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Proof Complexity of Propositional Default Logic*

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Abstract. Default logic is one of the most popular and successful formalisms for non-monotonic reasoning. In 2002, Bonatti and Olivetti introduced several sequent calculi for credulous and skeptical reasoning in propositional default logic. In this paper we examine these calculi from a proof-complexity perspective. In particular, we show that the calculus for credulous reasoning obeys almost the same bounds on the proof size as Gentzen's system LK. Hence proving lower bounds for credulous reasoning will be as hard as proving lower bounds for LK. On the other hand, we show an exponential lower bound to the proof size in Bonatti and Olivetti's enhanced calculus for skeptical default reasoning.

1 Introduction

Trying to understand the nature of human reasoning has been one of the most fascinating adventures since ancient times. It has long been argued that due to its monotonicity, classical logic is not adequate to express the flexibility of commonsense reasoning. To overcome this deficiency, a number of formalisms have been introduced (cf. [20]), of which Reiter's default logic [21] is one of the most popular and widely used systems. Default logic extends the usual logical (first-order or propositional) derivations by patterns for default assumptions. These are of the form "in the absence of contrary information, assume ...". Reiter argued that his logic adequately formalizes human reasoning under the *closed world assumption*. Today default logic is widely used in artificial intelligence and computational logic.

The semantics and the complexity of default logic have been intensively studied during the last decades (cf. [7] for a survey). In particular, Gottlob [13] has identified and studied two reasoning tasks for propositional default logic: the *credulous* and the *skeptical* reasoning problem which can be understood as analogues of the classical problems SAT and TAUT. Due to the stronger expressibility of default logic, however, credulous and skeptical reasoning become harder than their classical counterparts—they are complete for the second level Σ_2^p and Π_2^p of the polynomial hierarchy, respectively [13].

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Less is known about the complexity of proofs in default logic. While there is a rich body of results for propositional proof systems (cf. [17]), proof complexity of non-classical logics has only recently attracted more attention, and a number of exciting results have been obtained for modal and intuitionistic logics [14–16]. Starting with Reiter's work [21], several proof-theoretic methods have been developed for default logic (cf. [1, 11, 18, 19, 22] and [9] for a survey). However, most of these formalisms employ external constraints to model non-monotonic deduction and thus cannot be considered purely axiomatic (cf. [10] for an argument). This was achieved by Bonatti and Olivetti [4] who designed simple and elegant sequent calculi for credulous and skeptical default reasoning. Subsequently, Egly and Tompits [10] extended Bonatti and Olivetti's calculi to first-order default logic and showed a speed-up of these calculi over classical first-order logic, *i.e.*, they construct sequences of first-order formulae which need long classical proofs but have short derivations using default rules.

In the present paper we investigate the original calculi of Bonatti and Olivetti [4] from a proof-complexity perspective. Apart from some preliminary observations in [4], this comprises, to our knowledge, the first comprehensive study of lengths of proofs in propositional default logic. Our results can be summarized as follows. Bonatti and Olivetti's *credulous default calculus* BO_{cred} obeys almost the same bounds to the proof size as Gentzen's propositional sequent calculus LK, *i.e.*, we show that upper bounds to the proof size in both calculi are polynomially related. The same result also holds for the proof length (the number of steps in the system). Thus, proving lower bounds to the size of BO_{cred} will be as hard as proving lower bounds to LK (or, equivalently, to Frege systems), which constitutes a major challenge in propositional proof complexity [5,17]. This result also has implications for automated theorem proving. Namely, we transfer the non-automatizability result of Bonet, Pitassi, and Raz [6] for Frege systems to default logic: BO_{cred}-proofs cannot be efficiently generated, unless factoring integers is possible in polynomial time.

While already BO_{cred} appears to be a strong proof system for credulous default reasoning, admitting very concise proofs, we also exhibit a general method of how to construct a proof system $Cred(P)$ for credulous reasoning from a propositional proof system P. This system $Cred(P)$ bears the same relation to P with respect to proof size as BO_{cred} does to LK. Thus, choosing for example P as extended Frege might lead to stronger proof systems for credulous reasoning.

For *skeptical reasoning*, the situation is different. Bonatti and Olivetti [4] construct two proof systems for this task. While they already show an exponential lower bound for their first skeptical calculus, we obtain also an exponential lower bound to the proof length in their enhanced skeptical calculus. This lower bound also holds if the enhanced calculus is augmented by further rules such as the cut rule.

This paper is organized as follows. In Sect. 2 we start with some background information on proof systems and default logic. The calculi of Bonatti and Olivetti [4] consist of four main ingredients: classical sequents, antisequents to refute non-tautologies, a residual calculus, and default rules. Thus we start our investigation in Sect. 3 by analyzing the preliminary antisequent and residual calculi. Our main results on the proof complexity of credulous and skeptical default reasoning follow in Sects. 4 and 5, respectively. In Sect. 6, we conclude with a discussion and some open questions.

2 Preliminaries

We assume familiarity with propositional logic and basic notions from complexity theory (cf. [17]). By $\mathcal{L}$ we denote the set of all propositional formulae over some fixed standard set of connectives. For $T \subseteq \mathcal{L}$, the set of all logical consequences of T will be denoted by $Th(T)$.

2.1 Proof Systems

Cook and Reckhow [8] defined the notion of a *proof system* for an arbitrary language L as a polynomial-time computable function f with range L. A string w with $f(w) = x$ is called an *f-proof* for $x \in L$. Proof systems for $L = \mathrm{TAUT}$ are called *propositional proof systems*. The sequent calculus LK of Gentzen [12] is one of the most important and best studied propositional proof systems. It is well known that LK and Frege systems mutually p-simulate each other(cf. [17]).

There are two measures which are of primary interest in proof complexity. The first is the minimal *size* of an f-proof for some given element $x \in L$. To make this precise, let $s_f(x) = \min\{|w| \mid f(w) = x\}$ and $s_f(n) = \max\{s_f(x) \mid |x| \leq n\}$. We say that the proof system f is *t-bounded* if $s_f(n) \leq t(n)$ for all $n \in \mathbb{N}$. If t is a polynomial, then f is called *polynomially bounded*. Another interesting parameter of a proof is the *length* defined as the number of proof steps. This measure only makes sense for proof systems where proofs consist of lines containing formulae or sequents. This is the case for LK and most systems studied in this paper. For such a system f, we let $t_f(\varphi) = \min\{k \mid f(\pi) = \varphi \text{ and } \pi \text{ uses } k \text{ steps}\}$ and $t_f(n) = \max\{t_f(\varphi) \mid |\varphi| \leq n\}$. Obviously, it holds that $t_f(n) \leq s_f(n)$, but the two measures are even polynomially related for a number of natural systems as extended Frege (cf. [17]).

For sequent calculi one distinguishes between dag-like and tree-like proofs where in the latter notion each derived sequent can be used at most once as a prerequisite of a rule. While for LK these two measures are equivalent [17], we will concentrate here only on the stronger dag-like model.

2.2 Default Logic

Default logic is an extension of classical logic that has been proposed by Reiter [21]. The logic is non-monotonic in the sense that an increase in information may decrease the number of consequences. A *default theory* $\langle W, D\rangle$ consists of a set W of propositional sentences and a set D of *defaults*. A default (rule) δ is an inference rule of the form $\frac{\alpha : \beta}{\gamma}$, where α and γ are propositional formulae and β is a set of propositional formulae. The *prerequisite* α is also referred to as $p(\delta)$, the formulae in β are called *justifications* (referred to as $j(\delta)$), and γ is the *conclusion* that is referred to as $c(\delta)$. *Stable extensions* are originally defined in terms of a fixed-point equation [21], but we use the following characterization as a starting definition:

Theorem 1 (Reiter [21]). *Let $E \subseteq \mathcal{L}$ be a set of formulae and $\langle W, D\rangle$ be a default theory. Furthermore let $E_0 = W$, and*

$$E_{i+1} = Th(E_i) \cup \{c(\delta) \mid \delta \in D, E_i \vdash p(\delta), \neg j(\delta) \cap E = \emptyset\},$$

where $\neg j(\delta)$ denotes the set of all negated sentences contained in $j(\delta)$. Then E is a (stable) extension *of $\langle W, D\rangle$ if and only if $E = \bigcup_{i\in\mathbb{N}} E_i$.*

A default theory $\langle W, D\rangle$ can have none or several stable extensions (cf. [2,13] for examples). A sentence $\psi \in \mathcal{L}$ is *credulously* entailed by $\langle W, D\rangle$ if ψ holds in *some* stable extension of $\langle W, D\rangle$. If ψ holds in *every* extension of $\langle W, D\rangle$, then ψ is *skeptically* entailed by $\langle W, D\rangle$.

Default rules with empty justification are called *residues*. Let $\mathcal{L}^{res} = \mathcal{L} \cup \left\{\frac{\alpha}{\gamma} \mid \alpha, \gamma \in \mathcal{L}\right\}$ be the set of all formulae and residues. Residues can be used to alternatively characterize stable extensions. For a set D of defaults and $E \subseteq \mathcal{L}$ let

$$RES(D, E) = \left\{ \frac{p(\delta)}{c(\delta)} \;\middle|\; \delta \in D,\ E \cap \neg j(\delta) = \emptyset \right\}.$$

Apparently, $RES(D, E)$ is a set of residues. We can then build stable extensions via the following closure operator. For a set R of residues we define $Cl_0(W, R) = W$ and

$$Cl_{i+1}(W, R) = Th(Cl_i(W, R)) \cup \left\{ \gamma \;\middle|\; \frac{\alpha}{\gamma} \in R, \alpha \in Th(Cl_i(W, R)) \right\}.$$

Let $Cl(W, R) = \bigcup_{i=0}^{\infty} Cl_i(W, R)$. Then for the sets E_i from Theorem 1 the following holds:

Proposition 2 (Bonatti, Olivetti [4]). *Let $\langle W, D\rangle$ be a default theory and let $E \subseteq \mathcal{L}$. Then $E_i = Cl_i(W, RES(D, E))$ for all $i \in \mathbb{N}$. In particular, E is a stable extension of $\langle W, D\rangle$ if and only if $E = Cl(W, RES(D, E))$.*

If D only contains residues, then there is an easier way of characterizing Cl:

Lemma 3 (Bonatti, Olivetti [4]). *For $D \subseteq \mathcal{L}^{res} \setminus \mathcal{L}$, $W \subseteq \mathcal{L}$, and for $i \in \mathbb{N}$ let*

$$C_0 = W \quad \text{and} \quad C_{i+1} = C_i \cup \left\{ \gamma \;\middle|\; \frac{\alpha}{\gamma} \in D, \alpha \in Th(C_i) \right\}.$$

Then $\gamma \in Cl(W, D)$ if and only if there exists $k \in \mathbb{N}$ with $\gamma \in Th(C_k)$.

3 Proof Complexity of the Antisequent and Residual Calculi

Bonatti and Olivetti's calculi for default logic use four main ingredients: usual propositional sequents and rules of LK, antisequents to refute formulae, residual rules, and default rules. In this section we will investigate the complexity of the antisequent calculus AC and the residual calculus RC.

We start with the definition of Bonatti's *antisequent calculus* AC from [3]. A related refutation calculus for first-order logic was previously developed by Tiomkin [23]. In AC we use *antisequents* $\Gamma \nvdash \Delta$, where $\Gamma, \Delta \subseteq \mathcal{L}$. Intuitively, $\Gamma \nvdash \Delta$ means that $\bigvee \Delta$ does not follow from $\bigwedge \Gamma$. Axioms of AC are all sequents $\Gamma \nvdash \Delta$, where Γ and Δ are disjoint sets of propositional variables. The inference rules of AC are shown in Fig. 1. For this calculus, Bonatti [3] shows:

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \neg\alpha \nvdash \Sigma}\ (\neg \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma \nvdash \Sigma, \neg\alpha}\ (\nvdash \neg)$$

$$\frac{\Gamma, \alpha, \beta \nvdash \Sigma}{\Gamma, \alpha \wedge \beta \nvdash \Sigma}\ (\wedge \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\ (\nvdash \bullet\wedge) \qquad \frac{\Gamma \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\ (\nvdash \wedge\bullet)$$

$$\frac{\Gamma \nvdash \Sigma, \alpha, \beta}{\Gamma \nvdash \Sigma, \alpha \vee \beta}\ (\nvdash \vee) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\ (\bullet\vee \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\ (\vee\bullet \nvdash)$$

$$\frac{\Gamma, \alpha \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \to \beta}\ (\nvdash\to) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \alpha \to \beta \nvdash \Sigma}\ (\bullet \to\nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \to \beta \nvdash \Sigma}\ (\to \bullet \nvdash)$$

Fig. 1. Inference rules of the antisequent calculus AC.

Theorem 4 (Bonatti [3]). *The antisequent calculus AC is sound and complete.*

Concerning the size of proofs in the antisequent calculus we observe:

Proposition 5. *The antisequent calculus AC is polynomially bounded.*

Proof. Observe that the calculus contains only unary inference rules, each of which reduces the logical complexity of one of the contained formulae (if perceived bottom-up). Thus each use of an inference rule decrements the size of the formulae by at least one. After a linear number of steps we end up with only propositional variables which we cannot reduce any further. Each antisequent is of linear size, hence the complete derivation has quadratic size. $\square$

The above observation is not very astounding, since, to verify $\Gamma \nvdash \Delta$ we could alternatively guess assignments to the propositional variables in Γ and Δ and thereby verify antisequents in NP.

We now turn to the *residual calculus* RC of Bonatti and Olivetti [4]. Its objects are *residual sequents* $\langle W, R\rangle \vdash \Delta$ and *residual antisequents* $\langle W, R\rangle \nvdash \Delta$ where $W, \Delta \subseteq \mathcal{L}$ and $R \subseteq \mathcal{L}^{res}$. The intuitive meaning is that Δ does (respectively does not) follow from W using the residues R. The rules of RC comprise of the inference rules from Fig. 2 together with the rules of LK and AC. However, the use of rules from LK and AC is restricted to purely propositional (anti)sequents. For this calculus, Bonatti and Olivetti [4] showed:

Theorem 6 (Bonatti, Olivetti [4]). *The residual calculus RC is sound and complete,* i.e., *for all default theories $\langle W, R\rangle$ with $R \subseteq \mathcal{L}^{res}$ and all $\Delta \subseteq \mathcal{L}$,*

$$(\mathbf{Re1})\ \frac{\Gamma \vdash \Delta}{\Gamma, \frac{\alpha}{\gamma} \vdash \Delta} \qquad (\mathbf{Re2})\ \frac{\Gamma \vdash \alpha \qquad \Gamma, \gamma \vdash \Delta}{\Gamma, \frac{\alpha}{\gamma} \vdash \Delta}$$

$$(\mathbf{Re3})\ \frac{\Gamma \nvdash \Delta \qquad \Gamma \nvdash \alpha}{\Gamma, \frac{\alpha}{\gamma} \nvdash \Delta} \qquad (\mathbf{Re4})\ \frac{\Gamma, \gamma \nvdash \Delta}{\Gamma, \frac{\alpha}{\gamma} \nvdash \Delta}$$

Fig. 2. Inference rules of the residual calculus RC.

1. *$\langle W, R\rangle \vdash \Delta$ is derivable in RC if and only if $\bigvee \Delta \in Cl(W, R)$;*
2. *$\langle W, R\rangle \nvdash \Delta$ is derivable in RC if and only if $\bigvee \Delta \notin Cl(W, R)$.*

To bound the lengths of proofs in this calculus we exploit the property that residues only have to be used at a certain level and are not used to deduce any formulae afterwards (cf. Lemma 3). Using this we prove that the complexity of RC is tightly linked to that of LK.

Lemma 7. *There exist a polynomial p and a constant c such that $s_{RC}(n) \leq p(n) \cdot s_{LK}(cn)$ and $t_{RC}(n) \leq p(n) \cdot t_{LK}(cn)$.*

Proof. The proof consists of two parts. First we will show the bounds stated above for sequents. In the second part we will then show that antisequents even admit polynomial-size proofs in RC.

Assume first that we want to derive the sequent $\langle W, R\rangle \vdash \Delta$, where $W, \Delta \subseteq \mathcal{L}$ and $R = \{r_1, \ldots, r_k\}$ is a set of residues with $r_i = \frac{\alpha_i}{\gamma_i}$. Let $R' \subseteq R$ be minimal with respect to the size $|R'|$ such that $\langle W, R'\rangle \vdash \Delta$. We may w.l.o.g. assume that $R' = \{r_1, \ldots, r_{k'}\}$ and $k' \leq k$. Furthermore, by Lemma 3, we may assume that the rules r_i are ordered in the way they are applied when computing the sets C_i. In particular, this means that for each $i = 1, \ldots, k'$,

$$W \cup \{\gamma_1, \ldots, \gamma_{i-1}\} \vdash \alpha_i$$

is a true propositional sequent for which we fix an LK-proof Π_i. We augment Π_i by $k' - i$ applications of rule ($\mathbf{Re1}$) to obtain

$$\langle W \cup \{\gamma_1, \ldots, \gamma_{i-1}\}, \{r_{i+1}, \ldots, r_{k'}\}\rangle \vdash \alpha_i \ .$$

Let us call the proof of this sequent Π'_i.

The proof tree depicted in Fig. 3 for deriving $\langle W, R\rangle \vdash \Delta$ unfurls as follows. We start with an LK-proof for the sequent $W \cup \{\gamma_1, \ldots, \gamma_{k'}\} \vdash \Delta$ and then apply k'-times the rule ($\mathbf{Re2}$) in the step

$$\frac{\langle W \cup \{\gamma_1, \ldots, \gamma_{i-1}\}, \{r_{i+1}, \ldots, r_{k'}\}\rangle \vdash \alpha_i \qquad \langle W \cup \{\gamma_1, \ldots, \gamma_i\}, \{r_{i+1}, \ldots, r_{k'}\}\rangle \vdash \Delta}{\langle W \cup \{\gamma_1, \ldots, \gamma_{i-1}\}, \{r_i, \ldots, r_{k'}\}\rangle \vdash \Delta}$$

to reach $\langle W, R'\rangle \vdash \Delta$. To derive the left prerequisite we use the proof Π'_i. Finally we use $k - k'$ applications of the rule ($\mathbf{Re1}$) to get $\langle W, R\rangle \vdash \Delta$.

Our proof for $\langle W, R\rangle \vdash \Delta$ uses at most $(k' + 1) \cdot t_{LK}(n) + \frac{k'(k'+1)}{2} + k$ steps, *i.e.*, $t_{RC}(n) \leq \mathcal{O}(n \cdot t_{LK}(n) + n^2)$. Each sequent is of linear size. Hence, $s_{RC}(n) \leq p(n) \cdot s_{LK}(n)$ for some polynomial p.

$$
\dfrac{\dfrac{\Pi'_1 \qquad \dfrac{\Pi'_2 \qquad \begin{array}{c} \dfrac{\Pi'_{k'} \qquad \langle W \cup \{\gamma_1, \ldots, \gamma_{k'}\}, \emptyset\rangle \vdash \Delta}{\vdots}\;(\mathbf{Re2}) \\ \langle W \cup \{\gamma_1, \gamma_2\}, \{r_3, \ldots, r_{k'}\}\rangle \vdash \Delta \end{array}}{\langle W \cup \{\gamma_1\}, \{r_2, \ldots, r_{k'}\}\rangle \vdash \Delta}\;(\mathbf{Re2})}{\langle W, R'\rangle \vdash \Delta}\;(\mathbf{Re2})}{\begin{array}{c} \vdots \\ \langle W, R\rangle \vdash \Delta \end{array}}\;(\mathbf{Re1})
$$

Fig. 3. Proof tree for the sequent $\langle W, R\rangle \vdash \Delta$ in the residual calculus.

In the second part of the proof we will now show that any true antisequent has an RC-proof of polynomial size, thus concluding the proof. Let $\langle W, R\rangle \nvdash \Delta$ be the antisequent we wish to prove. Again, let $R = \{r_1, \ldots, r_k\}$ with $r_i = \frac{\alpha_i}{\gamma_i}$, and let $\{i_1, \ldots, i_\ell\} = I \subseteq \{1, \ldots, k\}$ be a set of maximal cardinality such that $\left\langle W \cup \bigcup_{i \in I}\{\gamma_i\}\right\rangle \nvdash \Delta$ and let $I' = \{i_{\ell+1}, \ldots, i_k\} = \{1, \ldots, k\} \setminus I$.

Because of $\langle W, R\rangle \nvdash \Delta$, the set I contains all indices i with $\alpha_i \in Cl(W)$. Therefore, for each $j \in I'$ we have $W \cup \bigcup_{i \in I}\{\gamma_i\} \nvdash \alpha_j$. We fix a polynomial-size AC-proof Π_j of this antisequent. Augmenting these proofs with ℓ applications of (**Re4**) we obtain a proof Π'_j of $\left\langle W, \bigcup_{i \in I}\{r_i\}\right\rangle \nvdash \alpha_j$. Similarly, as $\left\langle W \cup \bigcup_{i \in I}\{\gamma_i\}\right\rangle \nvdash \Delta$ we get a polynomial-size proof Π'_{k+1} of $\left\langle W, \bigcup_{i \in I}\{r_i\}\right\rangle \nvdash \Delta$.

Now, the proof for $\langle W, R\rangle \nvdash \Delta$ ends with the following application of (**Re3**)

$$
\frac{\left\langle W, \{r_{i_1}, \ldots, r_{i_{k-1}}\}\right\rangle \nvdash \Delta \qquad \left\langle W, \{r_{i_1}, \ldots, r_{i_{k-1}}\}\right\rangle \nvdash \alpha_{i_k}}{\left\langle W, \{r_{i_1}, \ldots, r_{i_k}\}\right\rangle \nvdash \Delta}
$$

More generally, for all choices of s, t with $\ell < s < t \leq k+1$ we use the (**Re3**)-step

$$
\frac{\left\langle W, \{r_{i_1}, \ldots, r_{i_{s-1}}\}\right\rangle \nvdash \alpha_{i_t} \qquad \left\langle W, \{r_{i_1}, \ldots, r_{i_{s-1}}\}\right\rangle \nvdash \alpha_{i_s}}{\left\langle W, \{r_{i_1}, \ldots, r_{i_s}\}\right\rangle \nvdash \alpha_{i_t}}
$$

where we set $\alpha_{k+1} = \bigvee \Delta$. After all these steps, it remains to derive the antisequents $\langle W, \{r_{i_1}, \ldots, r_{i_\ell}\}\rangle \nvdash \alpha_{i_t}$ for $\ell < t \leq k+1$. But for these we have already built the proofs Π'_t. Therefore, we have constructed an RC-proof of $\langle W, R\rangle \nvdash \Delta$ which apart from the AC-proofs Π'_t uses only $\mathcal{O}(k^2)$ applications of (**Re3**) and (**Re4**). As each antisequent in the proof is of linear size, we obtain a polynomial-size RC-proof of $\langle W, R\rangle \nvdash \Delta$. □

Let us remark that while the RC-proof of $\langle W, R\rangle \vdash \Delta$ in Fig. 3 is tree-like, this is not true for our dag-like RC-proof of $\langle W, R\rangle \nvdash \Delta$ constructed in the second part of the proof of Lemma 7.

4 Proof Complexity of Credulous Default Reasoning

Now we turn to the analysis of Bonatti and Olivetti's calculus for credulous default reasoning. An essential ingredient of the calculus are *provability constraints* which resemble a necessity modality. Provability constraints are of the form $\mathbf{L}\alpha$ or $\neg\mathbf{L}\alpha$ with $\alpha \in \mathcal{L}$. A set $E \subseteq \mathcal{L}$ satisfies a constraint $\mathbf{L}\alpha$ if $\alpha \in Th(E)$. Similarly, E satisfies $\neg\mathbf{L}\alpha$ if $\alpha \notin Th(E)$.

We can now describe the calculus BO_{cred} of Bonatti and Olivetti [4] for credulous default reasoning. A *credulous default sequent* is a 3-tuple $\langle \Sigma, \Gamma, \Delta\rangle$, denoted by $\Sigma;\Gamma{\mid\!\sim}\Delta$, where $\Gamma = \langle W, D\rangle$ is a default theory, Σ is a set of provability constraints and Δ is a set of propositional sentences. Semantically, the sequent $\Sigma;\Gamma{\mid\!\sim}\Delta$ is true, if there exists a stable extension E of Γ which satisfies all of the constraints in Σ and $\bigvee \Delta \in E$. The calculus BO_{cred} uses such sequents and extends LK, AC, and RC by the inference rules in Fig. 4.

$$(\mathbf{cD1})\ \frac{\Gamma \vdash \Delta}{;\ \Gamma{\mid\!\sim}\Delta}\ (\Gamma \subseteq \mathcal{L}^{res})$$

$$(\mathbf{cD2})\ \frac{\Gamma \vdash \alpha \qquad \Sigma;\ \Gamma{\mid\!\sim}\Delta}{\mathbf{L}\alpha,\ \Sigma;\ \Gamma{\mid\!\sim}\Delta}\ (\Gamma \subseteq \mathcal{L}^{res}) \qquad (\mathbf{cD3})\ \frac{\Gamma \nvdash \alpha \qquad \Sigma;\ \Gamma{\mid\!\sim}\Delta}{\neg\mathbf{L}\alpha,\ \Sigma;\ \Gamma{\mid\!\sim}\Delta}\ (\Gamma \subseteq \mathcal{L}^{res})$$

$$(\mathbf{cD4})\ \frac{\mathbf{L}\neg\beta_i,\ \Sigma;\ \Gamma{\mid\!\sim}\Delta}{\Sigma;\ \Gamma, \frac{\alpha:\ \beta_1\ldots\beta_n}{\gamma}{\mid\!\sim}\Delta} \qquad (\mathbf{cD5})\ \frac{\neg\mathbf{L}\neg\beta_1\ldots\neg\mathbf{L}\neg\beta_n,\ \Sigma;\ \Gamma, \frac{\alpha}{\gamma}{\mid\!\sim}\Delta}{\Sigma;\ \Gamma, \frac{\alpha:\ \beta_1\ldots\beta_n}{\gamma}{\mid\!\sim}\Delta}$$

Fig. 4. Inference rules for the credulous default calculus BO_{cred}.

For this calculus Bonatti and Olivetti [4] show the following:

Theorem 8 (Bonatti, Olivetti [4]). *BO_{cred} is sound and complete,* i.e., *a credulous default sequent is true if and only if it is derivable in BO_{cred}.*

We now investigate lengths of proofs in BO_{cred}. Our next lemma shows that upper bounds on the proof size of RC can be transferred to BO_{cred}.

Lemma 9. *For any function $t(n)$, if RC is $t(n)$-bounded, then BO_{cred} is $p(n) \cdot t(n)$-bounded for some polynomial p. The same relation holds for the number of steps in RC and BO_{cred}.*

Proof. Let $\Sigma;\Gamma{\mid\!\sim}\Delta$ be a true credulous default sequent. We will construct a BO_{cred}-derivation of $\Sigma;\Gamma{\mid\!\sim}\Delta$ starting from the bottom with the given sequent. Observe that we cannot use any of the rules (**cD1**) through (**cD3**) as long as Γ contains proper defaults with nonempty justification. Thus we first have to reduce all defaults to residues plus some set of constraints using (**cD4**) or (**cD5**). As one of these rules has to be applied exactly once for each appearance of some default in Γ we end up with $\Sigma';\Gamma'{\mid\!\sim}\Delta$, where $|\Sigma'|$ is polynomial in $|\Gamma \cup \Sigma|$ and Γ' is equal to Γ on its propositional part and contains some of the corresponding residues instead of the defaults from Γ. From this point on we can only use rules (**cD2**) and (**cD3**) until we have eliminated all constraints and then finally apply rule (**cD1**) once. Thus, BO_{cred}-proofs look as shown in Fig. 5 where RC indicates a derivation in the residual calculus and σ is the remaining constraint from Σ after applications of (**cD2**) or (**cD3**). Hence we obtain the bounds on $s_{BO_{cred}}$ and $t_{BO_{cred}}$. $\square$

$$\dfrac{\dfrac{RC \qquad \dfrac{RC}{\Gamma' \mathrel{|\!\sim} \Delta}\,(\mathbf{cD1})}{\sigma; \Gamma' \mathrel{|\!\sim} \Delta}\,(\mathbf{cD2})\text{ or }(\mathbf{cD3})}{\vdots}\,(\mathbf{cD2})\text{ or }(\mathbf{cD3})$$

$$\dfrac{\dfrac{RC \qquad \Sigma''; \Gamma' \mathrel{|\!\sim} \Delta}{\Sigma'; \Gamma' \mathrel{|\!\sim} \Delta}\,(\mathbf{cD2})\text{ or }(\mathbf{cD3})}{\vdots}\,(\mathbf{cD4})\text{ or }(\mathbf{cD5})$$

$$\Sigma; \Gamma \mathrel{|\!\sim} \Delta$$

Fig. 5. The structure of the BO_{cred}-proof in Lemma 9

Combining Lemmas 7 and 9 we obtain our main result in this section stating a tight connection between the proof complexity of LK and BO_{cred}.

Theorem 10. *There exist a polynomial p and a constant c such that $s_{LK}(n) \leq s_{BO_{cred}}(n) \leq p(n) \cdot s_{LK}(cn)$ and $t_{LK}(n) \leq t_{BO_{cred}}(n) \leq p(n) \cdot t_{LK}(cn)$.*

In the light of this result, proving either non-trivial lower or upper bounds to the proof size of BO_{cred} seems very difficult—as such a result would mean a major breakthrough in propositional proof complexity (cf. [3, 17]).

4.1 On the Automatizability of BO_{cred}

Practitioners are not only interested in the size of a proof, but face the more complicated problem to actually construct a proof for a given instance. Of course, in the presence of super-polynomial lower bounds to the proof size this cannot be done in polynomial time. Thus, in proof search the best one can hope for is the following notion of automatizability:

Definition 11 (Bonet, Pitassi, Raz [6]). *A proof system P for a language L is* automatizable *if there exists a deterministic procedure that takes as input a string x and outputs a P-proof of x in time polynomial in the size of the shortest P-proof of x if $x \in L$. If $x \notin L$, then the behaviour of the algorithm is unspecified.*

For practical purposes automatizable systems would be very desirable. Searching for a proof we may not find the shortest one, but we are guaranteed to find one that is only polynomially longer. Unfortunately, for BO_{cred} there are strong limitations towards this goal as our next result shows:

Theorem 12. *BO_{cred} is not automatizable unless factoring integers is possible in polynomial time.*

Proof. First we observe that automatizability of BO_{cred} implies automatizability of Frege systems. For this let φ be a propositional tautology. By assumption, we can construct a BO_{cred}-proof of $\emptyset \mathrel{|\!\sim} \varphi$. This BO_{cred}-proof contains an LK-proof of $\emptyset \vdash \varphi$ by rule (**cD1**). As LK is polynomially equivalent to Frege systems [17], we can construct from this LK-proof a Frege proof of φ in polynomial time. By a result of Bonet, Pitassi, and Raz [6], Frege systems are not automatizable unless Blum integers can be factored in polynomial time (a Blum integer is the product of two primes which are both congruent 3 modulo 4). $\square$

4.2 A General Construction of Proof Systems for Credulous Default Reasoning

In this section we will explain a general method how to construct proof systems for credulous default reasoning. These proof systems arise from the canonical Σ_2^{p} algorithm for credulous default reasoning (Algorithm 1). Algorithm 1 first guesses a generating set G_{ext} for a potential stable extension and then verifies by the stage construction from Theorem 1 that G_{ext} indeed generates a stable extension which moreover contains the formula φ. Algorithm 1 is a Σ_2^{p} procedure, *i.e.*, it can be executed by a nondeterministic polynomial-time Turing machine M with access to a coNP-oracle. The nondeterminism solely lies in line 1 and the oracle queries are made in lines 6 and 11 to the coNP-complete problem of propositional implication $\text{IMP} = \{\langle \Psi, \varphi\rangle \mid \Psi \subseteq \mathcal{L},\ \varphi \in \mathcal{L},\ \text{and } \Psi \models \varphi\}$.

Algorithm 1 A Σ_2^{p} procedure for credulous default reasoning

Require: $\langle W, D\rangle$, φ
1: guess $D_0 \subseteq D$ and let $G_{\text{ext}} \leftarrow W \cup \left\{\gamma \mid \frac{\alpha:\beta}{\gamma} \in D_0\right\}$
2: $G_{\text{new}} \leftarrow W$
3: **repeat**
4: $\quad G_{\text{old}} \leftarrow G_{\text{new}}$
5: $\quad$ **for all** $\frac{\alpha:\beta}{\gamma} \in D$ **do**
6: $\quad\quad$ **if** $G_{\text{old}} \models \alpha$ and $G_{\text{ext}} \not\models \neg\beta$ **then**
7: $\quad\quad\quad G_{\text{new}} \leftarrow G_{\text{new}} \cup \{\gamma\}$
8: $\quad\quad$ **end if**
9: $\quad$ **end for**
10: **until** $G_{\text{new}} = G_{\text{old}}$
11: **if** $G_{\text{new}} = G_{\text{ext}}$ and $G_{\text{ext}} \models \varphi$ **then**
12: $\quad$ **return true**
13: **else**
14: $\quad$ **return false**
15: **end if**

Algorithm 1 can be converted into a proof system for credulous default reasoning as follows. We fix a propositional proof system P and define a proof system $\mathit{Cred}(P)$ for credulous default reasoning where proofs are of the form

$$\langle W, D, \varphi, \mathit{comp}, q_1, \dots, q_k, a_1, \dots, a_k\rangle \quad .$$

Here *comp* is a computation of M on input $\langle W, D, \varphi\rangle$ and $q_1, \dots, q_k$ are the queries to IMP during this computation. If the IMP-query $q_i = \langle \Psi_i, \varphi_i\rangle$ is answered positively, then a_i is a P-proof of $\left(\bigwedge_{\psi\in\Psi_i} \psi\right) \rightarrow \varphi_i$, otherwise a_i is an assignment falsifying this formula. For this proof system we obtain the following bounds:

Theorem 13. *Let P be a propositional proof system. Then $\mathit{Cred}(P)$ is a proof system for credulous default reasoning with $s_P(n) \leq s_{\mathit{Cred}(P)}(n) \leq \mathcal{O}(n^2 s_P(n))$.*

Proof. The first inequality holds because we can use $Cred(P)$ to prove propositional tautologies φ by choosing $W = D = \emptyset$.

For the second inequality, we observe that Algorithm 1 has quadratic running time. In particular, a computation of Algorithm 1 contains at most a quadratic number of queries to IMP. Each of these queries is of linear size because it only consists of formulae from the input. If the query is answered positively, then we have to supply a P-proof and there exists such a P-proof of size $\leq s_P(n)$. For a negative answer we just include an assignment of linear size. This yields $s_{Cred(P)}(n) \leq \mathcal{O}(n^2 s_P(n))$. □

Theorem 13 tells us that proving lower bounds for proof systems for credulous default reasoning is more or less the same as proving lower bounds to propositional proof systems. In particular, we get:

Corollary 14. *There exists a polynomially bounded proof system for credulous default reasoning if and only if there exists a polynomially bounded propositional proof system.*

5 Lower Bounds for Skeptical Default Reasoning

Bonatti and Olivetti [4] introduce two calculi for skeptical default reasoning. As before, objects are sequents of the form $\Sigma; \Gamma \mid\!\sim \Delta$, where Σ is a set of constraints, Γ is a propositional default theory, and Δ is a set of propositional formulae. But now, the sequent $\Sigma; \Gamma \mid\!\sim \Delta$ is true, if $\bigvee \Delta$ holds in *all* extensions of Γ satisfying the constraints in Σ.

The first calculus BO_{skep} consists of the defining axioms of LK and AC, the inference rules of LK, AC, RC, and the rules from Fig. 6. Bonatti and

$$\textbf{(sD1)}\ \frac{\Gamma \vdash \Delta}{\Sigma; \Gamma \mid\!\sim \Delta}\ (\Gamma \subseteq \mathcal{L}^{res})$$

$$\textbf{(sD2)}\ \frac{\Gamma \vdash \alpha}{\neg\mathbf{L}\alpha, \Sigma; \Gamma \mid\!\sim \Delta}\ (\Gamma \subseteq \mathcal{L}^{res}) \qquad \textbf{(sD3)}\ \frac{\Gamma \nvdash \alpha}{\mathbf{L}\alpha, \Sigma; \Gamma \mid\!\sim \Delta}\ (\Gamma \subseteq \mathcal{L}^{res})$$

$$\textbf{(sD4)}\ \frac{\neg\mathbf{L}\neg\beta_1, \ldots, \neg\mathbf{L}\neg\beta_n, \Sigma; \Gamma, \frac{\alpha}{\gamma} \mid\!\sim \Delta \qquad \mathbf{L}\neg\beta_1, \Sigma; \Gamma \mid\!\sim \Delta \quad \ldots \quad \mathbf{L}\neg\beta_n, \Sigma; \Gamma \mid\!\sim \Delta}{\Sigma; \Gamma, \frac{\alpha:\beta_1\ldots\beta_n}{\gamma} \mid\!\sim \Delta}$$

Fig. 6. Inference rules for the skeptical default calculus BO_{skep}.

Olivetti show that each true sequent is derivable in BO_{skep}, *i.e.*, the calculus is sound and complete. However, they already remark that proofs in BO_{skep} are of exponential size in the number of default rules in the sequent. This is due to the residual rules for they cannot be applied unless all defaults with nonempty justifications have been eliminated using rule (**sD4**).

To get more concise proofs, Bonatti and Olivetti [4] suggest an enhanced calculus BO'_{skep} where the rules (**sD1**) to (**sD3**) are replaced by rules (**sD1**$'$) to

($\mathbf{sD3'}$) and rule ($\mathbf{sD4}$) is kept (see Fig. 7). Bonatti and Olivetti prove soundness and completeness for BO'_{skep}. Moreover, they show that BO'_{skep} is exponentially separated from BO_{skep}, *i.e.*, there exist sequents $(S_n)_{n\geq 1}$ which require exponential-size proofs in BO_{skep} but have linear-size derivations in BO'_{skep}. In

$$(\mathbf{sD1'})\ \frac{\Sigma', \Gamma' \vdash \Delta}{\Sigma; \Gamma \mid\!\sim \Delta} \qquad (\mathbf{sD2'})\ \frac{\Sigma; \Gamma \mid\!\sim \alpha}{\neg\mathbf{L}\alpha, \Sigma; \Gamma \mid\!\sim \Delta} \qquad (\mathbf{sD3'})\ \frac{\Gamma'' \not\vdash \alpha}{\mathbf{L}\alpha, \Sigma; \Gamma \mid\!\sim \Delta}$$

$$(\mathbf{sD4})\ \frac{\neg\mathbf{L}\neg\beta_1, \ldots, \neg\mathbf{L}\neg\beta_n, \Sigma; \Gamma, \frac{\alpha}{\gamma} \mid\!\sim \Delta \qquad \mathbf{L}\neg\beta_1, \Sigma; \Gamma \mid\!\sim \Delta \quad \ldots \quad \mathbf{L}\neg\beta_n, \Sigma; \Gamma \mid\!\sim \Delta}{\Sigma; \Gamma, \frac{\alpha:\beta_1\ldots\beta_n}{\gamma} \mid\!\sim \Delta}$$

where $\Sigma' \subseteq \{\alpha \mid \mathbf{L}\alpha \in \Sigma\}$, $\Gamma' \subseteq \Gamma \cap \mathcal{L}^{res}$, and $\Gamma'' = (\Gamma \cap \mathcal{L}) \cup \left\{ \frac{p(\delta)}{c(\delta)} \,\middle|\, \delta \in \Gamma \right\}$.

Fig. 7. Inference rules for the enhanced skeptical default calculus BO'_{skep}.

our next result we will show an exponential lower bound to the proof length (and therefore also to the proof size) in the enhanced skeptical calculus BO'_{skep}.

Theorem 15. *The calculus BO'_{skep} has exponential lower bounds to the lengths of proofs. More precisely, there exist sequents S_n of size $\mathcal{O}(n)$ such that every BO'_{skep}-proof of S_n uses $2^{\Omega(n)}$ steps. Therefore, $s_{BO'_{skep}}(n), t_{BO'_{skep}}(n) \in 2^{\Omega(n)}$.*

Proof. (Sketch) We construct a sequence $(S_n)_{n\geq 1} = (\Sigma_n; \Gamma_n \mid\!\sim \psi_n)_{n\geq 1}$ such that for some constant c, every BO'_{skep}-proof of S_n has length at least $2^{\Omega(n)}$. We choose $\Sigma_n = \emptyset$, $\psi_n = A_{2n}$, and $\Gamma_n = \langle \emptyset, D_{2n} \rangle$, where D_{2n} consists of the defaults listed in Fig. 8. The default theory Γ_n possesses 2^{n+1} stable extensions. Observe that each of these contains A_{2n}, but that each pair of stable extensions differs in truth assigned to the propositional variables $A_0, \ldots, A_n$. We claim that every proof of S_n has exponential length in n. More precisely, we will show that rule ($\mathbf{sD4}$) has to be applied an exponential number of times.

We point out that our argument does not only work against tree-like proofs, but also rules out the possibility of sub-exponential dag-like derivations for $D_{2n} \mid\!\sim A_{2n}$. The lower bound is obtained from the fact that to derive A_{2n}, we have to derive A_i and $\neg A_i$ for each $n < i < 2n$, each of which can only be achieved from ancestors with mutually different proof constraints. This, by definition of BO_{skep}, leads to mutually disjoint sets of ancestor sequents.

The complete proof of the theorem is contained in the appendix. □

6 Conclusion

In this paper we have shown that with respect to lengths of proofs, proof systems for credulous default reasoning and for propositional logic are very close to each other. Although deciding credulous default sequents is presumably harder than deciding tautologies (the former is Σ_2^{p}-complete [13], while the latter is complete for coNP), the difference disappears when we want to prove these objects (Sect. 4.2).

$$
\begin{array}{cccc}
 & \dfrac{:\ A_0}{A_0} & \dfrac{:\ \neg A_0}{\neg A_0} & \\[2ex]
\dfrac{A_0\ :\ A_1}{A_1} & \dfrac{\neg A_0\ :\ A_1}{A_1} & \dfrac{A_0\ :\ \neg A_1}{\neg A_1} & \dfrac{\neg A_0\ :\ \neg A_1}{\neg A_1} \\[2ex]
 & \vdots & & \\[2ex]
\dfrac{A_{n-1}\ :\ A_n}{A_n} & \dfrac{\neg A_{n-1}\ :\ A_n}{A_n} & \dfrac{A_{n-1}\ :\ \neg A_n}{\neg A_n} & \dfrac{\neg A_{n-1}\ :\ \neg A_n}{\neg A_n} \\[2ex]
\dfrac{A_n\ :\ A_{n-1}}{A_{n+1}} & \dfrac{\neg A_n\ :\ A_{n-1}}{A_{n+1}} & \dfrac{A_n\ :\ \neg A_{n-1}}{\neg A_{n+1}} & \dfrac{\neg A_n\ :\ \neg A_{n-1}}{\neg A_{n+1}} \\[2ex]
 & \vdots & & \\[2ex]
\dfrac{A_{2n-2}\ :\ A_1}{A_{2n-1}} & \dfrac{\neg A_{2n-2}\ :\ A_1}{A_{2n-1}} & \dfrac{A_{2n-2}\ :\ \neg A_1}{\neg A_{2n-1}} & \dfrac{\neg A_{2n-2}\ :\ \neg A_1}{\neg A_{2n-1}} \\[2ex]
\dfrac{A_{2n-1}\ :\ A_0}{A_{2n}} & \dfrac{\neg A_{2n-1}\ :\ A_0}{A_{2n}} & \dfrac{A_{2n-1}\ :\ \neg A_0}{A_{2n}} & \dfrac{\neg A_{2n-1}\ :\ \neg A_0}{A_{2n}}
\end{array}
$$

Fig. 8. The defaults in D_{2n} in the proof of Theorem 15.

For skeptical reasoning this is less clear. While skeptical default reasoning has polynomially bounded proof systems if and only if this holds for TAUT, we leave open whether this equivalence extends to other bounds. However, in the light of our exponential lower bound for BO'_{skep} (Theorem 15), searching for natural proof systems for skeptical default reasoning with more concise proofs will be a rewarding task for future research.

In this direction Bonatti and Olivetti [4] themselves introduced two rules to supplement their enhanced calculus. These are the cut rule

$$\frac{\Sigma; \Gamma \mathrel{|\!\!\sim} \alpha \qquad \Sigma; \Gamma, \alpha \mathrel{|\!\!\sim} \Delta}{\Sigma; \Gamma \mathrel{|\!\!\sim} \Delta}\ (\mathbf{Cut})$$

and the following version of the rule (**sD4**)

$$\frac{\Sigma_0, \Sigma; \Gamma, \frac{\alpha}{\gamma} \mathrel{|\!\!\sim} \Delta \qquad \Sigma_1, \Sigma; \Gamma \mathrel{|\!\!\sim} \Delta \quad \dots \quad \Sigma_n, \Sigma; \Gamma \mathrel{|\!\!\sim} \Delta}{\Sigma; \Gamma, \frac{\alpha:\beta_1\dots\beta_n}{\gamma} \mathrel{|\!\!\sim} \Delta}\ (\mathbf{sD4'})$$

where $\Sigma_i = \mathbf{L}\neg\beta_{\pi(i)}, \neg\mathbf{L}\neg\beta_{\pi(i+1)}, \dots, \neg\mathbf{L}\neg\beta_{\pi(n)}$ for an arbitrary permutation π of $\{1, \dots, n\}$. While it is not hard to see that our lower bound in Theorem 15 still remains true if we add ($\mathbf{sD4'}$) to BO'_{skep}, we leave open the problem to show super-polynomial lower bounds in the presence of the cut rule.

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Technical Appendix

The appendix contains the full proof of Theorem 15 which was only briefly sketched in the main part of the paper.

Theorem 15. *The calculus BO'_{skep} has exponential lower bounds to the lengths of proofs. More precisely, there exist sequents S_n of size $\mathcal{O}(n)$ such that every BO'_{skep}-proof of S_n uses $2^{\Omega(n)}$ steps. Therefore, $s_{BO'_{skep}}(n), t_{BO'_{skep}}(n) \in 2^{\Omega(n)}$.*

Proof. We construct a sequence $(S_n)_{n\geq 1} = (\Sigma_n; \Gamma_n {\mid\!\sim} \psi_n)_{n\geq 1}$ such that for some constant c, every BO'_{skep}-proof of S_n has length at least $2^{\Omega(n)}$. We choose $\Sigma_n = \emptyset$, $\psi_n = A_{2n}$, and $\Gamma_n = \langle \emptyset, D_{2n} \rangle$, where D_{2n} consists of the defaults listed in Fig. 8. The default theory Γ_n possesses 2^{n+1} stable extensions. Observe that each of these contains A_{2n}, but that each pair of stable extensions differs in truth assigned to the propositional variables $A_0, \ldots, A_n$. We claim that every

$$\frac{:\ A_0}{A_0} \qquad \frac{:\ \neg A_0}{\neg A_0}$$

$$\frac{A_0\ :\ A_1}{A_1} \qquad \frac{\neg A_0\ :\ A_1}{A_1} \qquad \frac{A_0\ :\ \neg A_1}{\neg A_1} \qquad \frac{\neg A_0\ :\ \neg A_1}{\neg A_1}$$

$$\vdots$$

$$\frac{A_{n-1}\ :\ A_n}{A_n} \qquad \frac{\neg A_{n-1}\ :\ A_n}{A_n} \qquad \frac{A_{n-1}\ :\ \neg A_n}{\neg A_n} \qquad \frac{\neg A_{n-1}\ :\ \neg A_n}{\neg A_n}$$

$$\frac{A_n\ :\ A_{n-1}}{A_{n+1}} \qquad \frac{\neg A_n\ :\ A_{n-1}}{A_{n+1}} \qquad \frac{A_n\ :\ \neg A_{n-1}}{\neg A_{n+1}} \qquad \frac{\neg A_n\ :\ \neg A_{n-1}}{\neg A_{n+1}}$$

$$\vdots$$

$$\frac{A_{2n-2}\ :\ A_1}{A_{2n-1}} \qquad \frac{\neg A_{2n-2}\ :\ A_1}{A_{2n-1}} \qquad \frac{A_{2n-2}\ :\ \neg A_1}{\neg A_{2n-1}} \qquad \frac{\neg A_{2n-2}\ :\ \neg A_1}{\neg A_{2n-1}}$$

$$\frac{A_{2n-1}\ :\ A_0}{A_{2n}} \qquad \frac{\neg A_{2n-1}\ :\ A_0}{A_{2n}} \qquad \frac{A_{2n-1}\ :\ \neg A_0}{A_{2n}} \qquad \frac{\neg A_{2n-1}\ :\ \neg A_0}{A_{2n}}$$

Fig. 8. The defaults in D_{2n}.

proof of S_n has exponential length in n. More precisely, we will show that rule (**sD4**) has to be applied an exponential number of times.

To this end, let Π be a BO'_{skep}-proof of $D_{2n} {\mid\!\sim} A_{2n}$. We show that Π has to contain an application of (**sD4**) to a default rule deriving A_i or $\neg A_i$ for any sequent

$$\Sigma; D, R {\mid\!\sim} A_{2n} \tag{1}$$

such that Σ is consistent and D_{2n} can be partitioned into three sets I_1, I_2, I_3 satisfying

1. $\neg\mathbf{L}\neg j(\delta) \in \Sigma$ and $\frac{p(\delta)}{c(\delta)} \in R$ if $\delta \in I_1$,
2. $\mathbf{L}\neg j(\delta) \in \Sigma$ if $\delta \in I_2$,
3. $\delta \in D$ if $\delta \in I_3$, and
4. $\{A_i, \neg A_i\} \cap \{c(\delta) \mid \delta \in I_1\} = \emptyset$ for some $n < i \leq 2n$

To prove this claim, let $\Sigma; D, R \mid\!\sim A_{2n}$ be a sequent as stated above and $n < i \leq 2n$ be such that $\{A_i, \neg A_i\} \cap \{c(\delta) \mid \delta \in I_1\} = \emptyset$. Suppose that Π does not contain any application of (**sD4**) to default rules deriving A_i or $\neg A_i$. Consequently, $\Sigma; D, R \mid\!\sim A_{2n}$ is derived by an application of (**sD1'**), (**sD2'**), (**sD3'**) or (**sD4**) to a default rule not deriving A_i or $\neg A_i$. We distinguish among these possibilities.

(**sD1'**) Suppose $\Sigma; D, R \mid\!\sim A_{2n}$ were derived by an application of (**sD2'**), then Π had to contain the the sequent $\Sigma', R \vdash A_{2n}$, where $\Sigma \subseteq \{A_{2n-k}, \neg A_{2n-k} \mid n \leq k \leq 2n\}$. By the fourth condition, $\{A_i, \neg A_i\} \cap \{c(\delta) \mid \delta \in I_1\} = \emptyset$. Hence, R cannot contain any of the residual rules $\frac{\alpha_{i-1}}{\alpha_i}$ with $\alpha_j \in \{A_j, \neg A_j\}$. Consequently, $\Sigma'; R \mid\!\sim A_{2n}$ cannot be closed.

(**sD2'**) If $\Sigma; D, R \mid\!\sim A_{2n}$ were derived by an application of (**sD2'**), then Π had to contain the antecedent $\Sigma', D, R \mid\!\sim \neg\alpha_{2n-k}$, where $\alpha_{2n-k} \in \{A_{2n-k}, \neg A_{2n-k}\}$ with $n \leq k \leq 2n$ and $\Sigma' := \Sigma \setminus \{\neg\mathbf{L}\neg\alpha_{2n-k}\}$. However, $\Sigma'; D, R \mid\!\sim \neg\alpha_{2n-k}$ could in turn only be closed by using either of the default rules

$$\frac{A_{2n-k-1} \ : \ \tilde{\alpha}_{2n-k}}{\tilde{\alpha}_{2n-k}}, \quad \frac{\neg A_{2n-k-1} \ : \ \tilde{\alpha}_{2n-k}}{\tilde{\alpha}_{2n-k}},$$

where $\tilde{\alpha}_{2n-k} \equiv \neg\alpha_{2n-k}$: no other rules derives $\neg A_{2n-k}$. Say that Π contains an application of the first rule. By consistency of Σ, $\frac{A_{2n-k-1}:\tilde{\alpha}_{2n-k}}{\tilde{\alpha}_{2n-k}}$ has to be contained in D. Suppose w.l.o.g. that Π contains this application in the previous step of Π. Then we obtain as the right ancestor sequent $\Sigma'; \mathbf{L}\neg\tilde{\alpha}_{2n-k}; D', R \mid\!\sim \tilde{\alpha}_{2n-k}$, where $D' := D \setminus \left\{\frac{A_{2n-k-1}:\tilde{\alpha}_{2n-k}}{\tilde{\alpha}_{2n-k}}\right\}$. But $\Sigma', \mathbf{L}\neg\tilde{\alpha}_{2n-k}; D', R \mid\!\sim \tilde{\alpha}_{2n-k}$ cannot be closed: The only default rule being able to derive $\tilde{\alpha}_{2n-k}$ remaining in D' has a premise that is contradictory to the premise of $\frac{A_{2n-k-1}:\tilde{\alpha}_{2n-k}}{\tilde{\alpha}_{2n-k}}$. By soundness of BO'_{skep}, the ability to close this sequent would therefore contradict the consistency of D_{2n}. The case that Π contains an application of the second rule, $\frac{\neg A_{2n-k-1}:\tilde{\alpha}_{2n-k}}{\tilde{\alpha}_{2n-k}}$, is completely analogous.

(**sD3'**) Similarly, if the sequent $\Sigma; D, R \mid\!\sim A_{2n}$ were derived by an application of the rule (**sD3'**), then Π contained the sequent $D', R \nvdash \neg\alpha_l$ for some α_l such that $\mathbf{L}\neg\alpha_l \in \Sigma$, where $D' = \left\{\frac{p(\delta)}{c(\delta)} \,\middle|\, \delta \in D\right\}$. But if $D', R \nvdash \neg\alpha_l$ were true, then there had to exist an $0 \leq j \leq l$ such that neither of the rules $\frac{A_{j-1}:\alpha_j}{\alpha_j}$, $\frac{\neg A_{j-1}:\alpha_j}{\alpha_j}$, where $\alpha_j \in \{A_j, \neg A_j\}$ for $j < l$, nor one of their residues could be contained in $D \cup R$. Consequently, Π would again have to contain the proof constraints $\mathbf{L}\neg\alpha_j, \mathbf{L}\neg\neg\alpha_j \in \Sigma$, contradictory to the consistency of Σ.

(**sD4'**) Suppose that $\Sigma; D, R \mid\!\sim A_{2n}$ is derived by an application of (**sD4**) to the default rule $\frac{\alpha_{k-1}:\alpha_{2n-k}}{\alpha_k} \in D$ with $\alpha_j \in \{A_j, \neg A_j\}$ and $n < k \neq i$. Then Π contains the two ancestor sequents $\Sigma, \neg\mathbf{L}\neg\alpha_{2n-k}; D, R, \frac{\alpha_{k-1}}{\alpha_k} \mid\!\sim A_{2n}$ and $\Sigma, \mathbf{L}\neg\alpha_{2n-k}; D, R \mid\!\sim A_{2n}$. But as $\Sigma, \neg\mathbf{L}\neg\alpha_{2n-k}; D, R, \frac{\alpha_{k-1}}{\alpha_k} \mid\!\sim A_{2n}$ still does not contain any residual rule deriving A_i or $\neg A_i$, the same arguments as for $\Sigma; D, R \mathrel{|\!\!\sim} A_{2n}$ apply.

Concluding, the containment of $\Sigma; D, R \mathrel{|\!\!\sim} A_{2n}$ in Π enforces an application of (**sD4**) to a default rule with conclusion A_i or $\neg A_i$. This yields the ancestor sequents $\Sigma, \neg \mathbf{L} \neg \alpha_{2n-i}; D, R, \frac{\alpha_{i-1}}{\alpha_i} \mathrel{|\!\!\sim} A_{2n}$ and $\Sigma, \mathbf{L} \neg \alpha_{2n-i}; D, R \mathrel{|\!\!\sim} A_{2n}$. The latter of these still satisfies the requirements of (1). Thus, by the same arguments as above, Π has to contain an application of (**sD4**) to a default rule $\frac{\alpha'_{i-1}:\alpha'_{2n-i}}{\alpha'_i}$, where $\alpha'_i \equiv \neg \alpha_i$ and $\alpha'_{2n-i} \equiv \neg \alpha_{2n-i}$. Each of these applications yields a sequent satisfying (1) unless for these $\{A_i, \neg A_i\} \cap \{c(\delta) \mid \delta \in I_1\} \neq \emptyset$ holds for all $n < i \leq 2n$; however, with mutually different proof constraints.

Summing up, to prove $D_{2n} \mathrel{|\!\!\sim} A_{2n}$, Π has to contain 2^{2n-i+1} applications of (**sD4**) to default rules with conclusion A_i or $\neg A_i$. Therefore, every proof of S_n has length at least $2^{\Omega(n)}$. $\square$

We point out that the above argument does not only work against tree-like proofs, but also rules out the possibility of sub-exponential dag-like derivations for $D_{2n} \mathrel{|\!\!\sim} A_{2n}$. The lower bound is obtained from the fact that to derive A_{2n}, we have to derive residual rules concluding A_i and $\neg A_i$ for each $n < i \leq 2n$, each of which can only be achieved from ancestors with mutually different proof constraints. This, by definition of BO'_{skep}, leads to mutually disjoint sets of ancestor sequents.