^{i} \hat{h}_1^2(j)\hat{h}_2(i-j) \quad (53)$$

The criterion of eqn (52) exploits the full information contained in the first and second order correlation functions and eliminates many of the errors incurred by previous algorithms.

5.2 Errors introduced in the identification of the nonlinear-element

Once the parameters associated with the linear subsystems of the general model have been estimated the nonlinear element can be identified by fitting either a polynomial approximation using a least squares algorithm or by fitting a series of straight line segments using a minimisation procedure[5] or a histogram based method. The errors associated with each of these approaches are investigated below.

If the nonlinear element is to be identified as a power series nonlinearity, a simple least squares algorithm can be implemented using the input/output eqn

$$Z = \hat{\phi}\hat{\theta} \quad (54)$$

where Z is a vector of output measurements, $\hat{\theta}$ is a vector of unknown polynomial coefficients and $\hat{\phi}$ is a matrix of elements computed from the estimates of the linear subsystems $\phi(i,j) = \mu_2 \sum_{k=o}^{\ell} h_2(k)\hat{q}^j(i-k)$.

The least squares estimate of the unknown parameters $\hat{\theta} = (\hat{\phi}^T\hat{\phi})^{-1}\hat{\phi}^T Z$ will be unbiased providing no input noise sources exist and the output measurement noise has zero mean and is independent of the input and noise free output. Biased estimates will be obtained if input noise sources are present since identifiability conditions are violated[15].

If the nonlinearity is to be approximated by fitting a series of straight line segments this can be achieved by minimising[16]

$$J = \sum_{i=1}^{N} (z(i)-\hat{z}(i))^2$$

where

$$\hat{z}(i) = \mu_2 \sum_{j=1}^{M} \hat{h}_2(j) f\{\hat{q}(i-j)\}$$

$$\hat{q}(i) = \mu_1 \sum_{j=1}^{N} \hat{h}_1(j) u(i-j) \qquad (55)$$

and $f\{q(i)\}$ is the function describing the nonlinearity. The advantage of this approach compared with the least squares method based on a polynomial approximation is that very violent nonlinearities such as deadzone, saturation etc can be easily characterized by estimating the slopes and breakpoints of the straight line segments.

Alternatively, if $\hat{H}_2(z^{-1})$ is minimum phase, $\hat{y}(i)$ can be estimated from $\hat{y}(i) = \hat{H}_2^{-1}(z^{-1})z(i)$, $\hat{q}(i)$ is given by eqn (55) and an amplitude histogram relating $\hat{y}(i)$ and $\hat{q}(i)$ can be produced. This can be obtained digitally by dividing the range of $\hat{q}(i)$ into an appropriate number of class intervals and averaging the corresponding values of $\hat{y}(i)$. If $\hat{H}_2(z^{-1})$ is minimum phase this approach will provide the experimenter with a graphical estimate of the shape of the nonlinear element and should indicate whether a polynomial fit or a straight line segment approximation is appropriate.

6. Simulation Results

A general model consisting of a linear system

$$H_1(z^{-1}) = \frac{0.5z^{-1}}{1-1.5z^{-1}+0.65z^{-2}} \tag{56}$$

in cascade with the nonlinear element

$$y(i) = q(i)+2.0q^2(i)+3.0q^3(i)+4.0q^4(i) \tag{57}$$

followed by a second linear system

$$H_2(z^{-1}) = \frac{0.1z^{-1}}{1-1.4z^{-1}+0.55z^{-2}} \tag{58}$$

was simulated on a PDP-10 digital computer. The variation of the variances of the first and second order correlation functions with respect to the mean level b of the Gaussian white input process were evaluated from 20 independent measurements for each value of b. A total of 3000 data points were used for each measurement with the standard deviation of the input maintained at a constant level of 2.0. The results, which are illustrated in Fig.3 clearly show the significant influence of b on the variance of the estimates. As expected from the theoretical analysis the selection of b to minimise the variance of the first and second order correlation functions is often a compromise. Inspection of Fig.3 shows that for the example considered b should be in the range 0.4-0.5 to achieve the best results.

The variance of the first and second order correlation functions plotted as a function of the standard deviation of the input signal with a constant zero mean level is illustrated in Fig.4. Inspection of Fig.4 clearly shows the importance of selecting the standard deviation of the input to ensure that the variance of the estimates is as small as possible.

A comparison of the estimated correlation functions $\hat{\phi}_{uz'}(\sigma)$ and $\hat{\phi}_{uuz'}(\sigma,\sigma)$ with $\psi_1(\sigma)$ and $\psi_2(\sigma)$ eqn (53) computed using the original multistage least squares and the modified algorithm eqn (52) are illustrated in Fig.5. The improvement in the estimates using the criterion of eqn (52) is clearly evident from Fig.5 especially for the second order correlation function.

Estimates of the nonlinear element are illustrated in Fig.6. Since $H_2(z^{-1})$ is minimum phase the estimates were obtained by constructing an amplitude histogram. A summary of the identification results is given in Table 1.

7. Conclusions

Estimation errors involved in the identification of a class of nonlinear systems have been analysed. Expressions for the variance of the first and second order correlation function estimates have been derived and shown to be dependent upon the record length, mean level and power of the input and the structure of the system under investigation. Whilst the record length should be as long as possible to reduce the variances it is difficult to formulate simple rules to determine the optimum value of the mean level and power of the input because of the nonlinear nature of the expressions and the dependence upon the specific system under investigation. The simulation results do however indicate that suitable selection of these parameters can result in a considerable reduction in the variance of the correlation functions. In practice the investigator must conduct some short experiments to determine the influence of λ and b on the estimates.

The use of band-limited Gaussian white noise in the estimation procedure introduces both statistical deviations in the values of the estimates and a bias due to imperfect deconvolution. Increasing the input bandwidth reduces the convolution errors but increases the statistical errors, and hence a compromise must be sought. This can be achieved by selecting the input bandwidth to be just larger than the system bandwidth and making the record length as long as possible.

Additive noise on either the input or the output measurements can induce bias in the estimates but providing the noise is not transmitted through the system and the input and noise are independent this bias tends to zero. Additional errors can be induced by the decomposition algorithms associated with the general model but these can often be minimised by using a suitable algorithm.

The identification of nonlinear systems is a very difficult task and careful design of experiments and selection of input parameters is needed to reduce the statistical variations in the estimates and increase the probability of success.

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9. Appendices

9.1 Variance of the second order kernel $\sigma_1 \neq \sigma_2$

The constants k_i, $i = 1,\ldots 4$ associated with the variance of the estimate of the second order kernel, eqn (14) are defined as

$$k_1 = 8\int g_{2v}{}^2(\tau,\sigma_2)d\tau + 7\int g_{2v}{}^2(\tau,\sigma_1)d\tau + 3\int g_{2v}(\tau+\sigma_2-\sigma_1,\sigma_1)$$

$$\cdot\, g_{2v}(\tau,\sigma_2)d\tau + 3\int g_{2v}(\tau+\sigma_1-\sigma_2,\sigma_2)g_{2v}(\tau,\sigma_1)d\tau$$

$$+\ 2\int g_{2v}(\tau+\sigma_1-\sigma_2,2\sigma_1-\sigma_2)g_{2v}(\tau,2\sigma_2-\sigma_1)d\tau$$

$$+\ \int g_{2v}(\tau+\sigma_2-\sigma_1,2\sigma_2-\sigma_1)g_{2v}(\tau,2\sigma_1-\sigma_2)d\tau +$$

$$\int g_{2v}(2\sigma_1-\sigma_2,\tau+\sigma_2-\sigma_1)g_{2v}(\tau,2\sigma_2-\sigma_1)d\tau$$

$$+\ \int g_{2v}(2\sigma_2-\sigma_1,\tau+\sigma_1-\sigma_2)g_{2v}(\tau,2\sigma_1-\sigma_2)d\tau$$

$$+\ \int g_{2v}(\sigma_1,\sigma_2)g_{2v}(\sigma,\tau)d\tau + \int g_{2v}(3\sigma_2-2\sigma_1,2\sigma_2-\sigma_1)g_{2v}(2\sigma_1-\sigma_2,\tau)d\tau$$

$$+\ 2\int g_{2v}(2\sigma_2-\sigma_1,\tau)g_{2v}(2\sigma_1-\sigma_2,\tau+\sigma_1-\sigma_2)d\tau$$

$$+\ 4\int g_{2v}(\sigma_1+\xi,\sigma_2)g_{2v}(\sigma_2,\sigma_1-\xi)d\xi + 4\int g_{2v}(\sigma_1+\xi,\sigma_1)g_{2v}(\sigma_2,\sigma_2-\xi)d\xi$$

$$+\ 4\int g_{2v}(\sigma_1+\xi,\sigma_2+\xi)g_{2v}(\sigma_1-\xi,\sigma_2-\xi)d\xi$$

$$+\ 4\int g_{2v}(\sigma_2-\xi,\sigma_1)g_{2v}(\sigma_2+\xi,\sigma_1)d\xi + 4\int g_{2v}(\sigma_1-\xi,\sigma_1)g_{2v}(\sigma_2+\xi,\sigma_2)d\xi$$

$$+\ \int g_{2v}(\tau+\sigma_1-\sigma_2,\sigma_1)g_{2v}(\sigma_2,\tau)d\tau \qquad (59)$$

$$k_2 = 2g_{1v}{}^2(\sigma_1) + 3g_{1v}(\sigma_1)g_{1v}(\sigma_2) + g_{1v}{}^2(\sigma_2) + g_{1v}(2\sigma_1-\sigma_2)g_{1v}(\sigma_1)$$

$$+\ g_{1v}(2\sigma_2-\sigma_1)g_{1v}(2\sigma_1-\sigma_2) \qquad (60)$$

$$k_3 = \iint 2g_{2v}{}^2(\tau_1,\tau_2)d\tau_1 d\tau_2 \qquad (61)$$

$$k_4 = \int g_{1v}{}^2(\tau)d\tau \qquad (62)$$

where all the integrations are between the limits of $-\infty$ to $+\infty$.

9.2 Variance of the second order kernel $\sigma_1 = \sigma_2$

The constants α_i, i = 1...8 associated with eqn (18) are defined as

$$\alpha_1 = 8g_{1v}^2(\sigma) \quad (63)$$

$$\alpha_2 = 2\int g_{1v}(\sigma)g_{1v}(\sigma+\xi)d\xi+2\int g_{1v}(\sigma)g_{1v}(\sigma-\xi)d\xi \quad (64)$$

$$\alpha_3 = \int\int g_{1v}(\tau-\xi)g_{1v}(\tau)d\tau d\xi \quad (65)$$

$$\alpha_4 = 2\int g_{1v}^2(\tau)d\tau \quad (66)$$

$$\alpha_5 = 32\int g_{2v}^2(\tau,\alpha)d\tau+16\int g_{2v}(\sigma,\sigma-\xi)g_{2v}(\sigma,\sigma+\xi)d\xi$$

$$+ 4\int g_{2v}(\sigma-\xi,\sigma-\xi)g_{2v}(\sigma+\xi,\sigma+\xi)d\xi \quad (67)$$

$$\alpha_6 = \int_{-T}^{T}\int_{-\infty}^{\infty}(4g_{2v}(\tau-\xi,\sigma)g_{2v}(\tau,\sigma+\xi)+3g_{2v}(\sigma,\tau)g_{2v}(\sigma+\xi,\tau+\xi)$$

$$+ 4g_{2v}(\tau-\xi,\sigma-\xi)g_{2v}(\tau,\sigma)+4g_{2v}(\tau,\sigma-\xi)g_{2v}(\sigma,\tau+\xi)$$

$$+ g_{2v}(\sigma-\xi,\sigma)g_{2v}(\sigma+\xi,\tau))d\tau d\xi \quad (68)$$

$$\alpha_7 = 2\int_{-T}^{T}\int\int_{-\infty}^{\infty} g_{2v}(\tau_1-\xi,\tau_2-\xi)g_{2v}(\tau_1,\tau_2)d\tau_1 d\tau_2 d\xi \quad (69)$$

$$\alpha_8 = 2\int\int g_{2v}^2(\tau_1,\tau_2)d\tau_1 d\tau_2 \quad (70)$$

where all the limits of integration which are not included in the above eqns are between $-\infty$ and $+\infty$.

Figure Captions

Fig.1 The general model

Fig.2 Input noise sources

(i) Noise due to the statistical deviation of the input signal

(ii) Noise due to measurement errors

Fig.3 Normalised variances of the estimates of the 1st and 2nd order correlation functions, for = 1.1 sec, versus the mean level of the input signal, $x(t) = u(t)+b$; the standard deviation of $x(t)$ is constant (= 2.0)

Fig.4 Normalised variances of the estimates of the 1st and 2nd order correlation functions, for = 1.1 sec, versus the standard deviation of the input signal, $x(t) = u(t)+b$ where $b = 0.0$

Fig.5 Comparison of the estimated and computed cross-correlation functions

———————— estimated

——— - ——— computed, using multi-stage least squares results

—— —— —— computed, using optimised parameters

(i) First order correlation functions

(ii) Second order correlation functions

Fig.6 Nonlinear element characteristics estimated by using the Histogram Approach

PARAMETERS OF $H_1(z^{-1})$				PARAMETERS OF $H_2(z^{-1})$				PARAMETERS OF THE NONLINEARITY						normalised variance of the error = $\frac{var\{z(t)-\hat{z}(t)\}}{var\{z(t)\}}$
$N_{1,1}$	$N_{1,2}$	$d_{1,1}$	$d_{1,2}$	$N_{2,1}$	$N_{2,2}$	$d_{2,1}$	$d_{2,2}$		ξ_1	ξ_2	ξ_3	ξ_4	Bias	
0.5	0.0	-1.5	0.65	0.1	0.0	-1.4	0.55		1.0	2.0	3.0	4.0	-	-
								Least Squares Algorithm	1.488	5.364	3.086	-4.35	-	0.155
0.489	0.0908	-1.254	0.382	0.104	-0.011	-1.453	0.661	Iterative Algorithm	0.638	3.53	9.91	7.41	-	0.604
								Histogram Approach	0.988	2.05	3.99	4.57	0.185	0.178
								Least Squares Algorithm	1.189	1.78	3.89	4.75	-	0.057
0.736	-0.374	-1.569	0.712	0.058	0.031	-1.399	0.513	Iterative Algorithm	1.021	2.10	4.27	2.56	-	0.069
								Histogram Approach	1.062	2.89	4.47	3.98	-0.09	0.054

TABLE 1. SUMMARY OF THE IDENTIFICATION RESULTS

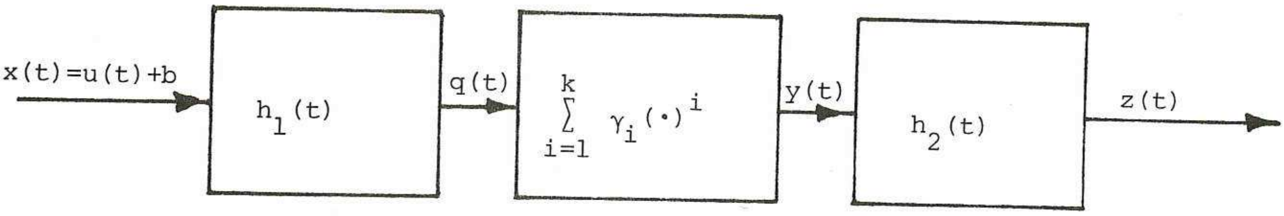
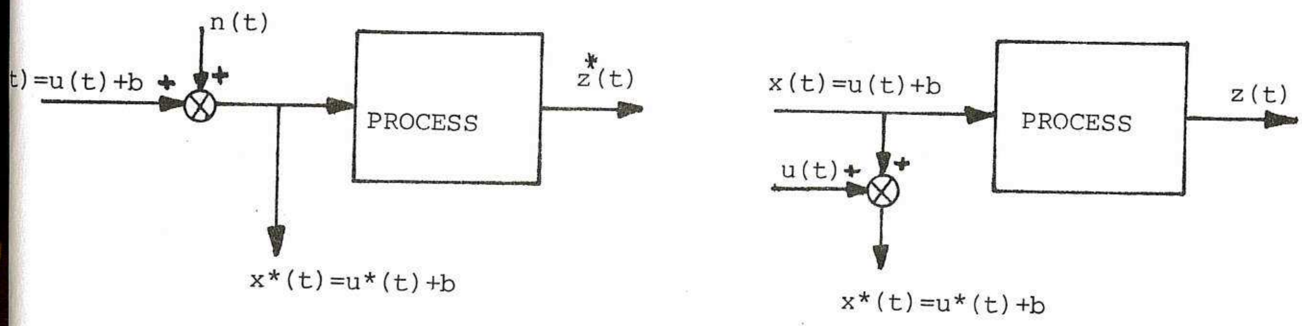


Fig. 1. The general model



Case (i) noise due to the statistical deviations of the input signal

Case (ii) noise due to measurement errors

Fig.2. Input noise sources

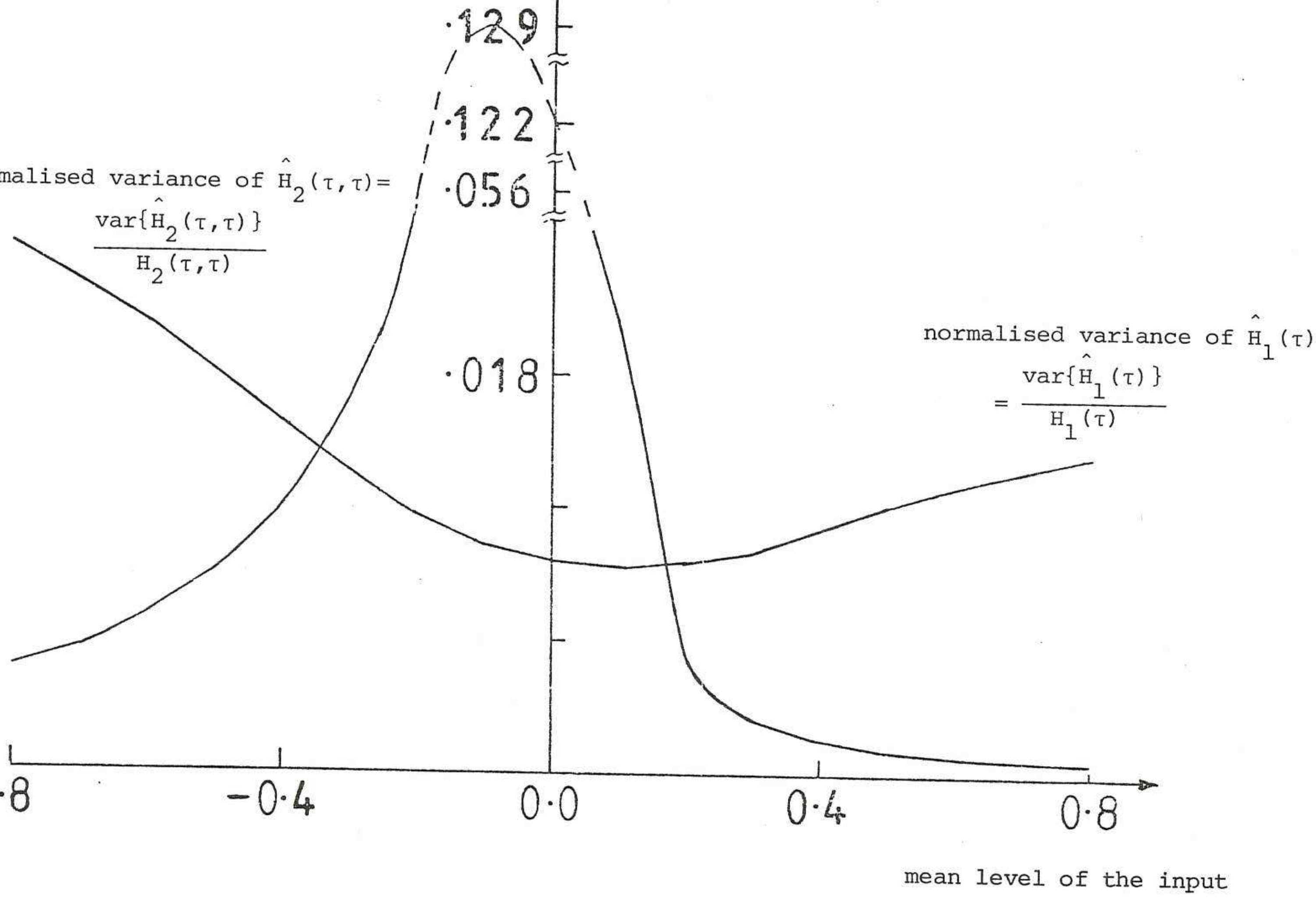


Fig.3. Normalised variances of the estimates of the 1st and 2nd order correlation functions, for τ = 1.1 sec., VS the mean level of the input signal, $x(t) = u(t)+b$; the standard deviation of $x(t)$ is constant (= 2.0)

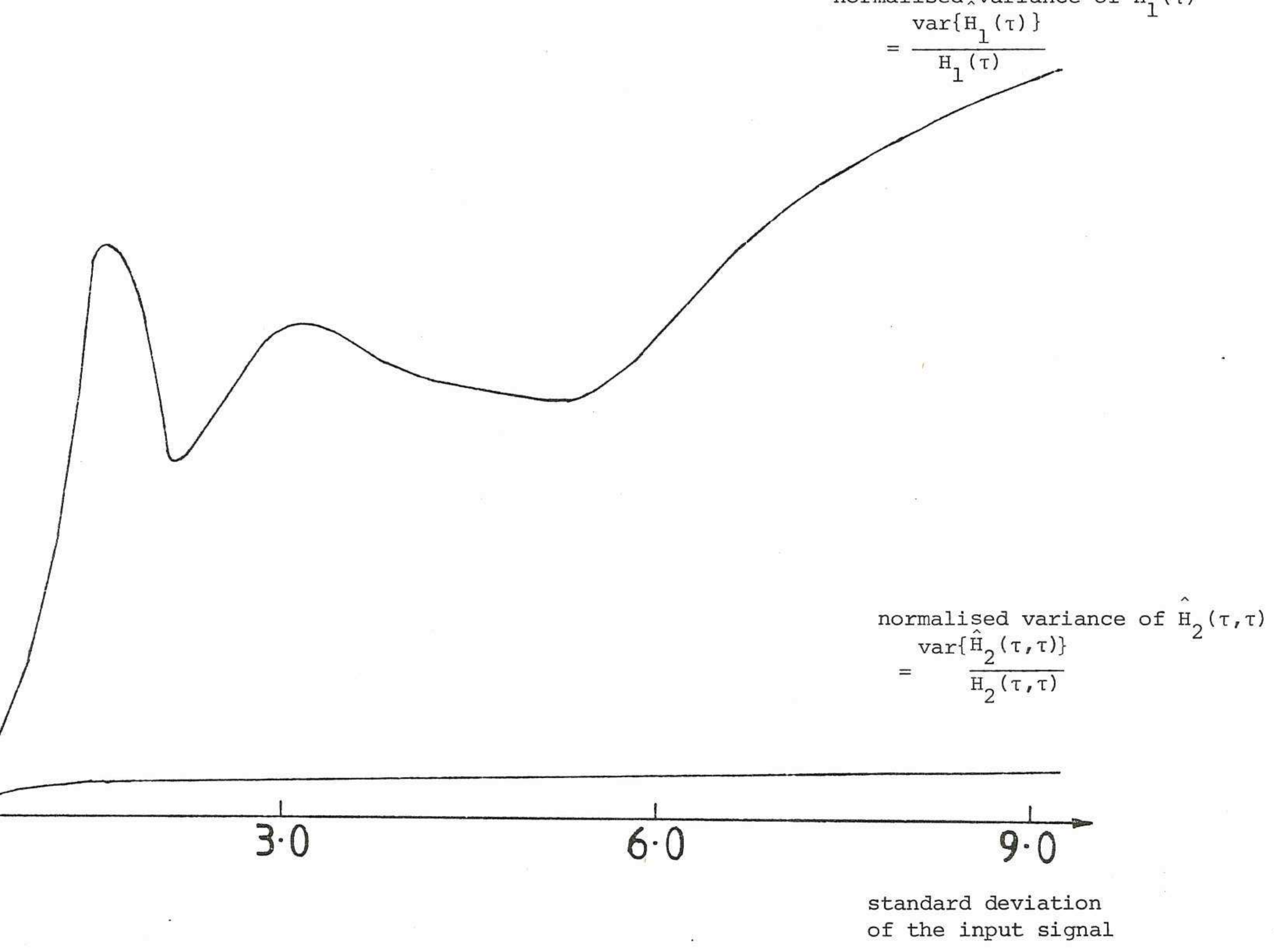


Fig.4. Normalised variances of the estimates of the 1st and 2nd order correlation functions, for τ = 1.1 sec., VS the standard deviation of the input signal, $x(t) = u(t)+b$, where $b = 0.0$.

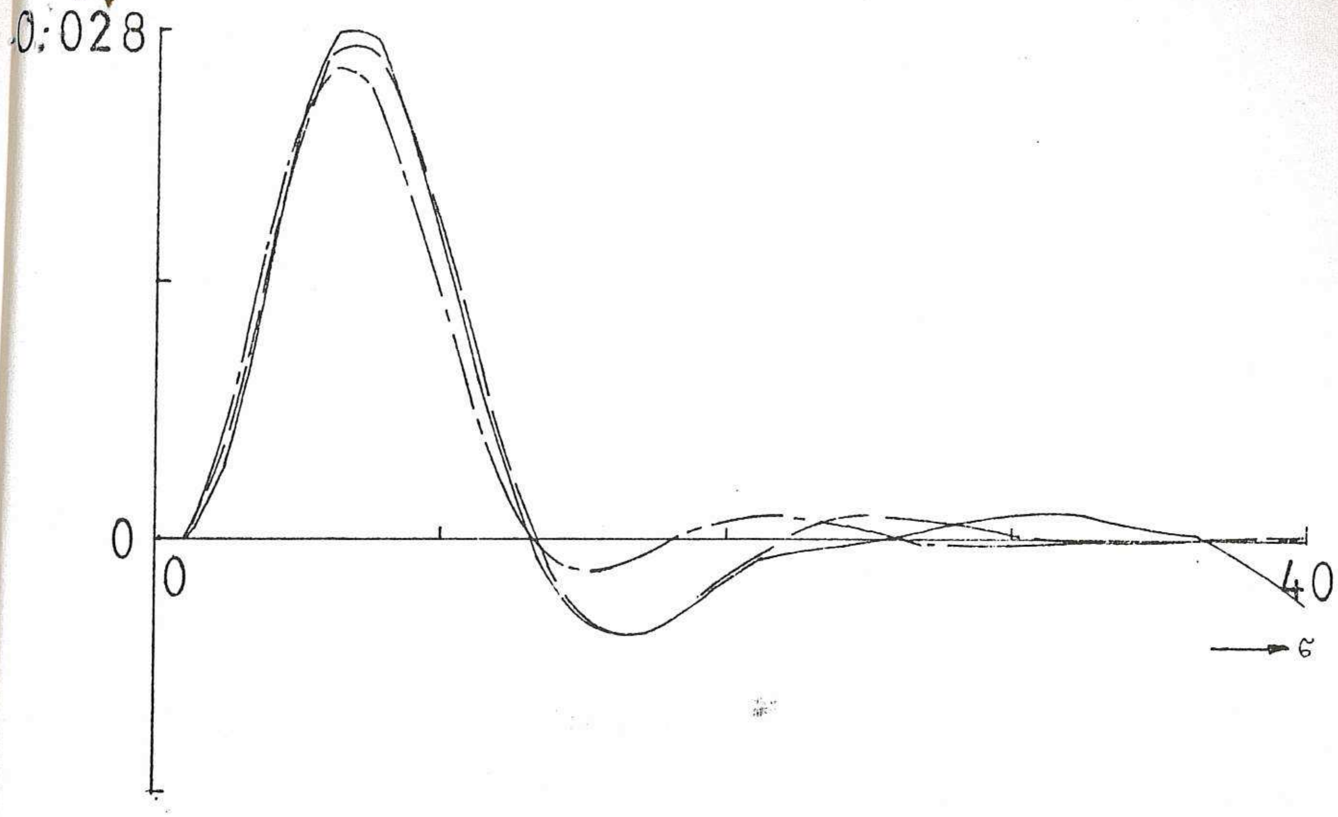


Fig.5(i). First order correlation functions

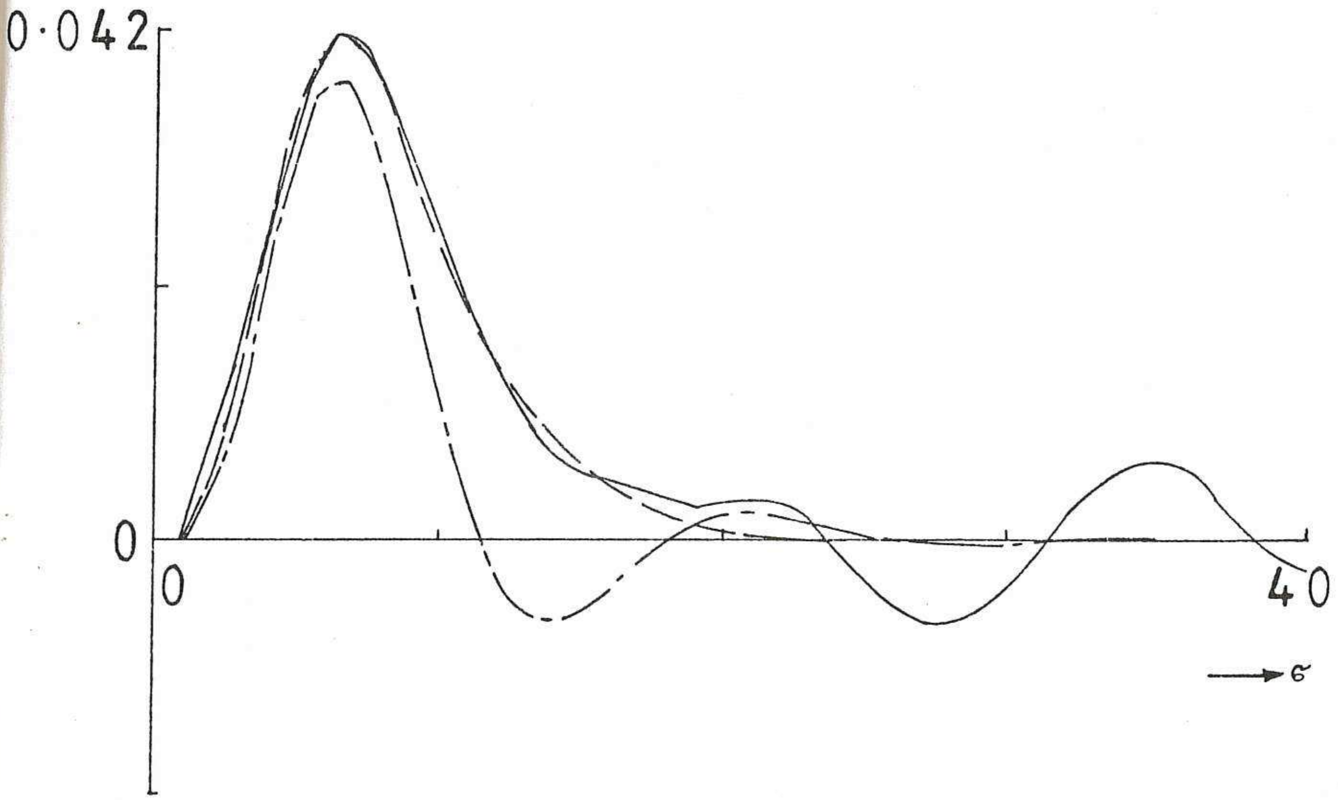


Fig.5(ii). Second order correlation functions

Fig.5. Comparison of the estimated and computed cross-correlation functions

——— estimated

— · — computed using multistage least squares results

— — — computed using optimised parameters

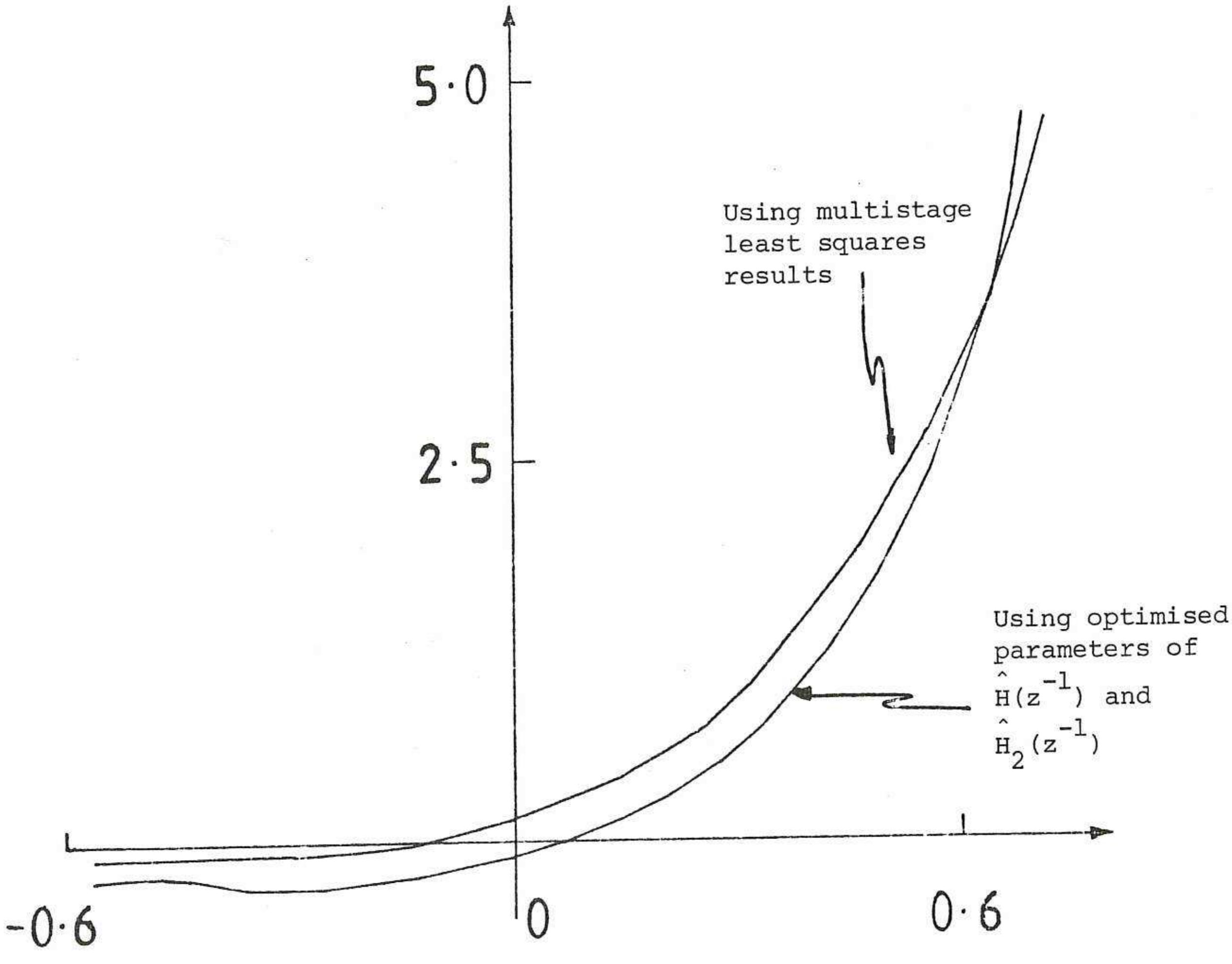


Fig.6. Non-linear element characteristics using the Histogram Approach